Levine: Mixed Motives

Conjectural Picture

Let k be a field

$\mathcal{M}_k \subseteq \mathcal{MM}_k \otimes \mathbb{Q}$

$\mathcal{MM}_k$ Beilinson's cat. of mixed motives ($\mathbb{Z}$-version)

Grothendieck's cat. of pure motives (num. eq.)

$\mathcal{M}_k$: constructed by SmProj/k, with Hom-sets

$$\mathrm{Hom}_k(X, Y) = \mathrm{Cor}_k^{num}(Y, X) \otimes \mathbb{Q}$$
$$:= Z^{\dim Y}(Y \times X)/_{num} \otimes \mathbb{Q}$$

$+ \ldots + \ldots$

Theorem (Jannsen) $\mathcal{M}_k$ is semisimple

$\mathcal{M}_k$ is the "universal Weil cohomology theory"

$\mathrm{SmProj} \ni X \mapsto H^*(\bar{X} := X \times_k \bar{k})$

e.g. $H^*(\bar{X}) = H^*_{et}(\bar{X}, \mathbb{Q})$

some properties of Weil coh: f^*, cycle classes,
Künneth formula
Poincaré duality, $H^1(\mathbb{P}^1)=0$

$\mathcal{MM}_k$ "universal Bloch-Ogus coh. thy on Sm/k"

Properties: f^*, Mayer-Vietoris, twisted cycle classes
$Z^d(X) \to H^{2d}(X, \Gamma(d))$
codimension

twisted purity $W \overset{i}{\hookrightarrow} X$ codim d
$$H^n(W, \Gamma(m)) \cong H^{n+2d}_W(X, \Gamma(m+d)),$$
homotopy invariance

$\mathcal{MM}_k$ is an abelian tensor category having objects
$\mathbb{Z}(n)$, $M(n) := M \otimes \mathbb{Z}(n)$.
$\mathbb{Z}(m) \otimes \mathbb{Z}(n) := \mathbb{Z}(m+n)$

$$h: Sm/k^{op} \longrightarrow D^b(\mathcal{MM}_k)$$

$$H^p_A(X, \mathbb{Z}(q)) := Ext^p(\mathbb{Z}, h(X)(q))$$
$$H^p_A(X, \mathbb{Z}(q)) \otimes \mathbb{Q} = K_{2q-p}(X)^{(q)}$$

$H^i(h(X))$ is in $\mathcal{M}_k$ for $X \in SmProj/k$

Consequence: $K_{2q-p}(X)^{(q)} = 0$ if $p<0$.
(weak Beilinson-Soulé vanishing conjecture)

II Algebraic cycles & correspondences

Def $X \in Sm/k$

a) $Z^q(X) := \mathbb{Z}[\{W \subseteq X \mid W \text{ closed irred, codim } q\}]$

b) ~~$W \in Z^q(X), Z^{q'}$~~ Z, Z' intersect properly in X if $|Z| \cap |Z'|$ has codim $q+q'$.

Here $Z = \sum_{n_i \neq 0} n_i W_i \Rightarrow |Z| = \bigcup W_i$

Note: This defines a (partially defined) associative & commutative product

$$Z^*(X) \otimes Z^*(X) \to Z^*(X).$$

Similar, there is partially defined

$$f^*: Z^r(X) \to Z^r(Y) \text{ for } f: Y \to X$$

well-defined $f_*: Z^*(X) \to Z^{*-d}(Y)$ for $f: X \to Y$ proj of codim d

Def: ~~Rational~~ Rational equivalence

$Z, Z' \in Z^q(X)$: $Z \sim_r Z'$ if ~~$\exists$~~

$\exists\, Z'' \in Z^q(X \times \mathbb{A}^1)$ with

$$Z - Z' = (i_0^* - i_1^*)(Z'').$$

i_0: 0-section, i_1: 1-section

$$CH^*(X) := Z^*(X)/\sim_r$$

Thm All the operations are well-defined on $CH^*(X)$.

Correspondence $X, Y \in \mathrm{SmProj}/k$

$$\mathrm{Cor}_k(X,Y) = CH^{\dim X}(X \times Y).$$

$$\mathrm{Hom}_k(Y,X) \to \mathrm{Cor}_k(X,Y)$$
$$f \longmapsto \Gamma_f^t$$

Composition Law: $(\Gamma', \Gamma) \mapsto p_{XZ*}((\Gamma \times Z)\cdot(X \times \Gamma'))$

Note: $\Gamma_{f\circ g}^t = \Gamma_g^t \circ \Gamma_f^t$.

III Motivic Complexes

A. Bloch's cycle complex

1. Cosimplicial scheme $\Delta^\bullet$

$$\Delta^n = \operatorname{Spec}\left(k[t_0, \ldots, t_n] / \textstyle\sum t_i = 1\right)$$
$$\cong \mathbb{A}^n$$

2. The cycle complex. Take $X \in$ ~~SmProj~~ Sm/k

$$Z^q(X \times \Delta^p) \supseteq Z^q(X,p) = \mathbb{Z}\Big[\{W \hookrightarrow X \times \Delta^p \mid W \text{ irred, } \operatorname{codim}_{X\times f} W \cap (X \times f) = q \ \forall \text{ faces } f \text{ of } \Delta^p\}\Big]$$

So δ_i^* is well-defined. Define $d := \sum (-1)^i \delta_i^*$,

which gives the complex $Z^q(X,*)$

<u>Def</u>: $CH^q(X,p) = H_p(Z^q(X,*))$

<u>Properties</u>
- $CH^q(X,0) = CH^q(X)$ $\left(CH^q(X) \otimes \mathbb{Q} = K_0(X)^{(q)} \right.$ Grothendieck-Riemann-Roch $\left.\right)$
- homotopy invariance
- functoriality (difficult)
- Mayer-Vietoris
- correct relation with K-theory
  $CH^q(X,p) \otimes \mathbb{Q} \cong K_p(X)^{(q)}$.

This suggests: $H^p(X, \mathbb{Z}(q)) = CH^p(X, 2q-p)$ ✓

<u>B Suslin's complex</u> $\rightsquigarrow H_n^{\text{Suslin}}(X, \mathbb{Z})$

<u>Dold-Thom-thm</u> $\pi_n S^\infty X \cong H_n(X, \mathbb{Z})$

<u>Def</u>: $X, Y \in \mathrm{Sm}/k$

$$\mathrm{SmCor}_k(Y, X) := \mathbb{Z}\left[\left\{ W \hookrightarrow Y \times X \;\middle|\; \begin{matrix} W \text{ irred}, \\ W \to Y \text{ finite} \\ \& \text{ surj. on some} \\ \text{component} \end{matrix} \right\}\right]$$

$$\underset{Z^{\bullet}(Y \times X)}{\cap|}$$

parametrizing a family of points of X over Y.

$$C_n^{\mathrm{Sus}}(X) = \mathrm{SmCor}_k(\Delta^n, X) \subseteq Z^{\dim X}(X, n)$$

$\rightsquigarrow$ chain complex $C_\bullet^{\mathrm{Sus}}(X)$

<u>Define</u> (Suslin '86) $H_n^{\mathrm{Sus}}(X) := H_n C_\bullet^{\mathrm{Sus}}(X)$

Problems: Mayer-Vietoris
relations to Bloch's complex

<u>IV Triangulated categories of motives</u>

A. Triangulated categories

It is an additive category $\mathcal{C}$, equivalence $T: \mathcal{C} \to \mathcal{C}$ (translation or shift $X \mapsto X[1]$) and collection of triangles $\mathcal{E}$:

$$A \longrightarrow B \to C \to A[1].$$

+ some axioms.

Consequence $\mathrm{Hom}_{\mathcal{C}}(X,-)$, $\mathrm{Hom}_{\mathcal{C}}(-,X)$ transform

dist. triangles to l.e. sequences.

<u>Examples</u> $\mathcal{A}$ additive cat, $K^{\circ}(\mathcal{A})$ homotopy cat. of complexes in $\mathcal{A}$

$\mathcal{A}$ abelian category, $D(\mathcal{A})$ derived category

SH stable homotopy cat.

<u>B Localization</u>: If $\mathcal{C}$ is Δ cat & a) S is a collection of morphisms (or or b) $\mathcal{B}$ is a collection of objects, one can form

$$\begin{array}{l} \mathcal{C} \longrightarrow S^{-1}\mathcal{C} \\ \quad\searrow \mathcal{C}/\mathcal{B} \end{array}$$

universal w.r. to
a) inverting S
b) killing $\mathcal{B}$.

C Motives: 1st construction

$DM_{gm}(k)$: let $SmCor_k$ be the cat. of smooth correspondences (use composition formula from above)

$$M : Sm/_k \longrightarrow SmCor_k$$
$$X \longmapsto X$$
$$f \longmapsto \Gamma_f$$

Form $K^b(SmCor_k)$ & localize w.r.t.

a) (homotopy invariance) $M(X \times \mathbb{A}^1) \to M(X)$

b) (Mayer-Vietoris)
$$\begin{array}{ccc} U \cap V & \hookrightarrow & V \\ \downarrow & \ulcorner & \downarrow \\ U & \hookrightarrow & X \end{array}$$

$$M(U \cap V) \to M(U) \oplus M(V) \to M(X)$$

$$\rightsquigarrow DM_{gm}^{eff}(k) \quad , \quad M(X) \otimes M(Y) := M(X \times Y)$$
$$\mathbb{Z}(1) = [M(\mathbb{P}^1) \to M(\mathrm{Spec}\, k)][-2]$$

Now invert $\otimes \mathbb{Z}(1)$ to form $DM_{gm}(k)$ from $DM^{eff}_{gm}(k)$

Thm 1: With

$$H^p(X, \mathbb{Z}(q)) := \mathrm{Hom}_{DM_{gm}(k)}(M(X), \mathbb{Z}(q)[p]),$$

$$H^p(X, \mathbb{Z}(q)) \cong CH^q(X, 2q-p)$$