April 26, 2004

Toen

$X/_k \rightsquigarrow \overline{H}^*(X)$ geometric part of cohomology comes with additional structures e.g. Hodge structures, continuous Galois action, crystalline, motivic

geom. part of $h(X)$?

If X ~~X~~ has a homotopy type $\overline{h}(X)$, $\overline{h}(X)$ should also have additional structure.

Tannakian formalism $\Rightarrow$ the structures on $\overline{H}^*(X)$ are encoded in an action of a pro algebraic group H_k.

$\Rightarrow \overline{h}(X)$ should come with an action of H_k, especially if one looks at the motivic homotopy type

Motivation for this: Study rational points via

$$X(k) \xrightarrow{\gamma} \pi_0 \overline{h}(X)^{hH_k}$$

hH_k ← homotopy fixed points

non-abelian Abel-Jacobi map (for γ)

Purpose of talk: define such $\overline{h}(X) \circlearrowleft H_k$.

<u>Plan</u>: 1) Schematic homotopy types (has nothing to do with $H(k)$ Morel-Voevodsky)

here $\longrightarrow$ (a particular case of ∞ <u>stacks</u>) ← no truncation

$\Pi(X)$ will live

2) Examples

3) 2 Questions

(not much to say about γ)

1) SHT K = base field (of coefficients)

Aff_K = affine schemes over K with ffqc topology

$\rightsquigarrow SPre(Aff_K)$ (not small, use universes) but we need all Aff, not just finite type, since we may not have points in any finite dim.

↑ model category

<u>Def 1</u>: Category of stacks := $Ho(SPre(Aff_K))$

Facts: • $Sh(Aff_K) \hookrightarrow Ho(Spre(Aff_K))$ is faithful

F s.t. $\pi_i(F,x)=0$ $\forall i>0$

• cat of 1-stacks ↗

F s.t. $\pi_i(F,x)=0$ $\forall i>1$

In fact, any reasonable cat of n-stacks will embed in $\mathrm{Ho}\,\mathrm{SPr}(\mathrm{Aff}_K)$

The model cat $\mathrm{SPre}(\mathrm{Aff}_K)$ is something like a homotopy topos. We have ~~homo~~ holim, hocolim, $\mathbb{R}\mathcal{H}om$ ... ("A model topos")

<u>Def 2</u>: A schematic homotopy type $/_K$ is a stack F s.t.

a) $\pi_0(F) = *$ (sheaf π_0)
(like being a gerbe)

b) $\forall\, K \hookrightarrow L$, $\forall\, s \in F(L)$,

$\pi_1(F, s)$ = sheaf of groups $/\mathrm{Aff}_L$
is affine (proalgebraic)

c) $\forall\, K \hookrightarrow L$, $\forall\, s \in F(L)$, $\pi_i(F, s)$ is a unipotent group scheme $/\mathrm{Aff}_L$ $\forall\, i > 1$.

(In char 0, this is a sequence of vector spaces)
(In char p, this is more interesting)

Def 2' A unipotent $SHT/_K$ is a SHT F such that $\pi_1(F,s)$ is <u>unipotent</u>,

so $USHT/_K \subset SHT/_K$.

↗ implies restriction on the action of π_1 on π_i: It will be nilpotent in char 0.

Basic **Examples**: • $K(\mathbb{G}_a, n): A \mapsto K(\mathbb{G}_a(A), n)$ $\in$ USHT

• $K(G,1)$, G pro algebraic group, $\in SHT$.

• general way of constructing ex: $F \in USHT$, G pro alg. group acting of on F, then $[F/_{uG}] \in SHT$

uG or G?

~~in char 0~~

Thm: K char 0: ∃ eq.

$Ho(\{\text{comm. cdl. } K\text{-dgas}\}) \longleftrightarrow USHT/_K$

↑ $H^0(A)=K$, nonnegative

The functor is

$$A \longmapsto \text{"Spec } A\text{"}$$

(you can guess what it is)

In char 0:

$\exists$ spectral sequence:

$F \in SHT/_k$ $\exists$ functorial "Curtis" spec seq

$$E_r^{p,q} \Rightarrow \pi_* F, \text{ with}$$

$E_1^{p,q}$ depending only on $H^*(F^0)$.

Here

$$F^0 \longrightarrow F \longrightarrow K(\pi_1^{red}(F), 1)$$

$F^0 \in USHT$; $F \longrightarrow K(\pi_1(F), 1) \longrightarrow K(\pi_1^{red}(F), 1)$ (Postnikov)

(It tends to deg. on E_2 when F comes from an alg. variety)

2) Examples (top): Let X be a top. space, $\pi_0 X = *$.

Thm: $\exists \left(X \otimes k\right)^{sch} \in SHT/_k$ s.t.

(it is the universal alg. envelope of X
" cts. containing X)

a) $\left\{ \begin{array}{l} \text{local systems of} \\ K\text{-vector spaces on } X \end{array} \right\} \hookrightarrow \left\{ \begin{array}{l} \text{vector bundles} \\ \text{on } (X \otimes K)^{sch} \end{array} \right\}$

$\{ \text{Rep of } \pi_1(F) \}$ ~~Rep~~

Vector bdl on 1-truncation, which is just a stack so this means "vector bdl. on stack"

b) $\forall$ L local system on X,

$$H^*(X, L) \cong H^*((X \otimes K)^{sch}, L)$$

In fact $[X^{ct}, F]_{SPr} \cong [(X \otimes K)^{sch}, F]_{SHT/K}$

Let $k = \mathbb{C}$, $X/\mathbb{C}$ sm proj var.

<u>Thm</u>: $\exists\ \overline{X}^{Hod} \in SHT/\mathbb{Q}$ s.t.

same as pure Hodge str.

a) $\overline{X}^{Hod}$ has action of $\mathbb{H}_{\mathbb{Q}}$ = Tannakian group of ~~pure~~ mixed $\mathbb{Q}$-Hodge structures

b) {Variations of ~~pure~~ mixed Hodge structures on $\mathbb{X}/\mathbb{Q}$}

$\longleftrightarrow \left\{ \text{vector bundles on } \bar{X}^{Hod} \right\}^{\mathbb{H}_q}$

c) $\forall L \in V.M.H.S.(X)$,

$H^*(X, L) \circlearrowleft \mathbb{H}_q$

$\cong$

$H^*(\bar{X}^{Hod}, L) \circlearrowleft \mathbb{H}_q$

a) actually encodes absolute Hodge cohomology of X

Now char $k = p$, prime to ℓ, X/k

<u>Thm</u> $\exists \left(\bar{X}^{et} \otimes \mathbb{Q}_\ell\right)^{sch} \in SHT/\mathbb{Q}_\ell$ ~~and~~

+ an ct action of $Gal(k)$.

s.t. a) $\{$lisse $\mathbb{Q}_\ell$-sheaves on X pure$\}$ = Lis

$\longleftrightarrow \left\{ \text{vector bdls on } \left(\bar{X}^{et} \otimes \mathbb{Q}_\ell\right)^{sch} \right\}^{Gal(k)}$

b) let $L \in$ Lis, then

$$H^*(\bar{X}, L) \cong H^*((\bar{X}^{et} \otimes \mathbb{Q}_\ell)^{sch}, L)$$

$\circlearrowleft$ Gal(k) $\qquad$ $\circlearrowleft$ Gal(k)

c) X sm proj $\Rightarrow$ Curtis spec seq deg. at E_2.

gen. of formalism theorem

There are other examples: crystalline (M. Olsson)

motivic one is missing due to standard conjectures

Should be universal one, all other ones are realizations of this one. It should be related to $\mathbb{A}^1$-homotopy category.

Back to rational points.

3) Rational points : $X|_k$,

$\bar{\pi}(X) \supseteq H_k$ is one of the SHT's (eg. l-adic one)

not enough idempotents $\rightarrow$ crystalline is not well suited for that

Now $\gamma : X(k) \longrightarrow \pi_0\left(\bar{h}(X)^{H_k}\right)$

(skipping details)

$$\bar{h}(X)^{H_k} = \mathrm{Map}_{H_k\text{-}\mathrm{Spre}}(*, \bar{h}(X))$$

$$\begin{array}{ccccc}
X(k) & \xrightarrow{\ \gamma\ } & \pi_0\left(\bar{h}(X)^{H_k}\right) & \Leftarrow & E_2^{p,q} = H^p(H_k, \pi_q(\bar{h}(X))) \\
\downarrow & & \downarrow \text{abelianization} & & \\
CH_0(X) & \xrightarrow[\text{cycle class}]{} & \pi_0\left(\left(C_*(\bar{h}(X))\right)^{H_k}\right) & & \\
 & & \Uparrow_{p-q} \text{ Leray spec. seq.} & & \\
 & & E_2^{p,q} = H^p(H_k, \bar{H}_q(X)) & &
\end{array}$$

not known to deg. (should cut

can be approached by Curtis spec. seq., which usually deg.

deg. when X is sm proj

3 Questions

① Give conditions on X so that γ is bijective?
(no one can do that)

② Take $X \hookrightarrow Y$ hyperplane section setup.
smooth proj. var$/k$, assume

$$\bar{h}(Y) = K(\pi_1, 1), \quad \gamma_Y \text{ bijective}$$

Then

$$\begin{array}{ccc} X(k) & \xrightarrow{\gamma_X} & \pi_0\left(\bar{h}(X)^{H_k}\right) \\ \downarrow & & \downarrow \\ Y(k) & \xrightarrow[\gamma_Y]{} & \pi_0\left(\bar{h}(Y)^{H_k}\right) \end{array}$$

first obstruction for

$x \in Y(k)$ to be in $X(k)$ is in

$H^n\left(H_k, \pi_{n-1}\left(\bar{h}(X)\right)\right)$, where $n = \dim Y$

is it non-zero?

if yes, the map P sees strictly more than what happens on π_1.

3) Use étale homotopy type.

$A_g / \mathbb{Q}$ is a "counter-ex" to Grothendieck anabelian prop.

(moduli stack of principal polarized abelian varieties of dim g)

In fact, topologically it is a $K(\pi,1)$ and a hyperbolic mfd. (so something like above)

But the action of $Gal(\bar{\mathbb{Q}})$ on $\pi_1^{et}(A_g/$ factors through $Gal(\mathbb{Q}) \to \hat{\mathbb{Z}}^{\times}$.

$\Rightarrow Aut_{Gal(\mathbb{Q})}\left(\pi_1^{et}(\bar{A_g})\right)$ is infinite $\neq Aut(A_g)$

But: $|\overline{A_g}|^{et}$ is ~~a~~ not a $K(\pi,1)$!

Because: $Sp_{2g}(\mathbb{Z})$ is not good.

What about

$$Aut_{Gal(\mathbb{Q}}\left(|\overline{A_g}|^{et}\right)?$$

$Gal(\mathbb{Q})$-action on $|\overline{A_g}|^{et}$ does not factor through $\widehat{\mathbb{Z}}^{*}$.

(so ~~maybe~~ one should not neglect higher homotopy information, which may fix the counter-ex to anabelian program)

Kahn: The non-abelian Abel-Jacobi map factors through

$$X(k) \longrightarrow [\mathrm{Spec}(k), X]_{H(k)}$$

And $X(k)/_{\text{rational}} \longrightarrow [\text{Spec } k, X]_{\mathbb{A}^1(k)}$

is conj. to be an eq. (Morel)
for X sm proj.

What happens here?

Toen: everything here is over a field, not over $\mathbb{Z}$.