MOTIVIC GALOIS GROUPS

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1. **Tannakian categories ([26], cf. [7])**

$K$ field of characteristic 0, $\mathcal{A}$ rigid tensor $K$-linear *abelian* category, $L$ extension of $K$.

**Definition 1.** An $L$-valued fibre functor is a tensor functor $\omega : \mathcal{A} \to \text{Vec}_L$ which is *faithful* and *exact*.

**Definition 2.** $\mathcal{A}$ is

- neutralised Tannakian if one is given a $K$-valued fibre functor
- neutral Tannakian if $\exists$ $K$-valued fibre functor
- Tannakian if $\exists$ $L$-valued fibre functor for some $L$.

**Example 1.** $G$ affine $K$-group scheme, $\mathcal{A} = \text{Rep}_K(G)$, $\omega : \mathcal{A} \to \text{Vec}_K$ the forgetful functor.
(\mathcal{A},\omega) neutralised Tannakian category: \( G_K := Aut^\otimes(\omega) \) is (canonically) the \( K \)-points of an affine \( K \)-group scheme \( G(\omega) \).

**Theorem 1** (Grothendieck-Saavedra [26]).

a) For \((\mathcal{A},\omega)\) as in Example 1, \( G(\mathcal{A},\omega) = G \).

b) In general \( \omega \) enriches into a tensor equivalence of categories

\[ \tilde{\omega} : \mathcal{A} \xrightarrow{\sim} \text{Rep}_K(G(\mathcal{A},\omega)) \].

\( c) \) **Dictionary** (special case): \( \mathcal{A} \) semi-simple \( \iff \) \( G \) proreductive.

When \( \mathcal{A} \) Tannakian but not neutralised, need replace \( G(\mathcal{A},\omega) \) by a *gerbe* (or a groupoid): Saavedra-Deligne [8].

**Theorem 2** (Deligne [8]). \( \mathcal{A} \) rigid \( K \)-linear abelian. Equivalent conditions:

- \( \mathcal{A} \) is Tannakian
- \( \forall M \in \mathcal{A}, \exists n > 0: \Lambda^n(M) = 0. \)
- \( \forall M \in \mathcal{A}, \dim_{\text{rigid}}(M) \in \mathbb{N}. \)
2. Are motives Tannakian?

Ideally, would like $Mot_{\text{num}}(k, \mathbb{Q})$ Tannakian, fibre functors given by Weil cohomologies $H$. Two problems:

- $Mot_{\text{num}}(k, \mathbb{Q})$ is never Tannakian because $\dim_{\text{rigid}}(X) = \chi(X)$ may be negative (e.g. $X$ curve of genus $g$: $\chi(X) = 2 - 2g$).

Second problem: matter of commutativity constraint – need modify it.

Yields Grothendieck’s *standard conjectures* ([13], cf. [20]):

- (HN) $\sim_H \sim_{\text{num}}$.
- (C) $\forall X$ the Künneth components of $H(\Delta_X)$ are algebraic.
Another conjecture (B) (skipped):

- (HN) $\Rightarrow$ (B) $\Rightarrow$ (C).
- (HN) $\iff$ (B) in characteristic 0.

**Theorem 3** (Lieberman-Kleiman [19]). *Conjecture (B) holds for abelian varieties.*

**Theorem 4** (Katz-Messing [18]). *Conjecture (C) is true if $k$ finite.*

**Corollary 1** (Jannsen [14]). *If $k$ finite, a suitable modification $\widetilde{\text{Mot}}_{\text{num}}(k, \mathbb{Q})$ is (abstractly) Tannakian.*

Apart from this, wide open!

**Definition 3.** When $\widetilde{\text{Mot}}_{\text{num}}(k, \mathbb{Q})$ exists, the gerbe that classifies it is called the [pure] motivic Galois group $GMot_k$. $H$ Weil cohomology with coefficients $K$: fibre of $GMot_k$ at $H$ is proreductive $K$-group $GMot_{H,k}$.

More generally, $\mathcal{A}$ thick rigid subcategory of $\text{Mot}_{\text{num}}$, get an “induced” Galois group $GMot(\mathcal{A})$ of $\mathcal{A}$, quotient of the motivic Galois group. E.g. $\mathcal{A}$ thick rigid subcategory generated by $h(X)$: get the motivic Galois group of $X$ $GMot_{H,k}(X)$(of finite type).
Examples 2.

1. $\mathcal{A} =$ Artin motives (generated by $h(\text{Spec } E), [E : k] < \infty)$: $\text{GMot}(\mathcal{A}) = G_k$.

2. $\mathcal{A} =$ pure Tate motives (generated by $L$ or $h(\mathbb{P}^1)$): $\text{GMot}(\mathcal{A}) = \mathbb{G}_m$.

3. $\mathcal{A} =$ pure Artin-Tate motives (put these two together): $\text{GMot}(\mathcal{A}) = G_k \times \mathbb{G}_m$.

4. $E$ elliptic curve over $\mathbb{Q}$, $H = H_{Betti}$.
   - $E$ not CM $\Rightarrow \text{GMot}_{H,\mathbb{Q}}(E) = GL_2$.
   - $E$ CM $\Rightarrow \text{GMot}_{H,\mathbb{Q}}(E) =$ torus in $GL_2$ or its normaliser.
Example 3. Suppose Conjecture (HN) true.

- **Characteristic 0**: Betti cohomology yields (several) $\mathbb{Q}$-valued fibre functors, as long as $\text{card}(k) \leq \text{card}(\mathbb{C})$: $\text{Mot}_{\text{num}}(k, \mathbb{Q})$ is neutral. Comparison isomorphisms $\Rightarrow$ isomorphisms between various motivic Galois groups.

- **Characteristic $p$**: $k \supseteq \mathbb{F}_{p^2}$ finite $\Rightarrow$ $\text{Mot}_{\text{num}}(k, \mathbb{Q})$ is *not* neutral: if $K \subseteq \mathbb{R}$ or $K \subseteq \mathbb{Q}_p$, no $K$-valued fibre functor (Serre: endomorphisms of a supersingular elliptic curve = quaternion $\mathbb{Q}$-algebra nonsplit by $\mathbb{R}, \mathbb{Q}_p$).
3. Connection with Hodge and Tate conjectures

3.1. Tate conjecture.

$k$ finitely generated, $G_k := \text{Gal}(\overline{k}/k)$, $H = H_l (l \neq \text{char} k)$: the $\otimes$-functor

$$H_l : \text{Mot}_H \rightarrow \text{Vec}_{\mathbb{Q}_l}^*$$

enriches into a $\otimes$-functor

$$\hat{H}_l : \text{Mot}_H \rightarrow \text{Rep}_{\mathbb{Q}_l}^{\text{cont}}(G_k)^*.$$

Tate conjecture $\iff \hat{H}_l$ fully faithful (it is faithful by definition).

**Proposition 1.** Tate conjecture $\Rightarrow$ Conjecture (B).

Hence under Tate conjecture, Conjecture (C) holds and can modify commutativity constraint:

$$\hat{H}_l : \hat{\text{Mot}}_H \rightarrow \text{Rep}_{\mathbb{Q}_l}^{\text{cont}}(G_k).$$
(\text{Rep}_{\mathbb{Q}_l}^{\text{cont}}(G_k), \text{forgetful functor}) \text{ neutralised Tannakian } \mathbb{Q}_l\text{-category with fundamental group } \Gamma_k: \text{ for } V \in \text{Rep}_{\mathbb{Q}_l}^{\text{cont}}(G_k), \Gamma_k(V) = \text{Zariski closure of } G_k \text{ in } GL(V).

\textbf{Proposition 2} (folklore, cf. [27], [17]). \textit{Assume Tate conjecture. Equivalent conditions:}

- Conjecture (HN);
- \text{Im}\tilde{H}_l \subseteq \text{Rep}_{\mathbb{Q}_l}^{\text{cont}}(G_k)_{ss} \text{ (full subcategory of semi-simple representations)}.

Under these conditions, Mot_{\text{num}} \text{Tannakian, reduce to } \Gamma_{k,ss} \text{ (for } \text{Rep}_{\mathbb{Q}_l}^{\text{cont}}(G_k)_{ss} \text{) proreductive and canonical epimorphism}

\[ \Gamma_{k,ss} \longrightarrow GMot_{H_l,k}. \]

In particular, \( \forall X, GMot_{H_l,k}(X) = \text{Zariski closure of } G_k \text{ in } GL(H_l(X)). \)

\textbf{Delicate question:} essential image of \( \tilde{H}_l \)? Conjectural answers for \( k \) finite (see below) and \( k \) number field (Fontaine-Mazur [11]).
3.2. Hodge conjecture.

$\sigma : k \hookrightarrow \mathbb{C}, H = H_\sigma$: this time enriches into $\otimes$-functor

$$\hat{H}_\sigma : \text{Mot}_{H_\sigma} \to PHS^*_\mathbb{Q}$$

(graded pure Hodge structures over $\mathbb{Q}$). Hodge conjecture $\iff \hat{H}_\sigma$ fully faithful.

**Proposition 3.** Hodge conjecture $\Rightarrow$ Conjecture (B) $\iff$ Conjecture (HN).

Hence, under Hodge conjecture, get modified fully faithful tensor functor

$$\tilde{H}_\sigma : \text{Mot}_{\text{num}} \to PHS_{\mathbb{Q}}.$$ 

Latter category semi-simple neutralised Tannakian (via forgetful functor). If extend scalars to $\mathbb{R}$, fundamental group = Hodge torus $S = \mathbb{R}_c/\mathbb{R}\mathbb{G}_m$. Over $\mathbb{Q}$ it is the Mumford-Tate group $MT$: for $V \in PHS_{\mathbb{Q}}$, $MT(V) = \mathbb{Q}$—Zariski closure of $S$ in $GL(V)$.

Hodge conjecture $\iff \forall X, GMot_{k,H_\sigma}(X) = MT(X) \subseteq GL(H_\sigma(X))$.

Sometimes gives proof of Hodge conjecture (for powers of $X$, $X$ abelian variety)!
4. **Unconditional motivic Galois groups**

Want an unconditional theory of motives (not assuming the unproven standard conjectures)

4.1. **First approach (Deligne, André).**

Both are in characteristic 0.

- **Deligne** [10]: replace motives by systems of compatible realisations: motives for **absolute Hodge cycles** (systems of cohomology classes corresponding to each other by comparison isomorphisms). Gives semi-simple Tannakian category.
  
  Hodge conjecture $\Rightarrow$ absolute Hodge cycles are algebraic so same category.

- **André** [3]: only adjoin to algebraic cycles the inverses of the Lefschetz operators: motives for **cycles**. Gives semi-simple Tannakian category.
  
  Conjecture (B) $\Rightarrow$ motivated cycles are algebraic so same category.

  (Hodge conjecture $\Rightarrow$ Conjecture (B) so cheaper approach!)
A abelian variety over number field:

**Theorem 5** (Deligne [9]). Every Hodge cycle on $A$ is absolutely Hodge.

**Corollary 2.** Tate conjecture $\Rightarrow$ Hodge conjecture on $A$.

Better:

**Theorem 6** (André [3]). Every Hodge cycle on $A$ is motivated.

**Corollary 3.** Conjecture (B) for abelian fibrations on curves $\Rightarrow$ Hodge conjecture on $A$.

Tannakian arguments:

**Theorem 7** (Milne [23]). Hodge conjecture for complex CM abelian varieties $\Rightarrow$ Tate conjecture for all abelian varieties over a finite field.

**Theorem 8** (André [4]). A abelian variety over a finite field: every Tate cycle is motivated.
4.2. Second approach (André-K): tensor sections.

A pseudo-abelian \( \mathbb{Q} \)-linear category, \( \mathcal{R} \) Kelly radical of \( \mathcal{A} \) (like Jacobson radical of rings): smallest ideal such that \( \mathcal{A}/\mathcal{R} \) semi-simple.

If \( \mathcal{A} \) tensor category, \( \mathcal{R} \) may or may not be stable under \( \otimes \). True e.g. if \( \mathcal{A} \) Tannakian.

**Theorem 9** (André-K [6]). Suppose that \( \mathcal{R} \) is \( \otimes \)-ideal, \( \mathcal{A}(1,1) = \mathbb{Q} \) and \( \mathcal{R}(M,M) \) nilpotent ideal of \( \mathcal{A}(M,M) \) for all \( M \). Then the projection functor

\[ \mathcal{A} \rightarrow \mathcal{A}/\mathcal{R} \]

has tensor sections, and any two are tensor-conjugate.
Application: 
$H$ classical Weil cohomology,

$$
\mathcal{A} = \text{Mot}_H^\pm(k, \mathbb{Q}) := \{ M \in \text{Mot}_H(k, \mathbb{Q}) \mid \text{sum of even K"unneth projectors of } M \text{ algebraic}\}.
$$

Then $\mathcal{A}$ satisfies assumptions of Theorem 9: in characteristic 0 by comparison isomorphisms, in characteristic $p$ by Weil conjectures.

**Theorem 10** (André-K [5]).

a) $\text{Mot}_\text{num}^\pm := \text{Im}(\text{Mot}_H^\pm \to \text{Mot}_\text{num}^\pm)$ independent of $H$.

b) Can modify commutativity constraints in $\text{Mot}_H^\pm$ and $\text{Mot}_\text{num}^\pm$, yielding $\widetilde{\text{Mot}}_H^\pm$ and $\widetilde{\text{Mot}}_\text{num}^\pm$.

c) Projection functor $\text{Mot}_H^\pm \to \widetilde{\text{Mot}}_\text{num}^\pm$ has tensor sections $\sigma$; any two are tensor-conjugate.

\[
\begin{array}{ccc}
\text{Mot}_H^\pm & \xrightarrow{H} & \text{Vec}_K^* \\
\downarrow & & \downarrow \sigma \\
\text{Mot}_\text{num}^\pm & \xrightarrow{H} & \widetilde{\text{Mot}}_\text{num}^\pm \\
\end{array}
\]

Variant with

$$
\text{Mot}_H^*(k, \mathbb{Q}) := \{ M \in \text{Mot}_H(k, \mathbb{Q}) \mid \text{all K"unneth projectors of } M \text{ algebraic}\}.
$$
5. Description of motivic Galois groups

Assume all conjectures (standard, Hodge, Tate).

5.1. In general:

Short exact sequence

\[ 1 \to GMot_{\bar{k}} \to GMot_k \to G_k \to 1 \]

Last morphism: \( G_k \) corresponds to motives of 0-dimensional varieties (Artin motives). The group \( GMot_{\bar{k}} \) is connected, hence \( = GMot_k^{0} \).

If \( k \subseteq k' \), \( GMot_{k'}^{0} \to GMot_k^{0} \) (but not iso unless \( k'/k \) algebraic: otherwise, “more” elliptic curves over \( k' \) than over \( k \)).

Conjecture (C) \( \Rightarrow \) weight grading on Mot_{num} \( \iff \) central homomorphism

\[ w : \mathbb{G}_m \to GMot_k. \]

On the other hand, Lefschetz motive gives homomorphism

\[ t : GMot_k \to \mathbb{G}_m \]

and \( t \circ w = 2 \) (\(-2\) with Grothendieck’s conventions).
5.2. Over a finite field:

Theorem 11 (cf. [22]). a) $\text{Mot}_{\text{num}}$ generated by Artin motives and motives of abelian varieties.

b) Essential image of $\tilde{\mathcal{H}}_l$: $l$-adic representations of $G_k$ whose eigenvalues are Weil numbers.

Uses Honda’s theorem [16]: every Weil orbit corresponds to an abelian variety.

Corollary 4. $G\text{Mot}^0_k = \text{group of multiplicative type determined by action of } G_{\mathbb{Q}} \text{ on Weil numbers.}$

Even though $\tilde{\text{Mot}}_{\text{num}}$ not neutral, $G\text{Mot}^0_k$ abelian so situation not so bad!
5.3. Over a number field:

\( S := (GMot_k^0)^{ab} \): the Serre protorus: describe its character group \( X(S) \):

\[
\mathbb{Q}^{cm} = \bigcup \{ E \mid E \text{ CM number field} \}
\]

Complex conjugation \( c \) central in \( Gal(\mathbb{Q}^{cm}/\mathbb{Q}) \) (largest Galois subfield of \( \overline{\mathbb{Q}} \) with this property).

**Definition 4.** \( f : Gal(\mathbb{Q}^{cm}/\mathbb{Q}) \to \mathbb{Z} \) CM type if \( f(s) + f(cs) \) independent of \( s \). \( G_{\mathbb{Q}} \) acts on CM types by \( \tau f(s) = f(\tau s) \).

**Theorem 12** ([24]). \( X(S) = \mathbb{Z}[CM \text{ types}] \).

Can also describe the centre \( C \) of \( GMot_k^0 \) (pro-isogenous to \( S \)), etc.: cf. [25].
6. Mixed (Tate) motives

Expect Tannakian category of mixed motives

$$\text{Mot}_{\text{num}}(k, \mathbb{Q}) \subset \text{MMot}(k, \mathbb{Q})$$

with socle $\text{Mot}_{\text{num}}(k, \mathbb{Q})$, classifying non smooth projective varieties. Corresponding motivic Galois group extension of $G\text{Mot}_k$ by a pro-unipotent group (or gerbe).

Constructions of MMot:

- Conjecturally, heart of “motivic $t$-structure” on $DM$ (Deligne, Beilinson: cf. Hana- mura [15]).
- In characteristic 0: explicit category constructed by Nori.
- Over a finite field: Tate conjecture $\Rightarrow \text{Mot}_{\text{num}} = \text{MMot}$ (cf. [22]).
- Can settle for subcategory: mixed Tate motives $T\text{MMot}_k$. Exists unconditionally if $k$ number field (cf. Levine’s talk and [21]).
Goncharov [12]: TM\text{Mot}_\mathbb{Z} \text{ (mixed Tate motives over } \mathbb{Z}) \text{ defined as full subcategory of } \text{TM}\text{Mot}_\mathbb{Q} \text{ by non-ramification conditions.}

Γ the motivic Galois group corresponding to TM\text{Mot}_\mathbb{Z}: \text{Proreductive quotient of } Γ \text{ is } \mathbb{G}_m \text{ (see above).}

**Theorem 13 (Goncharov [12]).** Action of \( \mathbb{G}_m \) on prounipotent kernel \( U \) yields a grading on \( \text{Lie}(U) \): for this grading, \( \text{Lie}(U) \) is free with one generator in every odd degree \( \leq -3 \).