PURE MOTIVES

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This and the next file are slightly revised versions of my talks at the Palo Alto workshop. I have basically added references.

1. Equivalence relations on algebraic cycles [4]

k field, $\mathrm{SmProj}(k)$ category of smooth projective varieties; $X \in \mathrm{SmProj}(k)$ has $\mathcal{Z}(X)$, group of algebraic cycles on X:

$$\mathcal{Z}^n(X) = \mathbf{Z}[X^{(n)}]$$

$$X^{(n)} = \{\text{points of codimension } n\}.$$

$\mathcal{Z}(X)$ is

- contravariant for flat morphisms
- covariant for all morphisms (with change of codimension).

But:

- not contravariant for arbitrary morphisms
- intersection product not well-behaved.

Both problems: codimension does not behave well by pull-back. Classically solved by *moving cycles*:

Proposition 1 (Chow [1]). *Z, Z' cycles on X. Then there exists a cycle $\tilde{Z}$ on $X \times \mathbf{P}^1$ such that*

- $\tilde{Z}(0) = Z$
- $\tilde{Z}(\infty)$ *meets Z' properly.*

If two cycles meet properly, their intersection product is well-defined.

Definition 1 (Samuel [9]). *Adequate pair*: a pair $(A, \sim)$, A commutative ring, $\sim_X$ equivalence relation on $\mathcal{Z}^*(X) \otimes A$ for all X:

- compatible with A-linear structure and gradation
- $\forall Z, Z' \in \mathcal{Z}^*(X) \otimes A$, $\exists Z_1 \sim_X Z$: Z_1 and Z' meet properly
- $\forall Z \in \mathcal{Z}^*(X) \otimes A$, $\forall \gamma \in \mathcal{Z}^*(X \times Y) \otimes A$ meeting $Z \times Y$ properly, $Z \sim_X 0 \Rightarrow \gamma_*(Z) := p_Y^{XY}(\gamma \cdot (Z \times Y)) \sim_Y 0$.

$(A, \sim)$ adequate pair: get groups $\mathcal{Z}^*_\sim(X, A)$ contravariant for all morphisms, covariant (with codim shift) for all morphisms and with intersection products.

Examples 1 (from finest to coarsest)**.**
Rational equivalence: parametrize with $\mathbf{P}^1$
Algebraic equivalence: parametrize with curves
Smash-nilpotence equivalence (Voevodsky [11]): Z smash-nilpotent on $X \iff Z^{\otimes n} \sim_{\text{rat}} 0$ on X^n for $n \gg 0$
Homological equivalence: see below
Numerical equivalence: $Z \sim_{\text{num}} 0 \iff \deg(Z \cdot Z') = 0 \, \forall Z'$ of complementary codimension (meeting Z properly)

Rational equivalence finest adequate equivalence relation and numerical equivalence coarsest if A is a field.

Usual notation: $\mathcal{Z}^*_{\text{rat}}(X, \mathbf{Z}) = CH^*(X)$ (Chow groups).

Homological equivalence involves a *Weil cohomology theory*:

Definition 2. A Weil cohomology theory with coefficients in a field K is a functor

$$H^* : \mathrm{SmProj}(k)^{op} \to Vec_K^* \text{ (fd graded vector spaces)}$$

with

- $\dim H^2(\mathbf{P}^1) = 1$
- Künneth formula $H^*(X \times Y) \simeq H^*(X) \otimes H^*(Y)$
- Multiplicative trace map $Tr : H^{2d}(X) \to K$ if $\dim X = d$ inducing
- Poincaré duality
- Multiplicative, contravariant and normalised cycle class maps

$$cl : \mathcal{Z}^n(X) \otimes A \to H^{2n}(X)$$

(given homomorphism $A \to K$)

(Normalised means: degree and trace are compatible.)

Then: $Z \sim_H 0 \iff cl(Z) = 0$

1.1. Examples of Weil cohomologies:

(1) In all characteristics: l-adic cohomology $H_l(X) = H^*_{et}(\bar{X}, \mathbf{Q}_l)$, $l \neq \operatorname{char} k$. ($K = \mathbf{Q}_l$.)
(2) In characteristic p, k perfect: crystalline cohomology $H_{cris}(X)$. ($K = Quot(W(k))$.)
(3) In characteristic 0:
   (a) algebraic de Rham cohomology $H_{dR}(X) = \mathbb{H}^*(X, \Omega^{\cdot}_X)$. ($K = k$.)
   (b) Betti cohomology: given $\sigma : k \hookrightarrow \mathbf{C}$, $H_\sigma(X) = H^*_{Betti}(\sigma X(\mathbf{C}), \mathbf{Q})$. ($K = \mathbf{Q}$.)

These are the *classical* Weil cohomologies.

Given an adequate pair $(A, \sim)$, get a category of *pure motives* as end of string of functors:

$$\begin{array}{ccccccc} \text{varieties} & & \text{correspondences} & & \text{effective motives} & & \text{motives} \\ \mathrm{SmProj}(k) & \longrightarrow & \mathrm{Cor}_\sim(k, A) & \xrightarrow{\text{ps-ab envelope}} & \mathrm{Mot}^{\mathrm{eff}}_\sim(k, A) & \xrightarrow{\text{invert } L} & \mathrm{Mot}_\sim(k, A) \\ X & \mapsto & [X] & \mapsto & h(X) & \mapsto & h(X) \\ f & \mapsto & [\Gamma_f] & & & & \end{array}$$

$$h(\operatorname{Spec} k) =: \mathbf{1}$$

$$h(\mathbf{P}^1) = \mathbf{1} \oplus L$$

2. Algebraic correspondences [4]

X, Y smooth projective, $\dim Y = d$:

Definition 3. $\mathrm{Cor}_\sim([X],[Y]) = \mathcal{Z}^d_\sim(X \times Y, A)$.

2.1. Composition of correspondences:

X, Y, Z 3 varieties, $\alpha \in \mathrm{Cor}_\sim([X],[Y])$, $\beta \in \mathrm{Cor}_\sim([Y],[Z])$:

$$\begin{array}{ccccc} & & X \times Y \times Z & & \\ & \swarrow^{p_{XY}} & \downarrow^{p_{XZ}} & \searrow^{p_{YZ}} & \\ X \times Y & & X \times Z & & Y \times Z \\ & & & & \\ \alpha & & \beta \circ \alpha & & \beta \end{array}$$

$$\beta \circ \alpha = (p_{XZ})_*(p_{XY}^*\alpha \cdot p_{YZ}^*\beta).$$

Then $\mathrm{Cor}_\sim(k, A)$ is an A-linear category and $f \mapsto [\Gamma_f]$ (graph) is a functor.

Warning 1. Here this functor is covariant as in Fulton and Voevodsky; it is contravariant with Grothendieck and his school.

$$
\begin{array}{ccccccc}
\text{varieties} & & \text{correspondences} & & \text{effective motives} & & \text{motives} \\
\mathrm{SmProj}(k) & \longrightarrow & \mathrm{Cor}_{\sim}(k, A) & \xrightarrow{\text{ps-ab envelope}} & \mathrm{Mot}^{\mathrm{eff}}_{\sim}(k, A) & \xrightarrow{\text{invert } L} & \mathrm{Mot}_{\sim}(k, A) \\
X & \mapsto & [X] & \mapsto & h(X) & \mapsto & h(X) \\
f & \mapsto & [\Gamma_f] & & & &
\end{array}
$$

$$h(\operatorname{Spec} k) =: \mathbf{1}$$

$$h(\mathbf{P}^1) = \mathbf{1} \oplus L$$

3. Effective motives

Definition 4. $\mathcal{A}$ additive category: $\mathcal{A}$ is *pseudo-abelian* if every idempotent endomorphism has a kernel (hence also an image).

An additive category $\mathcal{A}$ has a *pseudo-abelian envelope* $\natural : \mathcal{A} \to \mathcal{A}^\natural$: $\mathcal{A}^\natural$ pseudo-abelian, $\natural$ additive and universal for additive functors to pseudo-abelian categories. $\mathcal{A}$ A-linear $\Rightarrow$ $\mathcal{A}^\natural$, $\natural$ A-linear.

3.1. Description of $\mathcal{A}^\natural$:

- Objects: pairs (M, p), $M \in \mathcal{A}$, $p = p^2 \in End(M)$.
- Morphisms: $Hom((M, p), (N, q)) = qHom(M, N)p$.

The functor $\natural$ is fully faithful.

Definition 5. $\mathrm{Mot}^{\mathrm{eff}}_\sim(k, A) = \mathrm{Cor}_\sim(k, A)^\natural$.

$$\begin{array}{ccccccc}
\text{varieties} & & \text{correspondences} & & \text{effective motives} & & \text{motives} \\
\mathrm{SmProj}(k) & \longrightarrow & \mathrm{Cor}_\sim(k, A) & \xrightarrow{\text{ps-ab envelope}} & \mathrm{Mot}^{\mathrm{eff}}_\sim(k, A) & \xrightarrow{\text{invert } L} & \mathrm{Mot}_\sim(k, A) \\
X & \mapsto & [X] & \mapsto & h(X) & \mapsto & h(X) \\
f & \mapsto & [\Gamma_f] & & & &
\end{array}$$

$$h(\operatorname{Spec} k) =: \mathbf{1}$$

$$h(\mathbf{P}^1) = \mathbf{1} \oplus L$$

L is the Lefschetz motive.

4. Tensor structure

The symmetric monoidal structure $(X, Y) \mapsto X \times Y$ on $\mathrm{SmProj}(k)$ extends to an A-linear unital symmetric monoidal structure (:= tensor structure) on $\mathrm{Cor}_\sim(k, A)$ (unit: $[\operatorname{Spec} k]$).

$\mathcal{A}$ tensor category $\Rightarrow \mathcal{A}^\natural$ tensor category and $\natural$ tensor functor.

$\mathcal{A}$ category, $L : \mathcal{A} \to \mathcal{A}$ endofunctor: universal construction

$$\mathcal{A} \to \mathcal{A}[L^{-1}]$$

such that $M \mapsto L(M)$ becomes equivalence of categories.

4.1. Description of $\mathcal{A}[L^{-1}]$:

- Objects: pairs (M, m), $M \in \mathcal{A}$, $m \in \mathbf{Z}$.
- Morphisms: $Hom((M, m), (N, n)) = \varinjlim Hom(L^{k+m}(M), L^{k+n}(N))$.

If $\mathcal{A}$ tensor category and $L \in \mathcal{A}$, apply this to $L(M) = M \otimes L$ and get $\mathcal{A}[L^{-1}]$.

Lemma 1 (Voevodsky)**.** *$\mathcal{A}[L^{-1}]$ is tensor if and only if the cycle (123) acts on $L^{\otimes 3}$ as the identity.*

5. Motives

Definition 6. $\mathrm{Mot}_\sim(k, A) = \mathrm{Mot}^{\mathrm{eff}}_\sim(k, A)[L^{-1}]$ (L the Lefschetz motive).

$T := L^{-1}$ the Tate motive.

Notation 1. $M(n) = M \otimes L^{\otimes n}$.

Warning 2. Grothendieck writes $M(-n)$ instead of $M(n)$.

Projective bundle formula $\Rightarrow$ $M \mapsto M(1)$ fully faithful on $\mathrm{Mot}^{\mathrm{eff}}_\sim(k, A) \Rightarrow \mathrm{Mot}^{\mathrm{eff}}_\sim(k, A) \to \mathrm{Mot}_\sim(k, A)$ fully faithful.

$$
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\text{varieties} & & \text{correspondences} & & \text{effective motives} & & \text{motives} \\
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X & \mapsto & [X] & \mapsto & h(X) & \mapsto & h(X) \\
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$$h(\operatorname{Spec} k) =: \mathbf{1}$$

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6. Duals and rigidity

Definition 7 (Dold-Puppe [3]). $\mathcal{A}$ tensor category.
a) $M \in \mathcal{A}$: M has a dual if $\exists M^* \in \mathcal{A}$, $\eta_M : \mathbf{1} \to M^* \otimes M$, $\varepsilon_M : M \otimes M^* \to \mathbf{1}$ such that both compositions

$$M \xrightarrow{1_M \otimes \eta_M} M \otimes M^* \otimes M \xrightarrow{\varepsilon_M \otimes 1_M} M$$

$$M^* \xrightarrow{\eta_M \otimes 1_{M^*}} M^* \otimes M \otimes M^* \xrightarrow{1_{M^*} \otimes \varepsilon_M} M^*$$

equal the identity.
b) $\mathcal{A}$ is rigid if every object has a dual.

Proposition 2 (not difficult). $\mathrm{Mot}_\sim(k, A)$ *is rigid.*

Dual of $h(X)$: $h(X)(-\dim X)$; η, ε both given by $\Delta_X \in \mathcal{Z}^{\dim X}_\sim(X \times X)$.

7. Traces

$\mathcal{A}$ tensor category, $M \in \mathcal{A}$ has a dual: $\forall N \in \mathcal{A}$, isomorphism

$$\iota_{M,N} : Hom(\mathbf{1}, M^* \otimes N) \longrightarrow Hom(M, N)$$
$$\iota_{M,N}(f) = (\varepsilon_M \otimes 1_N) \circ (1_M \otimes f)$$
$$\iota^{-1}_{M,N}(g) = (1_{M^*} \otimes g) \circ \eta_M$$

Definition 8. a) $f \in End(M)$:

$$tr(f) \in End(\mathbf{1})$$

defined by composition

$$\mathbf{1} \xrightarrow{\iota^{-1}_{M,M}(f)} M^* \otimes M \xrightarrow{\text{switch}} M \otimes M^* \longrightarrow \mathbf{1}.$$

b) $\dim M := tr(1_M)$.

$H : \mathcal{A} \longrightarrow \mathcal{B}$ tensor functor: $tr(H(f)) = H(tr(f))$ (obvious) $\Rightarrow$ if $End_{\mathcal{A}}(\mathbf{1}) \hookrightarrow End_{\mathcal{B}}(\mathbf{1})$, may compute $tr(f)$ via H.

7.1. Application: the trace formula.

H Weil cohomology with coefficients K, $A \hookrightarrow K$: take $\mathcal{A} = \mathrm{Mot}_{\mathrm{rat}}(k, A)$, $\mathcal{B} = Vec_K^*$, $H = H$. For X smooth projective and $f \in \mathrm{Cor}_\sim([X], [X]) = \mathrm{Mot}_\sim(h(X), h(X))$,

$$tr(f) = tr(H(f)).$$

This is the trace formula:

- Left hand side $= f \cdot \Delta_X$
- Right hand side $= \sum_{i=0}^{2d} (-1)^i Tr(f \mid H^i(X))$.

Corollary 1. *$\sum_{i=0}^{2d} (-1)^i Tr(f \mid H^i(X))$ independent of H. In particular, $\dim_{rigid} h_H(X) = \chi_H(X)$ independent of H.*

Corollary 2. *$f \in \mathrm{Mot}_{\mathrm{num}}(h(X), h(X))$: may compute $tr(f)$ by lifting f to H-equivalence (for some H) and computing the trace via H. E.g. $\dim_{rigid} h_{\mathrm{num}}(X) = \dim_{rigid} h_H(X) = \chi_H(X)$.*

How about the Betti numbers of X themselves?

7.1.1. *In characteristic 0:* Comparison theorems

- Betti-de Rham: $H^i_\sigma(X) \otimes_{\mathbf{Q}} \mathbf{C} \simeq H^i_{dR}(X) \otimes_k \mathbf{C}$ (period isomorphisms, Grothendieck [5])
- Betti-l-adic: $H^i_\sigma(X) \otimes_{\mathbf{Q}} \mathbf{Q}_l \simeq H^i_l(X)$ (Grothendieck-Artin [12])

7.1.2. *In characteristic p:* Weil conjectures

- Deligne [2]: $\forall i \ \det(1 - tF \mid H^i_l(X))$ independent of l
- Katz-Messing [7]: also true for $H^i_{cris}(X)$.

In particular, the ranks are all equal...

Much deeper than for Euler-Poincaré characteristic!

7.1.3. *Cheaper approach:* Chow-Künneth decomposition

- Šermenev [10]: X abelian variety of dimension $d \Rightarrow h_{\mathrm{rat}}(X) \simeq \bigoplus_{i=0}^{2d} h^i(X)$ with $H(h^i(X)) = H^i(X)$ for any Weil cohomology.
- Murre [8]: true for any X if $d \leq 2$.

In both cases, Betti numbers only depend on X for any Weil cohomology, not only classical ones. Same for trace of an endomorphism. (Independence of l in characteristic p!)

Conjecturally true for any X.

8. Jannsen's theorem

Theorem 1 (Jannsen [6])**.** *For any k,* $\mathrm{Mot}_{\mathrm{num}}(k, \mathbf{Q})$ *is abelian semi-simple. Moreover* num *is the only adequate equivalence relation with this property.*

Proof not really difficult but uses existence of a Weil cohomology.

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