AIM Workshop on Moment Maps and Surjectivity in Various Geometries (August 9–13, 2004): Conjectures and Further Questions

Contents

INTRODUCTION

The following are a set of the questions, conjectures, and possible further research directions that were discussed at the AIM Workshop on Moment Maps and Surjectivity in Various Geometries that took place August 9–13, 2004. The questions are roughly grouped into topics. Moreover, the name of the workshop participant who offered or posed a particular question, insight, or comment has (as much was possible) been preserved.

This set of notes was prepared for the purpose of preserving and offering to a wide audience (including, but not limited to, the workshop participants) a sampling of questions which the participants feel are interesting and important for future work in this field. We hope that those who search here will find interesting projects and directions to pursue (and perhaps Ph.D. dissertation topics for graduate students).

Comments on these notes can be sent to `workshops@aimath.org`

Good luck! — Megumi Harada

1 HyperKähler Kirwan surjectivity

We first explain briefly some history. Frances Kirwan, in an unpublished manuscript [Kir85], gave a proof of Kirwan surjectivity for hyperKähler manifolds under certain conditions on the moment map and the gradient flow. In this manuscript, she gave two different proofs of this result: one is contained in Sections 3 and 4, and the other is contained in Section 5. She later found an error in her argument in Section 4 (Lemma 4.2), and she never published the manuscript.

The existence of this unpublished manuscript was known to some participants at the AIM workshop, and unknown to others. There were some lively discussions regarding Kirwan's manuscript in the AIM discussion sessions. We summarize below what is stated in the unpublished manuscript and the results of the discussions at the AIM workshop.

Conjecture 1.1 *Let X be a hyperKähler manifold with a hyperHamiltonian action of a compact Lie group G. Let f denote the norm-square of the hyperKähler moment map. Suppose that for every $x \in X$ the forward trajectory of x under the negative gradient flow of f is contained in a compact subset of X. Then there is a surjection*

$$H_K^*(X) \to H_K^*(\mu_{HK}^{-1}(\mathbf{0}))$$

in K-equivariant cohomology.

Comment 1.2 Kirwan, in stating her theorems, actually requires that one of the following hold:

1. The original hyperKähler manifold is compact.
2. The norm-square of the hyperKähler moment map is proper.
3. For every $x \in X$, the forward trajectory of x under the negative gradient flow of the norm-square of the hyperKähler moment map is contained in a compact subset of X.

(Note that 1 implies 2 implies 3.) The first and second hypotheses are almost never fulfilled, so the only relevant hypothesis is the last one. We have therefore stated Conjecture 1.1 using this last hypothesis.

Comment 1.3 Section 3 of Kirwan's manuscript deals with the case where the group is abelian. The participants at the AIM workshop tentatively agreed that in this case, under the hypotheses of the conjecture above, the argument seems to be identical to the one given by Kirwan for the original surjectivity (for the symplectic case). The only difference is that we replace the moment map by the hyperKähler moment map. However, we (the AIM participants) did not check the details in this case.

Comment 1.4 Kirwan's third hypothesis (that the gradient flows are contained in compact sets) holds for linear G-actions on $T^*\mathbb{C}^n$, by an argument similar to the one given by Sjamaar [Sja98] for linear G-actions on $\mathbb{C}^n$. Here, G can be nonabelian. Thus, if the conjecture is correct, we have Kirwan surjectivity for quiver varieties.

Comment 1.5 Section 5 of Kirwan's manuscript, which gives her second proof of hyperKähler Kirwan surjectivity, uses the "plus construction" of Carrell-Goresky [CG83] when $\mu_{\mathbb{C}}^{-1}(0)$ is singular. (In almost all examples of interest, $\mu_{\mathbb{C}}^{-1}(0)$ is indeed singular.) The workshop participants were confused on the following points: first, the results in Carrell-Goresky are stated for homology, not cohomology. Second, their results are not stated equivariantly. Regarding the first point, S. Wu believes that having the result for homology is not a problem; as long as there is a filtration, there are spectral sequences for both homology and cohomology. Regarding the second, S. Wu and J. Weitsman had a discussion during the workshop in which they seemed to conclude that the flow of the square of the moment map seems to define an $\mathbb{R}^2$ action rather than a $\mathbb{C}^*$ action, so we did not understand why the theorem in [CG83] can be applied. The workshop participants hope to come to a better understanding on these points.

2 3-Sasakian surjectivity

Conjecture 2.1 (C. Boyer) *Let S be a complete compact 3-Sasakian manifold with a connected compact Lie group $G \subset Aut(S)$ acting on S. Consider the 3-Sasakian reduction S_{red} (reduction at 0). Then the 3-Sasakian Kirwan map*

$$H_G^*(S) \to H^*(S_{red})$$

is surjective up to the middle dimension. Here we take either $\mathbb{Q}$ or $\mathbb{R}$ coefficients.

Comment 2.2 (C. Boyer) We know the conjecture is true for S^1-reductions of spheres. Note also that in odd dimension, "up to the middle dimension" is all we can hope for. Boyer, Galicki, and Piccinni [BGP02] have constructed some examples of 3-Sasakian quotients for which they don't know how to compute cohomology, and Kirwan surjectivity would be a nice way to get at some cohomology.

Comment 2.3 (C. Boyer and T. Hausel) We might need some additional hypotheses for the nonabelian case, but T. Hausel has some ideas about how to prove something like this for abelian reductions.

Namely, the surjectivity in the 3-Sasakian case should be related to surjectivity for hyperKähler quotients.

We consider a more specific example. Suppose there is an action of $Sp(n+1)$ acting on S^{4n+3}. Restrict to an action of a compact subgroup G of this $Sp(n+1)$. Consider the 3-Sasakian reduction $S^{4n+3}/\!\!/\!\!/G$, and additionally assume that this is orbifold. We conjecture that, in this case, the 3-Sasakian Kirwan map is surjective to the middle dimension of the quotient.

Here is a tentative idea of the proof. Look at the correponding hyperKähler quotient. Consider the same group G acting now on $\mathbb{H}^{n+1}$, which containes S^{4n+3} as its sphere. Let X denote now the hyperKähler quotient. The space X might be singular since G might not have a center. But now suppose there exists some ξ in the center of G so that the hyperKähler quotient is orbifold. Assume also that the quotient is hypercompact (see Definition 6.9), so the core is middle-dimensional. Then, in this case, the surjectivity of the hyperKähler Kirwan map should imply the surjectivity (up to middle dimension) of the 3-Sasakian Kirwan map. T. Hausel thinks this should be an elementary argument. C. Boyer and T. Hausel will be pursuing this line of thought after the AIM Workshop ends.

Question 2.4 (T. Hausel) Is there a Martin theorem for 3-Sasakian quotients? What about for Sasakian quotients? Getting a Martin-type formula requires Kirwan surjectivity, but perhaps surjectivity "up to middle dimension" would be enough to get a Martin-type theorem for the 3-Sasakian case.

Comment 2.5 (J. Munn) Martin's theorem probably should be extendable analytically in the polysymplectic case. Since 3-Sasakian manifolds can arise as boundaries of hyperKähler manifolds, we can look at compact cohomology based on the 3-Sasakian manifold (regarded as a boundary). Here we can ignore cone points and do integration theory.

Question 2.6 (C. Boyer) In the case of 3-Sasakian reduction by abelian groups, would it be possible to extract information about the cohomology of the reductions by using fixed points of smaller tori?

The question is motivated by the fact that some cases of 3-Sasakian reductions of spheres by S^1 is well-understood. Suppose there is a circle action on S^{4n+3}, where the action is locally free on the whole sphere, and free on the zero level set. Then one can see explicitly that $H^*_{S^1}(S^{4n+3};\mathbb{Q})$ surjects onto $H^*(S^{4n+3}/\!\!/\!\!/S^1;\mathbb{Q})$, up to the middle dimension.

The idea would be to look at the fixed point set M^H for subtori $H \subset T$, where we start with the abelian group T acting on the 3-Sasakian manifold M. Note that the fixed point sets M^H are contact, and for the Sasakian case, known examples of M^H are spheres.

Comment 2.7 (C. Boyer) Main references for 3-Sasakian toral reductions are [BG99, BGM94, BGMR98].

3 Kirwan surjectivity for contact quotients

Question 3.1 (E. Lerman) Kirwan surjecitivty cannot work for contact quotients. Here is a counterexample: consider $S^3 \subset \mathbb{C}^2$ with the action of S^1 given by $\lambda \cdot (z_1, z_2) = (\lambda z_1, \lambda^{-1} z_2)$. The S^1-equivariant cohomology of this S^3 is the ordinary cohomology of $\mathbb{P}^1$. On the other hand, the contact moment map is $\mu : (z_1, z_2) \mapsto |z_1|^2 - |z_2|^2$. Hence the contact quotient is $\mu^{-1}(0)/S^1 = S^1$. There is no surjective map from $H^*(\mathbb{P}^1)$ to $H^*(S^1)$.

Nevertheless, there are still interesting questions to ask: what is the kernel? What is the cokernel?

Comment 3.2 (C. Boyer) It would also be interesting to ask under what conditions a contact Kirwan surjectivity would hold. There may be a class of spaces for which surjectivity would hold in the 3-Sasakian case (although perhaps with some dimension condition, as in the "up to middle dimension" clause in the 3-Sasakian surjectivity conjecture above).

4 Orbifold cohomology and surjectivity

The starting point of this discussion is the theorem of Goldin, Holm, and Knutson.

Theorem 4.1 (Goldin-Holm-Knutson) *Let (M, ω) be a compact Hamiltonian T-space with moment map μ. Let α be a regular value of μ. Then the direct sum*

$$\oplus_{g \in T} H^*_T(M^g; \mathbb{Q})$$

has a ring structure such that there exists a natural ring map

$$\oplus_{g \in T} H^*_T(M^g; \mathbb{Q}) \to H^*_{orb}(M /\!\!/_\alpha T; \mathbb{Q})$$

is a surjection. Here M^g denotes the points in M fixed by the element $g \in T$, and orbifold cohomology H^*_{orb} is in the sense of Chen and Ruan.

Comment 4.2 (E. Lerman) A more standard definition of $H_o rb^*$ is due to Haefliger; it precedes Chen-Ruan's definition by at least a decade or two.

Question 4.3 (M. Pflaum) Is there an analogous theorem for compact non-abelian Lie group G?

Comment 4.4 (M. Pflaum) The generalization to non-abelian G would be interesting from the point of view of quantization of singular reduced spaces and cross product algebras, as they appear in symplectic orbifold theory.

Comment 4.5 (R. Goldin, T. Holm) The main problem would be in defining the product structure on the ring in the LHS of the statement of the theorem. In the theorem of Goldin-Holm-Knutson, they use in a fundamental way the commutativity of the group structure when defining this product structure.

Comment 4.6 (R. Goldin, T. Holm, M. Pflaum) We expect a theorem of this nature to be true if we take only the additive structure.

Note also that it is clear that we can't simply replace T by G naively. One idea for how to adjust the theorem would be to take the product in the LHS over conjugacy classes of G instead of all elements of G. Another idea would be to take the product on the LHS to be

$$\prod_{g \in G} H_G^*(\cup_{h \in G} M^{hgh^{-1}})$$

instead.

This question on a non-abelian version of the theorem of Goldin-Holm-Knutson was a topic during the small group discussion sessions at the AIM workshop. We now present some of the conclusions and comments arising from this discussion (R. Goldin, T. Holm, M. Pflaum).

One possible approach to this problem would be to use the groupoid language for orbifold cohomology. In other words, given an orbifold which is a symplectic reduction by a torus, we may translate the theorem of Goldin-Holm-Knutson into the language of proper étale Lie groupoids. We may then try to use the approach of Moerdijk on descriptions of orbifolds by proper étale Lie groupoids [MM03]. This approach could give a new description of the ring structure of the orbifold cohomology – at least, of orbifolds appearing as global quotients by a torus. It is possible that this new description would also suggest a product structure for the non-abelian case.

Comment 4.7 (C. Boyer) There may be some subtleties regarding Morita equivalent groupoids representing the same orbifold.

Another approach would be to use methods from Hochschild and cyclic homology theory. One can use

$$HC_*(\mathcal{C}^\infty \rtimes G)$$

to obtain the orbifold cohomology $H^*_{orb}(X, \mathbb{C})$.

Still another approach would be to use crepant resolutions (i.e. the crepant resolution conjecture for orbifold cohomology).

5 Topological aspects of moment map theory

Question 5.1 (E. Lerman) Kirwan surjectivity is not really a "symplectic" result — it's more a topological result which relies on certain *topological* properties of the moment map. So a similar result can be obtained for maps other than a moment map. See [LT97] for an example, where the topology of a small resolution of singular symplectic reduced space was computed using a map which was not a moment map. Can one prove similar results in the hyperKähler case?

There is a theory of "abstract moment maps" developed in [GGK02] which isolates those properties of moment maps that make the familiar theorems hold. The setting is roughly as follows; see [GGK02] for details. Suppose given an action of G on a manifold M. Note that M does not necessarily have a symplectic structure. An abstract moment map is a map $\mu : M \to \mathfrak{g}^*$ satisfying G-equivariance as well as some conditions involving subgroups H of G and its fixed points M^H. However, a valid example of an abstract moment map is the *constant* function on $\mathbb{R}^2$, where the group S^1 acts by rotation. Thus, to get a function suitable for Morse-Bott theory, we must add some condition of non-degeneracy.

We define, following [GGK02], an abstract moment map to be *non-degenerate* if its components are Morse-Bott, and for each component, the critical sets for that component correspond to the fixed point set M^G.

Conjecture 5.2 *(G. Landweber) Kirwan surjectivity holds under these conditions.*

Comment 5.3 (M. Harada, T. Holm, L. Mare) For S^1-actions, this is shown (under the additional assumption that M is compact) in [GGK02]. The proof for T-actions in [GGK02] contains an error.

Comment 5.4 (G. Landweber) Given an action of G on M with abstract moment map, can one find symplectic forms at least locally? If so, can we do the necessary geometry just using these local symplectic forms?

Comment 5.5 (M. Harada) The first question in the comment above is addressed in Theorem G.22 of [GGK02]. Roughly, it says that if the abstract moment map is non-degenerate and symplectic slices admit an invariant complex structure, then there is a symplectic form in a neighborhood of an orbit associated to the abstract moment map.

Question 5.6 (G. Landweber) In standard Morse theory, every function is arbitrarily close to a Morse function. In the space of abstract moment maps, is every abstract moment map arbitrarily close to a non-degenerate abstract moment map?

Question 5.7 (G. Landweber) We can model the local structure on M using abstract moment maps. Is there an analogous statement for contact moment maps? Is there a theory of abstract contact moment maps?

Question 5.8 (E. Lerman) Can we prove Kirwan surjectivity without Morse theory? The motivation for this question comes from the fact that two fundamental results in the theory of symplectic moment maps — connectedness and convexity, which were originally proved using Morse theory have an alternative proof [CDM88]. This alternative approach works well in the contact setting (equivalently in the setting of symplectic cones) where Morse theory fails. The reasons for the failure in the contact setting are due to the fact are that the contact moment maps are not Morse and that there is no relationship between critical points and isotropy groups. In the equivalent setting of symplectic cones the moment maps are not proper.

Comment 5.9 (E. Lerman) As mentioned in Question 5.8 above, for contact moment maps, one can show the convexity of the contact moment map image [Ler02] for tori of high enough dimension using methods of [CDM88]. A quick note on what "high enough" means: convexity fails for 2-tori and connectedness fails for circles. However, both are true for tori of dimension 3 and higher, as long as zero level set of the moment map is empty (which is true in the toric case).

Comment 5.10 (C. Boyer). There exists a version of Morse theory "through a range," as in the instanton moduli space. Perhaps this is applicable in the contact case.

6 The cohomology of hyperKähler quotients

This section is an edited version of T. Hausel's talk given at AIM during the workshop, in which he listed many conjectures and open problems. The main examples of hyperKähler manifolds considered in his talk were the moduli spaces of Higgs bundles on a Riemann surface, Nakajima's quiver varieties, and hypertoric manifolds (introduced by Bielawski-Dancer). Note that all examples hyperKähler quotients that T. Hausel considers here take the form $T^*A////G$, i.e. they are hyperKähler reductions of cotangent bundles to an affine space.

6.1 Generators for the cohomology ring of the quotient

A version of this conjecture was already presented as Conjecture 1.1, but we state it again here in T. Hausel's formulation, for good measure:

Conjecture 6.1 *(T. Hausel) The hyperkähler Kirwan map* $\kappa : H^*_G(T^*A) \cong H^*(BG) \to H^*(T^*A////_\xi G)$ *is surjective.*

Comment 6.2 Here is what is known about the three main examples (moduli spaces, quiver varieties, and hypertoric varieties).

- It is known for $\mathcal{M}^1_{Dol}(GL(2,\mathbb{C}))$ by (Hausel-Thaddeus 2000), for $\mathcal{M}^d_{Dol}(GL(n,\mathbb{C}))$ by (Markman 2001)
- For quiver varieties it is conjectured by (Nakajima 2002), for hyperpolygon spaces it is proven in (Konno 2000, Hausel-Proudfoot 2003)
- For hypertoric manifolds $\mathcal{M}(A,\xi)$ it is known by (Konno 2000, Hausel-Sturmfels 2002)

6.2 Integration theory on hyperKähler manifolds

To state the conjectures here, we must first make a few definitions.

Definition 6.3 A smooth oriented manifold M with a circle action $U(1)$ on M is *circle-compact* if the set of fixed points $M^{U(1)}$ is compact.

Note that the natural circle action on the fiber directions of T^*A induces a natural circle action on the hyperkähler quotients. With this circle action all of our examples (moduli spaces, quiver varieties, hypertoric varieties) are circle-compact.

Definition 6.4 (T. Hausel, N. Proudfoot) Let M be an oriented manifold with a $U(1)$ action such that M is circle-compact. Then the rationalized $U(1)$ equivariant cohomology is defined as the vector space

$$\hat{H}^*_{U(1)}(M) := H^*_{U(1)}(M) \otimes_{\mathbb{Q}[u]} \mathbb{Q}(u),$$

over the field $\mathbb{Q}(u)$ of rational functions. For $\alpha \in \hat{H}^*_{U(1)}(M)$ we define

$$\int_M \alpha := \sum_F \int_F \frac{i_F^*(\alpha)}{E(N_F)} \in \mathbb{Q}(u)$$

Comment 6.5 (S. Wu) There is related work of E. Prato and S. Wu [PW94] which takes the approach of interpreting the right-hand side of the definition above as a tempered distribution.

Comment 6.6 (T. Hausel) It would be interesting to make sense of this definition in terms of equivariant differential forms.

Question 6.7 (M. Libine) Here is another way to view integration theory. Let M be a noncompact symplectic manifold which is real algebraic. Let T be a compact torus, acting on M Hamiltonianly with a proper moment map μ. Assume that μ is semi-algebraic. Suppose M^T is compact. Let α be an equivariant form on M which is semi-algebraic as a map from $\mathfrak{g}$ to the space of differential forms on M (smooth, not necessarily polynomial). Then it should be possible to define the integral

$$\int_M \alpha$$

geometrically, so that the localization theorem holds. The integration takes values in smooth functions on the Lie algebra. The hypotheses given here should not be very restrictive.

The first result is

Theorem 6.8 *(Hausel-Proudfoot 2003) The pairing on $\hat{H}^*_{U(1)}(M)$ given by*

$$\int_M \alpha \wedge \beta$$

is non-degenarate.

This gives us a "Poincaré duality" for this pairing and allows us to do kernel computations.

Definition 6.9 A hyperkähler manifold M is *hypercompact* for the complex structure I, if there is a ω_I-Hamiltonian circle action on M, with proper moment map with finitely many critical points and a minimum, such that the holomorphic symplectic form $\omega_{\mathbb{C}} := \omega_J + i\omega_K$, for $\lambda \in \mathbb{C}^\times$ satisfies $\lambda^*\omega_{\mathbb{C}} = \lambda\omega_{\mathbb{C}}$.

Using this definition, we make the following

Conjecture 6.10 *(T. Hausel 2003) Let M^{4n} be a hyper-compact hyperkähler manifold and $\sigma(M)$ denote the signature (corresponding to the ordering, given by the sign of the leading term) of the pairing on $\hat{H}^*_{U(1)}(M)$. Then*

$$(-1)^n \sigma(M) \geq 0.$$

Comment 6.11 (T. Hausel) This conjecture is known in the hypertoric case, where it is in fact the Brown-Colbourn inequality for the the h-numbers of a matroid, which was first proved in the context of reliability of computer networks.

6.3 Abelianization

We now state some theorems and conjectures related to the *abelianization* procedure, given in the symplectic case by S. Martin.

Theorem 6.12 *(Hausel-Proudfoot 2003) In the construction of hyperkähler quotients let A be finite dimensional and G compact. Let $T \subset G$ be a maximal torus of G. Suppose that $T^*A////G$ and $T^*A////T$ are both circle compact. If $\alpha \in \widehat{H}^*_{U(1)\times G}(T^*A)$, then*

$$\int_{T^*A////G} \hat{\kappa}_G(\alpha) = \frac{1}{|W|} \int_{T^*A////T} \hat{\kappa}_T(\alpha) \wedge e,$$

where

$$e = \prod_{a\in\Delta} a(u-a) \in (\mathrm{Sym}\, \mathfrak{t}^*)^W \otimes \mathbb{Q}[u] \cong H_{U(1)\times G}(pt).$$

Theorem 6.13 *(Hausel–Proudfoot 2003) Suppose that $T^*A////G$ and $T^*A////T$ are equivariantly formal, circle compact, and that the Kirwan map $\kappa_G : H^*(T^*A) \to H^*(T^*A////G)$ is surjective. Then*

$$H^*_{U(1)}(T^*A////G) \cong \frac{H^*_{U(1)}(T^*A////T)^W}{Ann(e)}.$$

The ring $H^*_{U(1)}(T^*A////T)$ has been calculated in (Harada-Proudfoot 2002). Thus surjectivity of the hyperkähler Kirwan map $\Rightarrow$ description of the cohomology ring of the hyperkähler quotient. This program has been completed only in the case of the hyperpolygon spaces of Konno by (Hausel-Proudfoot 2003), obtaining the circle equivariant cohomology ring of the hyperpolygon space of (Harada–Proudfoot 2003).

Conjecture 6.14 *(Hausel 2000) Suppose that both $T^*A////G$ and $T^*A////T$ are hypercompact. Then*

$$H^*(T^*A////G) \cong \frac{H^*(T^*A////T)^W}{Ann(\tilde{e})},$$

where

$$\tilde{e} = \prod_{a\in\Delta} a \in (\mathrm{Sym}\, \mathfrak{t}^*)^W \cong H_T(pt)^W.$$

6.4 Equivariant intersection numbers on $\mathcal{M}^d_{Dol}(SL(n,\mathbb{C}))$

The following theorems and conjectures are motivated by similar work by Witten on other moduli spaces.

Theorem 6.15 *(Hausel-Szenes 2003) Let $\alpha \in H^2_{U(1)}(\mathcal{M}^1_{Dol}(SL(2,\mathbb{C})) \cong \mathbb{Z}$ be the positive integral generator, $\mathcal{M} := \mathcal{M}^1_{Dol}(SL(2,\mathbb{C}))$*

$$BA(y) = \left(\frac{2}{u}\right)^{g-1} \frac{\left(\frac{2}{(1-y/u)^2} + u\right)^g}{\left(e^{y\frac{u+y}{u-y}} - e^{-y\frac{u-y}{u+y}}\right) y^{2g-2}(u^2-y^2)^{g-1}},$$

we have

$$\int_{\mathcal{M}} e^{\alpha} = \operatorname*{Res}_{y=0} BA(y) + \operatorname*{Res}_{y=-u} BA(y) + \operatorname*{Res}_{y=u} BA(y)$$
$$= -\sum_{b} \operatorname*{Res}_{y=b} BA(y),$$

where the last sum is taken over the solutions of the Bethe-Ansatz equations:

$$e^{b}\frac{u+b}{u-b} = e^{-b}\frac{u-b}{u+b}$$

Conjecture 6.16 *(Nekrasov–Shatasvili–Moore 1998, Hausel–Szenes 2004) The equivariant volume of $\mathcal{M}^d_{Dol}(SL(n,\mathbb{C}))$:*

$$\int_{\mathcal{M}} e^{\alpha} = \sum_{F} \int_{F} \frac{e^{\alpha}}{E(N_F)}$$

(and indeed all equivariant intersection numbers of $\mathcal{M}^d_{Dol}(SL(n,\mathbb{C}))$), can be described as an iterated residue of a certain expression

$$BA(y_1, y_2, \ldots, y_n).$$

Some of the poles of the expression BA are in one-to-one correspondence with ordered partitions of n, and the rest are the solutions of certain Bethe Ansatz equations. The iterated residue taken at a pole corresponding to the ordered partition $n = \lambda_1 + \cdots + \lambda_k$, agrees with the contribution to the equivariant volume by the components of type $(\lambda_1, \lambda_2, \ldots, \lambda_k)$. Moreover minus of the sum of the residues at the Bethe poles give the equivariant volume of the Higgs moduli space.

6.5 Arithmetic approach

Fix a compact Riemann surface Σ and a pair of relatively prime integers n and d. Carlos Simpson's nonabelian Hodge theory provides a diffeomorphism between $\mathcal{M}^d_{Dol}(GL(n,\mathbb{C}))$, the moduli space of rank n degree d Higgs bundles, and $\mathcal{M}^d_B(GL(n,\mathbb{C}))$, the moduli space of twisted n-dimensional representations of $\pi_1(\Sigma)$:

$$\mathcal{M}^d_B(GL(n,\mathbb{C})) := \{A_1, B_1, \ldots, A_g, B_g \in GL(n,\mathbb{C})|$$
$$A_1^{-1}B_1^{-1}A_1B_1 \ldots A_g^{-1}B_g^{-1}A_gB_g = \xi_n Id\}/GL(n,\mathbb{C})$$

The strategy of (Hausel–Rodriguez-Villegas 2003) for getting the Betti numbers of $\mathcal{M}^d_B(GL(n,\mathbb{C}))$ is to count the rational points of the variety over a finite field $\mathbb{F}_q$. Thus we have to count points of

$$\mathcal{M}_B(GL(n,\mathbb{F}_q)) := \{A_1, B_1, \ldots, A_g, B_g \in GL(n,\mathbb{F}_q)|$$
$$A_1^{-1}B_1^{-1}A_1B_1 \ldots A_g^{-1}B_g^{-1}A_gB_g = \xi_n Id\}/GL(n,\mathbb{F}_q),$$

for which (Frobenius–Schur 1907) gives:

$$\#\{\mathcal{M}_B(GL(n,\mathbb{F}_q))\} \quad = \sum_{\chi\in Irr(GL(n,\mathbb{F}_q))} \frac{|GL(n,\mathbb{F}_q)|^{2g-2}}{\chi(1)^{2g-1}}\chi(\xi_n)$$

Deligne's mixed Hodge structure for

$$M := \mathcal{M}_B^d(GL(n,\mathbb{C}))$$

gives two filtrations on the cohomology $H^k(M,\mathbb{C})$ whose associated graded is

$$\bigoplus_{p,q} H^{p,q;k}(M),$$

we denote by $h^{p,q;k}$ the dimension of $H^{p,q;k}(M)$.

$$H_n(x,y,t) := \sum_{p,q,k} h^{p,q;k}(M)x^p y^q t^k,$$

is the mixed Hodge polynomial.

Theorem 6.17 *(Hausel–Rodriguez-Villegas 2003)*

$$\begin{aligned} H_n(\sqrt{q},\sqrt{q},-1) &= \#\{\mathcal{M}_B(GL(n,\mathbb{F}_q))\} \\ &= \sum_{\chi\in Irr(GL(n,\mathbb{F}_q))} \frac{|GL(n,\mathbb{F}_q)|^{2g-2}}{\chi(1)^{2g-1}}\chi(\xi_n) \end{aligned}$$

We now define a function $H_n(q,t)$ which will conjecturally be related to the mixed Hodge numbers. We define

$$V_n(q,t) = H_n(q,t)\frac{(qt^2)^{(1-g)n(n-1)}}{(qt^2-1)(q-1)},$$

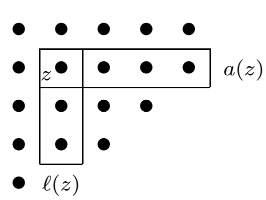
and

$$Z_n(q,t,T) = \exp\left(\sum_{r\geq 1} V_n(q^r, -(-t)^r)\frac{T^r}{r}\right).$$

Then let

$$\mathcal{H}_g^\lambda(q,t) = \prod_{z\in d(\lambda)} \frac{(qt^2)^{(2-2g)\ell(z)}(1+q^{h(z)}t^{2\ell(z)+1})^{2g}}{(1-q^{h(z)}t^{2\ell(z)+2})(1-q^{h(z)}t^{2\ell(z)})}.$$

Here $\ell(z)$ denotes the leg length and $h(z)$ is the hook length, and $h(z) = a(z)+\ell(z)-1$, where $a(z)$ is the arm length, as in the figure below.



We finally define $H_n(q,t)$, in generating function form:

$$\prod_{n=1}^{\infty} Z_n(q,t,T^n) = \sum_{\lambda\in\mathcal{P}} \mathcal{H}_g^\lambda(q,t)T^{|\lambda|}.$$

Using this notation, we have the following conjectures.

Conjecture 6.18 *(Hausel–Rodriguez-Villegas 2004) The mixed Hodge polynomial of $\mathcal{M}_B^d(GL(n,\mathbb{C}))$, is given by*

$$H_n(\sqrt{q},\sqrt{q},t) = H_n(q,t)$$

Example 6.19 The case $n=2$ follows from (Hausel–Thaddeus 2000):

$$\begin{aligned} H_2(\sqrt{q},\sqrt{q},t)/(qt+1)^{2g} &= \frac{(q^2t^3+1)^{2g}}{(q^2t^2-1)(q^2t^4-1)} + \frac{q^{2g-2}t^{4g-4}(q^2t+1)^{2g}}{(q^2-1)(q^2t^2-1)} \\ &\quad -\frac{1}{2}\frac{q^{2g-2}t^{4g-4}(qt+1)^{2g}}{(qt^2-1)(q-1)} - \frac{1}{2}\frac{q^{2g-2}t^{4g-4}(qt-1)^{2g}}{(q+1)(qt^2+1)}, \end{aligned}$$

and when $g=3$:

$$\begin{aligned} H_2(\sqrt{q},\sqrt{q},t)/(qt+1)^6 &= t^{12}q^{12} + t^{12}q^{10} + 6\,t^{11}q^{10} + t^{12}q^8 + t^{10}q^{10} \\ &\quad +6\,t^{11}q^8 + 16\,t^{10}q^8 + 6\,t^9q^8 + t^{10}q^6 + t^8q^8 + 26\,t^9q^6 \\ &\quad +16\,t^8q^6 + 6\,t^7q^6 + t^8q^4 + t^6q^6 + 6\,t^7q^4 + 16\,t^6q^4 + \\ &\quad +6\,t^5q^4 + t^4q^4 + t^4q^2 + 6\,t^3q^2 + t^2q^2 + 1. \end{aligned}$$

Conjecture 6.20 *(T. Hausel) The Pure rings of $\mathcal{M}_{Dol}^d(GL(n,\mathbb{C}))$ and $\mathcal{N}^d(GL(n,\mathbb{C}))$ (the moduli space of rank n stable bundles of degree d), i.e. the subrings of the cohomology rings generated by the classes $a_2,\ldots,a_n$ are isomorphic. In particular, unlike the whole cohomology ring of $\mathcal{N}^d(GL(n,\mathbb{C}))$, it does not depend on d. Moreover the Poincaré polynomial $PP_n(t)$ of the pure ring is given by:*

$$PV_n(t) = PP_n(t)\frac{t^{2(1-g)n(n-1)}}{(t^2-1)},$$

$$PZ_n(t,T) = \exp\left(\sum_{r\geq 1} PV_n(t^r)\frac{T^r}{r}\right).$$

$$\mathcal{P}H_g^\lambda(t) = t^{4(1-g)n(\lambda')} \prod_{x\in d(\lambda);a(x)=0} \frac{1}{(1-t^{2h(x)})},$$

$$n(\lambda') := \sum_{z\in d(\lambda)} \ell(z).$$

$$\prod_{n=1}^{\infty} PZ_n(t,T^n) = \sum_{\lambda\in\mathcal{P}} \mathcal{P}H_g^\lambda(t)T^{|\lambda|}.$$

Example 6.21 For the case $n = 2$, the pure ring is generated by $\beta = a_2$, and $\beta^g = 0$, this is the famous Newstead conjecture for $\mathcal{N}^1(GL(2,\mathbb{C}))$, was first proved by (Kirwan 1992, Thaddeus 1992), while for $\mathcal{M}^1_{Dol}(GL(2,\mathbb{C}))$ it was proved in (Hausel–Thaddeus 2000)

Example 6.22 For the case $n > 2$, similar vanishings for the pure ring of $\mathcal{N}^d(GL(n,\mathbb{C}))$ was proved by (Earl–Kirwan 1999) using (Jeffrey–Kirwan 1998), which also follow from the conjecture.

$$\begin{aligned} PP_3(t) \;=\; & \frac{1}{(t^6-1)(t^4-1)} + t^{12g-12} \\ & -\frac{t^{8g-8}}{t^2-1} + \frac{1}{3}\frac{t^{12g-12}}{(t^2-1)^2} - \frac{1}{3}\frac{t^{12g-12}}{t^4+t^2+1} \\ & -\frac{t^{8g-8}}{(t^4-1)(t^2-1)} + \frac{t^{12g-12}}{t^2-1} \end{aligned}$$

7 Kernel computations for Kirwan maps

Suppose a compact Lie group G acts linearly on an affine space $\mathbb{C}^n$ Hamiltonianly. Consider the symplectic quotient $\mathbb{C}^n /\!\!/_\alpha G$, for α central and regular. We can also view this space as a GIT quotient

$$(\mathbb{C}^n \setminus \{\text{the } \alpha\text{-unstable locus}\})/G_{\mathbb{C}}.$$

Either way, we get a Kirwan map κ_α

$$H^*_G(\mathbb{C}^n) \cong H^*_G(pt) \to H^*(\mathbb{C}^n /\!\!/_\alpha G).$$

This gives us a different maps κ_α and different ideals $I_\alpha := \ker(\kappa_\alpha)$ depending on our choice of α.

Note that, from the viewpoint of the GIT quotient, the α-unstable locus for any central regular α contains points in $\mathbb{C}^n$ on which G fails to act locally freely. So instead, let's look at

$$X := (\mathbb{C}^n \setminus \{\text{ points where } G \text{ fails to act locally freely}\})/G_{\mathbb{C}}.$$

Using this space, we also get a map

$$f : H^*(\mathbb{C}^n) \cong H^*_G(pt) \to H^*(X/G),$$

and from the description of the space X it is clear that

$$\ker(f) \subset \ker(\kappa_\alpha), \qquad \forall \alpha.$$

Question 7.1 (N. Proudfoot) In this setting, is it true that we have the equality

$$\ker(f) = \cap_\alpha \ker(\kappa_\alpha).$$

where the intersection is over central regular values α?

Comment 7.2 (N. Proudfoot) When $G = T$ is abelian, the conjecture is true. The proof requires hyperKähler geometry, despite the fact that the statement doesn't involve anything hyperKähler. Is it possible to prove it directly?

Question 7.3 (R Goldin) Let G act on $\mathbb{C}^n$ as above, and X be as defined above. Then is the map

$$H^*_G(\mathbb{C}^n) \to H^*(X/G) = H^*_G(X)$$

surjective? What is the kernel?

Comment 7.4 (M. Libine) We should probably begin by looking at irreducible representations. Note also that it is possible that X is empty without some additional assumptions on the action of G.

8 Higgs bundles and relations to gauge theory

Question 8.1 (G. Daskalopolous) For a 3-manifold, the theorem of Corlette gives a correspondence between representations of the fundamental group in $SL(2, \mathbb{C})$ with the space of Higgs bundles. The question is if under the existence of a contact structure, there is any additional structure on the space of Higgs bundles as in the surface case. For example, is there a contact interpretation of the ($\mathcal{L}^2$) norm of the Higgs field? Is there a description of the critical points of the norm of the Higgs field? Can this be used in any way to show existence of representations of the fundamental group into $SU(2)$ or $PSL(2, \mathbb{R})$?

9 Intersection Cohomology

Question 9.1 (N. Proudfoot) The intersection cohomology of a singular hypertoric varieties has a "seemingly natural" ring structure (as a quotient of the equivariant cohomology $H^*_T(T^*\mathbb{C}^n)$ of the original space). Is there, in general, a natural ring structure on the intersection cohomology group of a singular hyperKähler quotient?

Comment 9.2 (N. Proudfoot) One setting in which one gets a ring structure on intersection cohomology is when one has a small resolution, but in the hypertoric examples, Nick is not aware of any such small resolution.

Question 9.3 (T. Holm) Is there a natural ring structure for the intersection cohomology of singular Kähler quotients?

Comment 9.4 (E. Lerman) In [LT00], E. Lerman and S. Tolman construct a small resolution for S^1-reduced spaces and thus obtain a ring structure on the intersection cohomology of the S^1-reduction, but their results are special to the S^1 case.

Comment 9.5 (R. Sjamaar) In their paper "ntersection cohomology of symplectic quotients by circle actions" (to be published in *J. London Math. Soc.*), Kiem and Woolf produce an example of a singular symplectic (in fact Kähler) quotient which has two small resolutions with distinct cohomology rings. So there appears to be no natural ring structure on the intersection homology of a singular quotient.

10 Computations over $\mathbb{Z}$

Problem 10.1 (R. Goldin, S. Tolman, J. Weitsman) Describe all the conditions under which Kirwan surjectivity holds for symplectic reductions over the integers. What are the most general conditions under which it holds?

Question 10.2 (R. Goldin) Consider the case of smooth toric varieties $\mathbb{C}^n /\!\!/ T$. Is it always true for all toric varieties?

Question 10.3 (R. Goldin) For which symplectic toric orbifolds [LT97] does the surjectivity hold over $\mathbb{Z}$? Actually, perhaps the more appropriate question involves looking at the analogous statement which uses the orbifold cohomology instead of ordinary cohomology. Also, what happens if we take $\mathbb{Z}/p\mathbb{Z}$ coefficients?

Comment 10.4 (E. Lerman) Note that in Question 10.3, the issue is not just one of surjectivity but also of the choice of the cohomology theory for the reduced space. I.e. should one take Chen-Ruan orbifold cohomology? Or that of Haefliger?

One approach to addressing the question would be to look for a good (= optimal) condition on critical sets of the norm-square of the moment map.

11 Localization formulas for non-compact groups

Question 11.1 (M. Libine) Let M be a compact manifold, and consider T^*M. Let σ be the canonical symplectic form on T^*M. For any other exact symplectic form $\omega = d\alpha$ on T^*M, with $\alpha|_M$ exact, is it possible to find a diffeomorphism on T^*M preserving M sending ω to $\pm\sigma$?

Question 11.2 (M. Libine) If there is no such diffeomorphism, how can we parametrize symplectic forms on T^*M up to this equivalence?

Comment 11.3 (M. Libine) Answering this question would provide further extensions to the Berligne-Vergne localization formula extended to non-compact (reductive) group actions.

12 Volume growth of hyperKähler manifolds

Question 12.1 (H. Konno) Let M be a connected noncompact hyperKähler manifold. Fix a point $p \in M$. Consider the open ball $B(p,r)$ of radius r around p in M. Describe the asymptotic behavior of the volume of the ball $Vol(B(p,r))$ as $r \to \infty$. (Fact: this is independent of the choice of $p \in M$.) It would be interesting to search for examples of hyperKähler manifolds with different volume growth.

13 Hodge theory

Question 13.1 (J. Weitsman) Suppose M is a compact Kähler manifold and a Hamiltonian G-space. Suppose all the ordinary cohomology is of type (p, p). Can we say anything about the cohomology of the quotient?

Comment 13.2 (R. Sjamaar) Usual quantization-commutes-with-reduction states that the cohomology $H^*(M; \mathcal{L})$ with coefficients in the sheaf of the prequantum line bundle $\mathcal{L}$ is a G-module, and further, that the G-invariant part is the cohomology of the GIT quotient with coefficients in the sheaf of the induced prequantum line bundle $\mathcal{L}_{red}$. Teleman [Tel00] says that something similar should work not just for the prequantum line bundle but also for a sheaf such as $\mathcal{L} \otimes \Omega^q$, where $\mathcal{L}$ is still the sheaf of the prequantum line bundle, and Ω^q is the sheaf of holomorphic q-forms. But then since $H^{p,q}(M) = H^p(M; \Omega^q)$, perhaps some quantization-commutes-with-reduction argument, using just the sheaf Ω^q instead of a sheaf tensored with $\mathcal{L}$, could also be used to show that the appropriate cohomology also vanishes downstairs.

Comment 13.3 (D. Burns, J. Weitsman) Burns and Weitsman have found one method for doing this, though it may already be implicit, as suggested by Sjamaar in Comment 13.2, in earlier work.

References

[BG99] C. P. Boyer and K. Galicki. 3-Sasakian manifolds. In *Surveys in differential geometry: essays on Einstein manifolds*, Surv. Differ. Geom., VI, pages 123–184. Int. Press, Boston, MA, 1999.

[BGM94] C. P. Boyer, K. Galicki, and B. M. Mann. The geometry and topology of 3-Sasakian manifolds. *J. Reine Angew. Math.*, 455:183–220, 1994.

[BGMR98] C. P. Boyer, K. Galicki, B. M. Mann, and E. G. Rees. Compact 3-Sasakian 7-manifolds with arbitrary second Betti number. *Invent. Math.*, 131(2):321–344, 1998.

[BGP02] C. P. Boyer, K. Galicki, and P. Piccinni. 3-Sasakian geometry, nilpotent orbits, and exceptional quotients. *Ann. Global Anal. Geom.*, 21(1):85–110, 2002.

[CDM88] M. Condevaux, P. Dazord, and P. Molino. Géométrie du moment. *Publ. Dép. Math. Nouvelle Sér. B 88-1*, pages 131–160, 1988.

[CG83] J. Carrell and R. Goresky. A decomposition theorem for integral homology of a variety. *Invent. math.*, 73:367 – 381, 1983.

[GGK02] V. Ginzburg, V. Guillemin, and Y. Karshon. *Moment maps, cobordisms, and Hamiltonian group actions.* Number 98 in Mathematical Surveys and Monographs. American Mathematical Society, 2002.

[Kir85] F. Kirwan. On the topology of hyperkahler quotients. *IHES preprint*, 1985. Unpublished.

[Ler02] E. Lerman. A convexity theorem for torus actions on contact manifolds. *Illinois J. Math.*, 46(1):171–184, 2002.

[LT97] E. Lerman and S. Tolman. Hamiltonian torus actions on symplectic orbifolds and toric varieties. *Trans. Amer. Math. Soc.*, 349(10):4201–4230, 1997.

[LT00] E. Lerman and S. Tolman. Intersection cohomology of circle symplectic quotients and small resolutions. *Duke Math. J.*, 103(1):79–99, 2000.

[MM03] I. Moerdijk and J. Mrcun. *Introduction to foliations and Lie groupoids*, volume 91. Cambridge Studies in Advanced Mathematics, Cambridge, 2003.

[PW94] E. Prato and S. Wu. Duistermaat-Heckman measures in a non-compact setting. *Compositio Math.*, 94(2):113–128, 1994.

[Sja98] R. Sjamaar. Convexity properties of the moment mapping re-examined. *Advances in Math.*, 138(1):46–91, 1998.

[Tel00] C. Teleman. The quantization conjecture revisited. *Ann. of Math. (2)*, 152(1):1–43, 2000.