Axiomatic higher torsion

and answers to questions

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Questions (for me)

- Can you explain higher Franz-Reidemeister (FR) torsion in a way that everyone can understand it? - NSF referee
- What does analytic torsion have to do with the moduli space?
- Why do the Pontrjagin classes not satisfy the axioms?
- What is even torsion and odd torsion?
- How do you extend the higher FR-torsion to the mapping class group?
- Can you get the other classes (cohomology classes in $B$ that are pushed down from $E$?)
- Where do $\zeta(2k + 1)$ and the polylogarithms come from?
Questions for audience

Take the Torelli group

$$\mathcal{I}^1_{g,n} = \ker (M^1_{g,n} \to Sp(2g, \mathbb{Z}))$$

of homologically framed surfaces $\Sigma$ with one (fixed) boundary component and $n$ marked points. Homological framing = fixed basis for $H_\ast(\Sigma)$

- Are $\psi_i^2 \in H^4(\mathcal{I}_{g,n}; \mathbb{Q})$ nonzero?
- Are $\kappa_{2k} \in H^{4k}(\mathcal{I}_{g,n}; \mathbb{Q})$ nonzero?
Def: A higher torsion invariant is a real characteristic class of unipotent smooth bundles satisfying two axioms: additivity and transfer.

Every higher torsion invariant has even and odd parts:

\[ \tau = \tau^+ + \tau^- \]

The even part \( \tau^+ \) is a scalar multiple of the (generalized) Miller-Morita-Mumford (MMM) class \( \kappa_{2k} \)

The odd part \( \tau^- \) is a scalar multiple of the odd part of the higher FR-torsion

Conjecture (S. Goette): Nonequivariant higher analytic torsion classes are odd torsion invariants.
Axioms

- **additivity** If \( E = E_1 \cup E_2 \) then

  \[
  \tau(E) = \frac{1}{2} \tau(DE_1) + \frac{1}{2} \tau(DE_2)
  \]

- **transfer** If \( S^n \to D \to E \) is an oriented linear sphere bundle. Then the torsion \( \tau_E(D) \), \( \tau_B(D) \) of \( D \) as a bundle over \( E, B \) resp. are related by:

  \[
  \tau_B(D) = \chi(S^n) \tau(E) + tr^E_B(\tau_E(D))
  \]

  \[
  tr^E_B(x) = p_*(x \cup e(E))
  \]

  where \( e(E) \in H^n(E) \) is the Euler class of the vertical tangent bundle.
Calculation of higher torsion

Parameters $s_1, s_2$ are given by

$$\tau(S^n(\lambda)) = 2s_n ch_{4k}(\lambda) \in H^{4k}(B; \mathbb{R})$$

where $S^n(\lambda)$ is an $S^1$ or $S^2$ bundle associated to a complex line bundle $\lambda$ over $B$.

Theorem: Let $f : E \to \mathbb{R}$ be a fiberwise Morse function with distinct critical values. Let $\xi_j, \eta_j$ be the negative and positive eigenspace bundles associated to the $j$th critical point. then

$$\tau(E) = \sum_j (-1)^i (s_1 + s_2) ch_{4k}(\eta_j) + (-1)^i (s_2 - s_1) ch_{4k}(\xi_j)$$

where $i$ is the index of the critical point.
The MMM classes, for closed fiber $F$, are given by

$$\kappa_{2k}(E) = tr_B^E ((2k)! ch_{4k}(T^v E))$$

where $ch_{4k}(T^v E) = \frac{1}{2} ch_{4k}(T^v E \otimes \mathbb{C})$.

This is a higher torsion invariant with

$$s_1 = 0$$

$$s_2 = (2k)!$$

Thm: Every even torsion invariant is a scalar multiple of $\kappa_{2k}$.
Higher Franz-Reidemeister torsion

Let $K(Z) = \mathbb{Z} \times BGL(\infty, \mathbb{Z})^+$. Then

$$H^*(\Omega K(Z); \mathbb{R}) \cong \mathbb{R}[b_2, b_4, \cdots]$$

$b_i$: Borel regulator (continuous cohomology) classes

$E \to B$ unipotent get: $C: B \to \Omega K(Z)$.

$$\tau^{FR}_{2k}(E) = C^*(b_{2k})$$

This is a higher torsion invariant with

$$s_i = \frac{1}{2}(-1)^{k+i}\zeta(2k + 1)$$

Thm: Every odd higher torsion invariant is a scalar multiple of the odd part of $\tau^{FR}_{2k}$.
Bismut-Lott analytic torsion

- Nonequivariant analytic torsion is a $4k$ form on $B$ which is closed when $\pi_1 B$ acts trivially on $H_\ast(F; \mathbb{R})$.

- Analytic torsion classes satisfy the transfer axiom (X. Ma).

- Additivity, an easy property of higher FR torsion, is unknown.

- Parameters are $s_2 = 0$,

$$s_1 = (-1)^k (2\pi)^{-2k} \frac{(4k + 1)!}{24k(2k)!} \zeta(2k + 1)$$

- Conjecture (which S. Goette now claims is true): $\tau_{2k}^{BL}$ is a scalar multiple of the odd part of $\tau_{2k}^{FR}$. 

Axiomatic higher torsion – p.9/11
Mapping class group

- Axiomatic higher torsion is only defined on (the classifying space $B\mathcal{I}_g$ of) the Torelli group.

- Higher FR torsion is defined on the mapping class group $\mathcal{M}_g$:

$$\tau_{2k}^{FR}(\mathcal{M}_g) = \frac{(-1)^k \zeta(2k + 1)}{2(2k)!} \kappa_{2k} \in H^{4k}(\mathcal{M}_g; \mathbb{R})$$

- Equivariant higher FR-torsion gives the odd kappa’s:

$$\tau_k^{FR}(\mathcal{M}_g; \zeta) = \frac{1}{2} m^k L_{k+1}(\zeta) \frac{1}{k!} \kappa_k \in H^{2k}(\mathcal{M}_g; \mathbb{R})$$

$$\zeta^m = 1, \quad L_{k+1}(\zeta) = \mathcal{R} \left( \frac{1}{i^k} \sum \zeta^n / n^{k+1} \right)$$
Outer automorphism group

- $F_n$: free group on $n$ letters.
- $Out(F_n) = Aut(F_n)/Inn(F_n)$: outer automorphism group.
- $BOut(F_n)$ is the moduli space of graphs $G \simeq \vee_n S^1$.
- $IOut(F_n)$ is defined to be the kernel:

$$1 \to IOut(F_n) \to Out(F_n) \to GL(n, \mathbb{Z}) \to 1$$

- Axiomatic torsion is defined on $IOut(F_n)$:

$$\tau_{2k}(IOut(F_n)) \in H^{4k}(IOut(F_n); \mathbb{R})$$

- $\phi_j : M_{g,n} \to Out(F_{2g})$, $\phi_j : \mathcal{I}_{g,n} \to IOut(F_{2g})$

$$\phi_j^*(\tau_{2k}) = \frac{-2s_1}{(2k)!} \left( \kappa_{2k} - \psi_{2k}^j \right) \in H^{4k}(\mathcal{I}_{g,n}; \mathbb{R}).$$