April 25, 2004

Levine II Mixed Tate Motives

- will be subcat of $DM_{gm} \otimes \mathbb{Q}$
- or modules over motivic cdga (Bloch, Bloch-Kriz, Kriz-May, Spitzweck)

Def DTM_k ← field

is the full triangulated subcat of $DM_{gm} \otimes \mathbb{Q}$ generated by Tate objects $\mathbb{Q}(n)$

does it contain motives of some varieties?

Since $\mathbb{Q}(n) \otimes \mathbb{Q}(m) = \mathbb{Q}(n+m)$, it is triangulated sub-tensor-category

Def $W_{\leq n} DTM_k$ is the full Δated subcat of

DTM_k gen. by $\mathbb{Q}(m)$'s, $m \geq -n$.

(a weight filtration) $(w(\mathbb{Q}(n)) = -2n)$

$W^{>n} DTM_k$ ... gen by $\mathbb{Q}(m)$'s, $m < -n$

Lemma $\mathrm{Hom}(W_{\leq n}, W^{>n}) = 0$.

Proof: check on generators $\mathbb{Q}(a)[b]$, $\mathbb{Q}(c)[d]$

here $a < -n$, $c \geq -n$. So $a < c$.

$$\mathrm{Hom}\left(\mathbb{Q}(c)[d],\ \mathbb{Q}(a)[b] \right)$$

$$\parallel$$

$$\mathrm{Hom}\left(\mathbb{Q},\ \mathbb{Q}(a-c)[b-d] \right) \qquad a-c < 0$$

$$\parallel$$

$$K(k)^{(a-c)}_{2(a-c)-(b-d)} \longleftarrow \text{since } (a-c) \text{ is negative,}$$

this K-group is 0. $\square$

<u>Prop</u> a) : The inclusion

$$w_{\leq n} DTM \xhookrightarrow{i_n} DTM$$

has a right adjoint r_n.

Set $w_{\leq n} = i_n r_n$. We have a can. dist. Δ

$$w_{\leq n} X \to X \to w^{>n} X \to w_{\leq n} X[1]$$

by def'n.

With $w^{>n} X \in w^{>n} DTM$.

b) $w^{>n}$ is a functor left adjoint to $w^{>n} DTM \xhookrightarrow{i_n'} DTM$.

c) For $m \le n$ $\exists$ natural map

$$w_{\le m} X \longrightarrow w_{\le n} X$$

$$w^{>m} X \longrightarrow w^{>n} X$$

making

$$\begin{tikzcd}
w_{\le n} X \arrow[r] & X \arrow[r] & w^{>n} X \arrow[r] & {} \\
w_{\le m} X \arrow[r] \arrow[u] & X \arrow[r] \arrow[u, "id"] & w^{>m} X \arrow[r] \arrow[u] & {} \arrow[u] \cdots
\end{tikzcd}$$

commute.

This gives us a "weight filtration"

$$\to w_{\le n-1} X \to w_{\le n} X \to w_{\le n+1} X \to \cdots \to X$$

$$w_{\le m} X = 0 \qquad\qquad X = w_{\le N} X$$

Note: Δated subcat by $\mathbb{Q}(n)$ is equivalent to

$$D^b(\mathbb{Q}^{f.d.}\text{-vector spaces})$$

$$\mathrm{Hom}(\mathbb{Q}(n)[a], \mathbb{Q}(n)[b]) = \mathrm{Hom}(\mathbb{Q}, \mathbb{Q}[b-a])$$

does this work integrally?

$$\cong K^{(0)}_{a-b}(k) = \begin{cases} 0 & \text{if } a \neq b \\ \mathbb{Q} & \text{if } a = b \end{cases}$$

Why is $w^{>n}X$ canonical?

$$\begin{tikzcd} w_{\leq n}X \arrow[r] & X \arrow[r] \arrow[d, equal] & w^{>n}X \arrow[r] \arrow[d] & w_{\leq n}X[1] \\ & X \arrow[r] & w^{>n}X & \end{tikzcd}$$

use Lemma.

Remark: $w_{\leq n}$ is exact

$$w_{\leq n-1}X \to w_{\leq n}X \to gr_n^W X$$

can do this in DM?

defines an exact functor

$$gr_n^W : DTM \to D^b(\text{f.d. } \mathbb{Q}\text{-vector spaces}) \cdot \mathbb{Q}(-n)$$

→ Beilinson-Soulé vanishing conj &
t-structures on DTM_k

Conj (B-S) $K_{2q-p}^{(q)}(k) = 0$ if $q > 0$, $p \leq 0$

(also if $q = 0$, $p < 0$)

$\Leftrightarrow \operatorname{Hom}_{DTM}(\mathbb{Q}, \mathbb{Q}(q)[p]) = 0$

if $q \neq 0$, $p \leq 0$

$q = 0$ $p \neq 0$

Def: $DTM_{\geq 0} = \{ M \mid H^i(gr^W_n M) = 0$ if $i < 0$ $\forall n \}$

$DTM_{\leq 0} = \{ M \mid H^i(gr^W_n M) = 0$ if $i > 0$ $\forall n \}$

$TM_k := \{ M \mid H^i(gr^W_n M) = 0$ $i \neq 0$ $\forall n \}$

Thm This is a t-structure on DTM if (and only if) BS conj. holds. The heart is TM_k.

Consequence Assume BS-conj. Then

- TM is an ab. subcat of DTM_k (closed under extension)
- $0 \to A \to B \to C \to 0$ in TM_k is exact $\Leftrightarrow$ extends to dist. Δ in DTM_k
- $\varphi_n : \mathrm{Ext}^n_{TM_k}(A, B) \to \mathrm{Hom}_{DTM_k}(A, B[n])$, φ_0, φ_1 iso's, φ_2 injective
- TM_k contains & is generated by the $\mathbb{Q}(n)$ $\forall n$.

Example Let k be a number field

$$\mathrm{Hom}(\mathbb{Q}, \mathbb{Q}(q)[p]) = K_{2q-p}(k)^{(q)} = 0 \text{ if } p \geq 2$$

(Borel, Borel-Yang)

$\Rightarrow$ BS conj. Moreover,

$$\mathrm{Hom}_{DTM_k}(M, N[p]) = 0 \text{ if } p \neq 0, 1,$$

$M, N \in TM_k$

$\Rightarrow \mathrm{Ext}^2_{TM}(M, N) = 0$

$\Rightarrow \mathrm{Ext}^p_{TM}(M, N) = 0 \quad \forall p \geq 2$.

And $\mathrm{Ext}^1_{TM}(\mathbb{Q}, \mathbb{Q}(q)) = K_{2q-1}(k)^{(q)}$

$$= \begin{cases} \mathbb{Q}^{r_1+r_2} & q \text{ odd} \geq 3 \\ \mathbb{Q}^{r_2} & q \text{ even} \geq 2 \\ k^\times \otimes \mathbb{Q} & q = 1 \end{cases}$$

$\rightsquigarrow$

Cor Let k be a number field. Then

TM_k is a Tannakian cat.

$\sum_n gr^W_n : TM_k \xrightarrow{\omega}$ fd $\mathbb{Q}$vs

is a fiber functor (faithful $\otimes$ functor)

Let $G^{mot}_k = \mathrm{Aut}(\omega)$ (a pro-algebraic group)

Then $TM_k \simeq \mathrm{Rep}(G^{mot}_k)$

$$
\begin{tikzcd}
TM \arrow[rr, "\omega = \sum_n gr_n^W"] \arrow[dr, "\omega_0"', "\bigoplus_n gr_n^W"] & & \text{f.d. } \mathbb{Q}\text{-vs} \\
& \text{graded f.d. } \mathbb{Q}\text{-vs} \arrow[ur, "\Sigma"'] &
\end{tikzcd}
$$

This factorization gives

$$
\begin{tikzcd}
1 \arrow[r] & \mathcal{U}_k^{mot} \arrow[r] & G_k^{mot} \arrow[r] & \mathbb{G}_m \arrow[r] \arrow[l, bend left, "s"] & 1
\end{tikzcd}
$$

$$\mathcal{U}_k^{mot} = \mathrm{Aut}(\omega_0)$$

pro-unipotent group,   grading

s is induced by graded f.d. $\mathbb{Q}$-vs $\to TM$.
So it's given by its Lie algebra

$\mathrm{Lie}(\mathcal{U}_k^{mot}(\mathbb{Q}))$ which is a free neg graded pro-Lie algebra

$$\mathrm{Lie}(\;)_{-q} = H^1(k, \mathbb{Q}(q))^\vee = [K_{2q-1}(k)^{(q)}]^\vee$$

Relation to Galois group $G_{\mathbb{Q}}$:

<u>étale realizations</u> Fix prime l, field k

$\exists$ functor $DM_{gm}^{(k)\otimes\mathbb{Q}} \xrightarrow{Re_{ét,\ell}} D^+\left(Sh^{ét}_{\mathbb{Q}_\ell}(k)\right)$

$\cup\!|$ $\qquad\qquad\qquad\qquad$ $\shortparallel$

$DTM_k \longrightarrow Rep_{\mathbb{Q}_\ell}(G_k)$ continuous fin.-d. rep's

$DTM_k \longrightarrow D^+$(quasi-unipotent $rep_{\mathbb{Q}_\ell}(G_k)$)

heart $\cup\!|$ $\qquad\qquad$ $|\cup$ heart

Ux
BS $\searrow$
~~As~~

$TM_k \longrightarrow$ quasi-unipotent $rep_{\mathbb{Q}_\ell}(G_k)$

$U_L^{mot}(\mathbb{Q}_\ell)$

Tannaka formalism $\Rightarrow$

$G_k \longrightarrow G_k^{mot}(\mathbb{Q}_\ell)$

$1 \to U_k \to G_k^{quasi\text{-}unipotent} \to \hat{\mathbb{Z}}^+ \to$

$G_k^{quasi\text{-}unipotent} \to G_m(\mathbb{Q}_\ell)$, $\hat{\mathbb{Z}}^+ \to G_m(\mathbb{Q}_\ell)$, $G_m(\mathbb{Q}_\ell) \to G_k^{mot}(\mathbb{Q}_\ell) \to G_m(\mathbb{Q}_\ell)$

The realization $Re_{ét,\ell}$ is a technical game generalizing cycle class map $Z(X) \to H_{ét}$.

Example: An interesting object in $TM_{\mathbb{Q}}$

(for $k=\mathbb{Q}$, $\mathrm{Lie}(U_{\mathbb{Q}})$ is free on $s_3, s_5, s_7, \dots$ in odd degree >1)

Beilinson's polylogarithm (work over $\mathbb{P}^1 - \{0,1,\infty\}$)

$$T := \mathbb{A}^1 - \{0\} = \mathbb{P}^1 - \{0,\infty\}$$

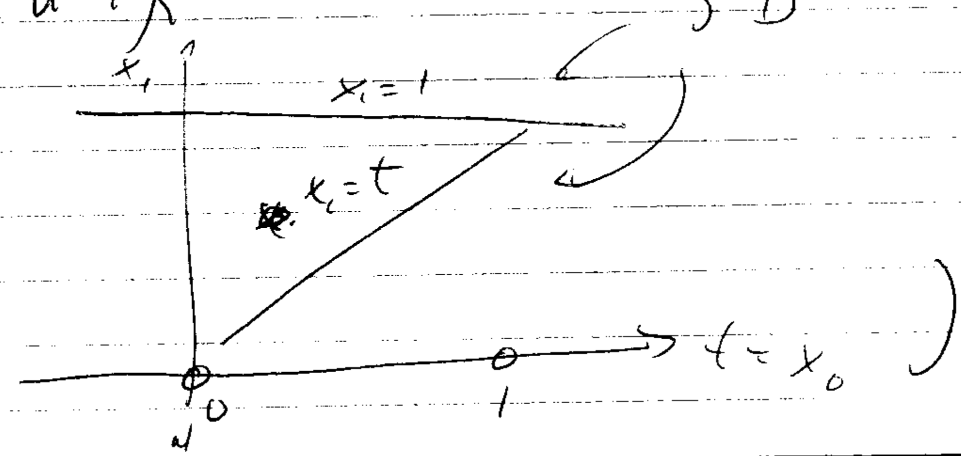
$$T \times T^n$$
$$(x_0, x_1, \dots, x_n)$$

$$D = \mathrm{sum}\begin{pmatrix} x_n = 1 \\ x_{n-1} = x_n \\ \vdots \\ x_0 = x_1 \end{pmatrix}$$

$$U = \mathbb{P}^1 - \{0,1,\infty\} \overset{j}{\hookrightarrow} T \quad \text{set } t := x_0$$

$$H^{n+1}(U \times T^n, j^*D, \mathbb{Q}(n+1))$$

(Picture $n=1$):



Note: $H^1(X, \mathbb{Z}(1)) = \mathcal{O}_X^*$ units.

~~map~~ cup product gives symbols

$\{1-t, x_1, \ldots, x_n\} = \mathcal{L}i_n$

Hodge realization is local system

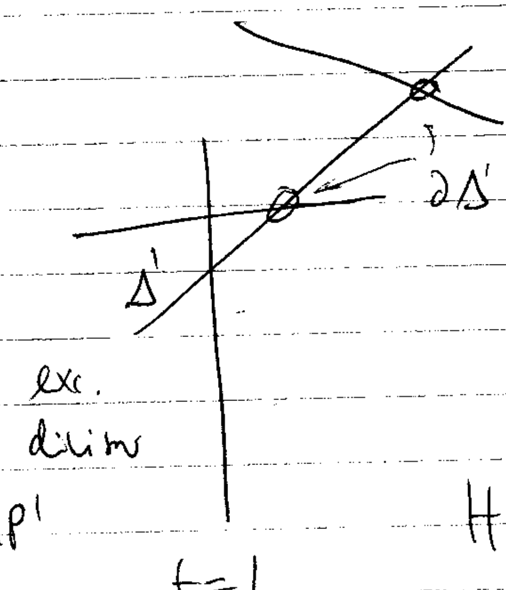
Li_n on $\mathbb{P}^1 - \{0,1,\infty\}$, $\nabla e_i = \begin{cases} e_{i+1} \frac{dt}{t} & 1 \le i \le n-1 \\ e_1 \frac{dt}{1-t} & i = 0 \end{cases}$

To see what happens motivically at $t=1$

blowup $(1,1,\ldots,1)$ & remove $[t=1]$

$\rightsquigarrow$

$n=1$



So you are in

$H^{n+1}(\Delta_n, \partial\Delta^n_n, \mathbb{Q}(n+1))$

$\downarrow$

$H^1(k, \mathbb{Q}(n+1)) = \begin{cases} \mathbb{Q} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$