This and the next file are slightly revised versions of my talks at the Palo Alto workshop. I have basically added references.
1. **Equivalence relations on algebraic cycles** [4]

$k$ field, $\text{SmProj}(k)$ category of smooth projective varieties; $X \in \text{SmProj}(k)$ has $\mathcal{Z}(X)$, group of algebraic cycles on $X$:

$$\mathcal{Z}^n(X) = \mathbb{Z}[X^{(n)}]$$

$$X^{(n)} = \{\text{points of codimension } n\}.$$  

$\mathcal{Z}(X)$ is

- **contravariant** for flat morphisms
- **covariant** for all morphisms (with change of codimension).

But:

- **not contravariant** for arbitrary morphisms
- **intersection product** not well-behaved.
Both problems: codimension does not behave well by pull-back. Classically solved by moving cycles:

**Proposition 1 (Chow [1]).** $Z, Z'$ cycles on $X$. Then there exists a cycle $\tilde{Z}$ on $X \times \mathbb{P}^1$ such that

- $\tilde{Z}(0) = Z$
- $\tilde{Z}(\infty)$ meets $Z'$ properly.

If two cycles meet properly, their intersection product is well-defined.
Definition 1 (Samuel [9]). Adequate pair: a pair $(A, \sim)$, $A$ commutative ring, $\sim_x$ equivalence relation on $\mathcal{Z}^*(X) \otimes A$ for all $X$:

- compatible with $A$-linear structure and gradation
- $\forall Z, Z' \in \mathcal{Z}^*(X) \otimes A$, $\exists Z_1 \sim_x Z$: $Z_1$ and $Z'$ meet properly
- $\forall Z \in \mathcal{Z}^*(X) \otimes A$, $\forall \gamma \in \mathcal{Z}^*(X \times Y) \otimes A$ meeting $Z \times Y$ properly, $Z \sim_x 0 \Rightarrow \gamma^*_Z(Z) := p^{XY}_Y(\gamma \cdot (Z \times Y)) \sim_Y 0$.

$(A, \sim)$ adequate pair: get groups $\mathcal{Z}^*(X, A)$ contravariant for all morphisms, covariant (with codim shift) for all morphisms and with intersection products.
Examples 1 (from finest to coarsest).

Rational equivalence: parametrize with $\mathbb{P}^1$

Algebraic equivalence: parametrize with curves

Smash-nilpotence equivalence (Voevodsky [11]): $Z$ smash-nilpotent on $X \iff Z^\otimes n \sim_{rat} 0$
on $X^n$ for $n \gg 0$

Homological equivalence: see below

Numerical equivalence: $Z \sim_{num} 0 \iff \deg(Z \cdot Z') = 0 \forall Z'$ of complementary codimension (meeting $Z$ properly)

Rational equivalence finest adequate equivalence relation and numerical equivalence coarsest if $A$ is a field.

Usual notation: $\mathcal{Z}_{rat}^*(X, \mathbb{Z}) = CH^*(X)$ (Chow groups).
Homological equivalence involves a **Weil cohomology theory**:

**Definition 2.** A Weil cohomology theory with coefficients in a field $K$ is a functor

$$H^* : \text{SmProj}(k)^{op} \rightarrow Vec^*_K \text{ (fd graded vector spaces)}$$

with

- $\dim H^2(\mathbb{P}^1) = 1$
- K"unneth formula $H^*(X \times Y) \simeq H^*(X) \otimes H^*(Y)$
- Multiplicative trace map $Tr : H^{2d}(X) \rightarrow K$ if $\dim X = d$ inducing
- Poincaré duality
- Multiplicative, contravariant and normalised cycle class maps

$$cl : \mathcal{Z}^n(X) \otimes A \rightarrow H^{2n}(X)$$

(given homomorphism $A \rightarrow K$)

(Normalised means: degree and trace are compatible.)

Then: $Z \sim_H 0 \iff cl(Z) = 0$
1.1. Examples of Weil cohomologies:

1. In all characteristics: $l$-adic cohomology $H_l(X) = H^*_{\text{et}}(\bar{X}, \mathbb{Q}_l), \ l \neq \text{char} \ k. \ (K = \mathbb{Q}_l).$
2. In characteristic $p$, $k$ perfect: crystalline cohomology $H_{\text{cris}}(X). \ (K = \text{Quot}(W(k))).$
3. In characteristic 0:
   a. algebraic de Rham cohomology $H_{dR}(X) = \mathbb{H}^*(X, \Omega^*_X). \ (K = k).$
   b. Betti cohomology: given $\sigma : k \hookrightarrow \mathbb{C}, \ H_\sigma(X) = H^*_{\text{Betti}}(\sigma X(\mathbb{C}), \mathbb{Q}). \ (K = \mathbb{Q}).$

These are the classical Weil cohomologies.
Given an adequate pair $(A, \sim)$, get a category of *pure motives* as end of string of functors:

<table>
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<th>varieties</th>
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<tr>
<td>$\text{SmProj}(k)$</td>
<td>$\text{Cor}_\sim(k, A)$</td>
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<td>$X$</td>
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<td>$f$</td>
<td>$[\Gamma_f]$</td>
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$h(\text{Spec } k) =: 1$

$h(\mathbb{P}^1) = 1 \oplus L$
2. **Algebraic correspondences** [4]

$X, Y$ smooth projective, $\dim Y = d$:

**Definition 3.** $\text{Cor}_{\sim}([X], [Y]) = Z^d_{\sim}(X \times Y, A)$.

### 2.1. Composition of correspondences:

$X, Y, Z$ 3 varieties, $\alpha \in \text{Cor}_{\sim}([X], [Y]), \beta \in \text{Cor}_{\sim}([Y], [Z])$:

![Diagram](attachment:image.png)

$$\beta \circ \alpha = (p_{XZ})_*(p_{XY}^* \alpha \cdot p_{YZ}^* \beta).$$

Then $\text{Cor}_{\sim}(k, A)$ is an $A$-linear category and $f \mapsto [\Gamma_f]$ (graph) is a functor.

**Warning 1.** Here this functor is **covariant** as in Fulton and Voevodsky; it is **contravariant** with Grothendieck and his school.
\[
\begin{array}{cccccc}
\text{varieties} & \text{correspondences} & \text{effective motives} & \text{motives} \\
\text{SmProj}(k) & \rightarrow & \text{Cor}_{\sim}(k, A) & \overset{\text{ps-ab envelope}}{\rightarrow} & \text{Mot}_{\sim}^{\text{eff}}(k, A) & \overset{\text{invert } L}{\rightarrow} & \text{Mot}_{\sim}(k, A) \\
X & \mapsto & [X] & \mapsto & h(X) & \mapsto & h(X) \\
f & \mapsto & [\Gamma_f] & & \\
\end{array}
\]

\[
h(\text{Spec } k) =: 1 \\
h(\mathbb{P}^1) = 1 \oplus L
\]
3. **Effective motives**

**Definition 4.** $\mathcal{A}$ additive category: $\mathcal{A}$ is *pseudo-abelian* if every idempotent endomorphism has a kernel (hence also an image).

An additive category $\mathcal{A}$ has a *pseudo-abelian envelope* $\mathcal{A}^\natural: \mathcal{A} \to \mathcal{A}^\sharp$: $\mathcal{A}^\sharp$ pseudo-abelian, $\mathcal{A}^\natural$ additive and universal for additive functors to pseudo-abelian categories. $\mathcal{A}$ $\mathcal{A}$-linear $\Rightarrow \mathcal{A}^\natural, \mathcal{A}^\sharp$ $\mathcal{A}$-linear.

3.1. **Description of $\mathcal{A}^\natural$:**

- **Objects:** pairs $(M, p)$, $M \in \mathcal{A}$, $p = p^2 \in \text{End}(M)$.
- **Morphisms:** $\text{Hom}((M, p), (N, q)) = q \text{Hom}(M, N)p$.

The functor $\mathcal{A}^\natural$ is fully faithful.

**Definition 5.** $\text{Mot}_{\sim}^\text{eff}(k, A) = \text{Cor}_{\sim}(k, A)^\natural$. 
$\text{SmProj}(k) \quad \xrightarrow{\text{varieties}} \quad \text{Cor}_\sim(k, A) \quad \xrightarrow{\text{correspondences}} \quad \text{Mot}_\sim(k, A) \quad \xrightarrow{\text{effective motives}} \quad \text{Mot}_\sim(k, A)$

$X \quad \xrightarrow{\text{varieties}} \quad [X] \quad \xrightarrow{\text{correspondences}} \quad h(X) \quad \xrightarrow{\text{effective motives}} \quad h(X)$

$f \quad \xrightarrow{\text{varieties}} \quad [\Gamma_f] \quad \xrightarrow{\text{correspondences}} \quad h(X) \quad \xrightarrow{\text{effective motives}} \quad h(X)$

$h(\text{Spec } k) =: 1$

$h(\mathbb{P}^1) = 1 \oplus L$

$L$ is the Lefschetz motive.
4. Tensor structure

The symmetric monoidal structure $(X, Y) \mapsto X \times Y$ on $\text{SmProj}(k)$ extends to an $A$-linear unital symmetric monoidal structure (:= tensor structure) on $\text{Cor}_{\sim}(k, A)$ (unit: $[\text{Spec } k]$).

A tensor category $\Rightarrow A^\sharp$ tensor category and $\sharp$ tensor functor.

A category, $L : A \to A$ endofunctor: universal construction

\[ A \to A[L^{-1}] \]

such that $M \mapsto L(M)$ becomes equivalence of categories.
4.1. Description of $\mathcal{A}[L^{-1}]$:

- **Objects**: pairs $(M, m)$, $M \in \mathcal{A}$, $m \in \mathbb{Z}$.
- **Morphisms**: $\text{Hom}((M, m), (N, n)) = \lim \rightarrow \text{Hom}(L^{k+m}(M), L^{k+n}(N))$.

If $\mathcal{A}$ tensor category and $L \in \mathcal{A}$, apply this to $L(M) = M \otimes L$ and get $\mathcal{A}[L^{-1}]$.

**Lemma 1 (Voevodsky).** $\mathcal{A}[L^{-1}]$ is tensor if and only if the cycle $(123)$ acts on $L^\otimes 3$ as the identity.
5. Motives

Definition 6. $\text{Mot}_\sim(k, A) = \text{Mot}^\text{eff}_\sim(k, A)[L^{-1}]$ ($L$ the Lefschetz motive).

$T := L^{-1}$ the Tate motive.

Notation 1. $M(n) = M \otimes L \otimes n$.

Warning 2. Grothendieck writes $M(-n)$ instead of $M(n)$.

Projective bundle formula $\Rightarrow M \mapsto M(1)$ fully faithful on $\text{Mot}^\text{eff}_\sim(k, A) \Rightarrow \text{Mot}^\text{eff}_\sim(k, A) \rightarrow \text{Mot}_\sim(k, A)$ fully faithful.
varieties $\text{SmProj}(k) \longrightarrow \text{Cor}_\sim(k, A)$

$X \mapsto [X]$ 

$f \mapsto [\Gamma_f]$ 

correspondences $\text{ps-ab envelope}$

effective motives $\text{Mot}_\sim^\text{eff}(k, A)$

$h(X)$ 

motives $\text{Mot}_\sim(k, A)$

$h(X)$

$h(\text{Spec } k) =: 1$

$h(\mathbb{P}^1) = 1 \oplus L$
6. **Duals and rigidity**

**Definition 7** (Dold-Puppe [3]). \( \mathcal{A} \) tensor category.

a) \( M \in \mathcal{A} \): \( M \) has a dual if \( \exists M^* \in \mathcal{A} \), \( \eta_M : 1 \to M^* \otimes M \), \( \varepsilon_M : M \otimes M^* \to 1 \) such that both compositions

\[
\begin{align*}
M & \xrightarrow{1_M \otimes \eta_M} M \otimes M^* \otimes M \xrightarrow{\varepsilon_M \otimes 1_M} M \\
M^* & \xrightarrow{\eta_M \otimes 1_{M^*}} M^* \otimes M \otimes M^* \xrightarrow{1_{M^*} \otimes \varepsilon_M} M^*
\end{align*}
\]
equal the identity.

b) \( \mathcal{A} \) is **rigid** if every object has a dual.

**Proposition 2** (not difficult). Mot\(\sim(k, A)\) is rigid.

Dual of \( h(X) \): \( h(X)(- \dim X) \); \( \eta, \varepsilon \) both given by \( \Delta_X \in \mathcal{C}^{\dim X}(X \times X) \).
7. Traces

A tensor category, $M \in \mathcal{A}$ has a dual: $\forall N \in \mathcal{A}$, isomorphism

$$\iota_{M,N} : Hom(1, M^* \otimes N) \rightarrow Hom(M, N)$$

$$\iota_{M,N}(f) = (\varepsilon_M \otimes 1_N) \circ (1_M \otimes f)$$

$$\iota_{M,N}^{-1}(g) = (1_{M^*} \otimes g) \circ \eta_M$$

Definition 8. a) $f \in End(M)$:

$$tr(f) \in End(1)$$

defined by composition

$$1 \xrightarrow{\iota_{M,M}^{-1}(f)} M^* \otimes M \xrightarrow{\text{switch}} M \otimes M^* \rightarrow 1.$$ 

b) dim $M := tr(1_M)$.

$H : \mathcal{A} \rightarrow \mathcal{B}$ tensor functor: $tr(H(f)) = H(tr(f))$ (obvious) $\Rightarrow$ if $End_A(1) \hookrightarrow End_B(1)$, may compute $tr(f)$ via $H$. 


7.1. **Application: the trace formula.**

$H$ Weil cohomology with coefficients $K$, $A \hookrightarrow K$: take $A = \text{Mot}_{\text{rat}}(k, A)$, $\mathcal{B} = Vec^*_K$, $H = H$. For $X$ smooth projective and $f \in \text{Cor}_\sim([X], [X]) = \text{Mot}_\sim(h(X), h(X))$, 

$$tr(f) = tr(H(f)).$$

This is the trace formula:

- Left hand side $= f \cdot \Delta_X$
- Right hand side $= \sum_{i=0}^{2d} (-1)^i Tr(f \mid H^i(X))$.

**Corollary 1.** $\sum_{i=0}^{2d} (-1)^i Tr(f \mid H^i(X))$ independent of $H$. In particular, $\dim_{\text{rigid}} h_H(X) = \chi_H(X)$ independent of $H$.

**Corollary 2.** $f \in \text{Mot}_{\text{num}}(h(X), h(X))$: may compute $tr(f)$ by lifting $f$ to $H$-equivalence (for some $H$) and computing the trace via $H$. E.g. $\dim_{\text{rigid}} h_{\text{num}}(X) = \dim_{\text{rigid}} h_H(X) = \chi_H(X)$.

How about the Betti numbers of $X$ themselves?
7.1.1. In characteristic 0: Comparison theorems

- Betti-de Rham: $H^i_\sigma(X) \otimes_\mathbb{Q} \mathbb{C} \simeq H^i_{dR}(X) \otimes_k \mathbb{C}$ (period isomorphisms, Grothendieck [5])
- Betti-$l$-adic: $H^i_\sigma(X) \otimes_\mathbb{Q} \mathbb{Q}_l \simeq H^i_l(X)$ (Grothendieck-Artin [12])

7.1.2. In characteristic $p$: Weil conjectures

- Deligne [2]: $\forall i \det(1 - tF \mid H^i_l(X))$ independent of $l$
- Katz-Messing [7]: also true for $H^i_{\text{cris}}(X)$.

In particular, the ranks are all equal...

Much deeper than for Euler-Poincaré characteristic!

7.1.3. Cheaper approach: Chow-Kühneth decomposition

- Šermenev [10]: $X$ abelian variety of dimension $d \Rightarrow h_{\text{rat}}(X) \simeq \bigoplus_{i=0}^{2d} h^i(X)$ with $H(h^i(X)) = H^i(X)$ for any Weil cohomology.
- Murre [8]: true for any $X$ if $d \leq 2$.

In both cases, Betti numbers only depend on $X$ for any Weil cohomology, not only classical ones. Same for trace of an endomorphism. (Independence of $l$ in characteristic $p$!)

Conjecturally true for any $X$. 
8. **Jannsen’s Theorem**

**Theorem 1** (Jannsen [6]). For any \( k \), \( \text{Mot}_{\text{num}}(k, \mathbb{Q}) \) is abelian semi-simple. Moreover \( \text{num} \) is the only adequate equivalence relation with this property.

Proof not really difficult but uses existence of a Weil cohomology.