THE MOTIVIC THOM ISOMORPHISM

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Abstract. The existence of a good theory of Thom isomorphisms in some rational category of mixed Tate motives would permit a nice interpolation between ideas of Kontsevich on deformation quantization, and ideas of Connes and Kreimer on a Galois theory of renormalization, mediated by Deligne's ideas on motivic Galois groups.

1. Introduction

This talk is in part a review of some recent developments in Galois theory, and in part conjectural; the latter component attempts to fit some ideas of Kontsevich about deformation quantization and motives into the framework of algebraic topology. I will argue the plausibility of the existence of liftings of (the spectra representing) classical complex cobordism and K-theory to objects in some derived category of mixed motives over $\mathbb{Q}$. In itself this is probably a relatively minor technical question, but it seems to be remarkably consistent with the program of Connes, Kreimer, and others suggesting the existence of a Galois theory of renormalizations.

1.1 One place to start is the genus of complex-oriented manifolds associated to the Hirzebruch power series

$$\frac{z}{\exp_\infty(z)} = z\Gamma(z) = \Gamma(1+z)$$

[25 §4.6]. Its corresponding one-dimensional formal group law is defined over the real numbers, with the entire function

$$\exp_\infty(z) = \Gamma(z)^{-1} : 0 \mapsto 0$$

as its exponential. I propose to take seriously the related idea that the Gamma function

$$\Gamma(z) \equiv z^{-1} \bmod \mathbb{R}[[z]]$$

defines some kind of universal asymptotic uniformizing parameter, or coordinate, at ∞ on the projective line, analogous to the role played by the exponential at the unit for the multiplicative group, or the identity function at the unit for the additive group.

1.2 The second point of reference is a classical conjecture of Galois theory. The cyclotomic closure $\mathbb{Q}_{\mathrm{cyc}}$ of the rationals, defined by adjoining all roots of unity to $\mathbb{Q}$, is the maximal extension of $\mathbb{Q}$ with commutative Galois group; that group,

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isomorphic to the multiplicative group $\hat{\mathbb{Z}}^\times$ of profinite integers, plays an important role in work of Quillen and Sullivan on the Adams conjecture and in differential topology. Shafarevich (cf. [27]) conjectures that the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}_{\mathrm{cyc}})$ is a **free** profinite group; in other words, the full Galois group of $\overline{\mathbb{Q}}$ over $\mathbb{Q}$ fits in an exact sequence

$$1 \to \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}_{\mathrm{cyc}}) \cong \widehat{\mathrm{Free}} \to \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gal}(\mathbb{Q}_{\mathrm{cyc}}/\mathbb{Q}) \cong \hat{\mathbb{Z}}^\times \to 1 \; .$$

What will be more relevant here is a related conjecture of Deligne [13 §8.9.5], concerning a certain motivic analog of the Galois group which I will denote $\mathrm{Gal}_{\mathrm{mot}}(\overline{\mathbb{Q}}/\mathbb{Q})$, which is not a profinite but rather a proalgebraic groupscheme over $\mathbb{Q}$; it is in some sense a best approximation to the classical Galois group in this category, which should contain the original group as a Zariski-dense subobject. [I should say that calling this object a Galois group is an abuse of terminology; it is more properly described (cf. §4.4) as the motivic Tate Galois group of Spec $\mathbb{Z}$ (without reference to $\overline{\mathbb{Q}}$).] In any case, this motivic group fits in a similar extension

$$1 \to \mathfrak{F}_{\mathrm{odd}} \to \mathrm{Gal}_{\mathrm{mot}}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{G}_m \to 1$$

of groupschemes over $\mathbb{Q}$, where $\mathbb{G}_m$ is the multiplicative groupscheme, and $\mathfrak{F}_{\mathrm{odd}}$ is the prounipotent groupscheme defined by a free graded Lie algebra $\mathfrak{f}_{\mathrm{odd}}$ with one generator of each odd degree greater than one; the grading is specified by the action of the multiplicative group on the Lie algebra. The generators of this Lie algebra are thought to correspond with the odd zeta-values (via Hodge realization, [14 §2.14]) which are expected to be transcendental numbers (and thus outside the sphere of influence of Galois groups of the classical kind).

Kontsevich introduced his Gamma-genus in the context of the Duflo - Kirillov theorem in representation theory. He argued that it lies in the same orbit, under an action of the motivic Galois group, as the analog of the classical $\hat{A}$-genus [3 §8.5]. How this group fits in the topological context is less familiar, and to a certain extent this paper is nothing but an attempt to find a place for that group in algebraic topology. The history of this question is intimately connected with Grothendieck's theory of anabelian geometry, and it enters Kontsevich's work through a conjectured Galois action on some form of the little disks operad. My impression these ideas are not yet very familiar to topologists, so I have included a very brief account of some of their history, with a few references, as an appendix below.

I should acknowledge here that Libgober and Hoffman [25,40] have studied a genus with related, but not identical, properties, and that an attempt to understand their work was instrumental in crystallizing the ideas behind this paper. I owe many mathematicians - including G. Carlsson, D. Christensen, F. Cohen, P. Deligne, A. Goncharov R. Jardine, T. Kohno, and T. Wenger - thanks for conversations about the material in this paper, and I am particularly indebted to a very knowledgeable and patient referee. In many cases they have saved me from mistakes and overstatements; but other such errors may remain, and those are the solely my responsibility. I also wish to thank the Newton Institute, and the Fields Institute program at Western Ontario, for support during the preparation of this paper.

2. The Gamma-genus

2.1 The Gamma-function is meromorphic, with simple poles at $z = 0, -1, -2, \ldots$; we might therefore hope for a Weierstrass product of the form

$$\Gamma(1+z)^{-1} \sim \prod_{n \geq 1} (1 + \frac{z}{n}) \ ,$$

from which we might hope to derive a power series expansion

$$\log \Gamma(1+z) \sim - \sum_{n \geq 1} \log(1 + \frac{z}{n}) \sim \sum_{n,k \geq 1} (-\frac{z}{n})^k$$

for its logarithm. Rearranging this carelessly leads to

$$\sum_{n,k \geq 1} \frac{(-z)^k}{k} \ \frac{1}{n^k} \sim \sum_{k \geq 1} \frac{\zeta(k)}{k} \ (-z)^k \ ,$$

which is unfortunately implausible since $\zeta(1)$ diverges. In view of elementary renormalization theory, however, we should not be daunted: we can add 'counter-terms' to conclude that

$$\log \prod_{n \geq 1} (1 + \frac{z}{n}) \ e^{-z/n} \sim - \sum_{k \geq 2} \frac{\zeta(k)}{k} (-z)^k \ ,$$

and with a little more care we deduce the correct formula

$$\Gamma(1+z) = \exp(-\gamma z + \sum_{k \geq 2} \frac{\zeta(k)}{k} \ (-z)^k) \ ,$$

where γ is Euler's constant. Reservations about the logic of this argument may perhaps be dispelled by observing that

$$\Gamma(1+z) \ \Gamma(1-z) = \exp(\sum_{k \geq 1} \frac{\zeta(2k)}{k} \ z^{2k}) \ ;$$

Euler's duplication formula implies that the left-hand side equals

$$z\Gamma(z) \ \Gamma(1-z) = \frac{\pi z}{\sin \pi z} \ ,$$

consistent with the familiar evaluation of ζ at positive even integers in terms of Bernoulli numbers.

2.2 From this perspective, the Hirzebruch series

$$\Gamma(1+z) = (\frac{\pi z}{\sin \pi z})^{\frac{1}{2}} \ \exp(-\gamma z + \sum \frac{\zeta(\mathrm{odd})}{\mathrm{odd}} \ z^{\mathrm{odd}}) \ ;$$

for Kontsevich's genus does in fact look like some kind of deformation of the $\hat{A}$-genus; its values on a complex-oriented manifold will be polynomials in odd zeta-values, with rational coefficients. Similarly, the Witten genus

$$\phi_W(x) = \frac{x/2}{\sinh x/2} \ \prod_{n \geq 1} [(1 - q^n u)(1 - q^n u^{-1})]^{-1}$$

([57], with $u = e^x$) can be written in the form

$$\exp(-2 \sum_{k \geq 1} g_k \frac{x^k}{k!}) \ ,$$

where the coefficients g_k are modular forms, with $g_{\rm odd} = 0$: it is also a deformation of $\hat{A}$, in another direction.

[Behind the apparent discrepancies in these formulae is the issue of complex versus oriented cobordism: there are several possible conventions relating Chern and Pontrjagin classes. Hirzebruch expresses the latter as symmetric functions of indeterminates x_i^2, and writes the genus associated to the formal series $Q(z)$ as $\prod Q(x_i^2)$; thus for the $\hat{A}$-genus,

$$Q(z) = \frac{\frac{1}{2}\sqrt{z}}{\sinh \frac{1}{2}\sqrt{z}} \; .$$

An alternate convention, used here, writes this symmetric function in the form

$$\prod ((\frac{x_i/2}{\sinh x_i/2})^{\frac{1}{2}} \cdot (\frac{-x_i/2}{\sinh(-x_i/2)})^{\frac{1}{2}}) \; .$$

The relation between the indeterminates x and z is a separate issue; I take z to be $2\pi i x$.]

Kontsevich suggests that the values of the zeta function at odd positive integers (expected to be transcendental) are subject to an action of the motivic group $\mathrm{Gal}_{\rm mot}(\overline{\mathbb{Q}}/\mathbb{Q})$, and that the $\hat{A}$-genus and his Γ genus lie in the same orbit of this action. One natural way to understand this is to seek an action of that group on genera, and thus on the complex cobordism ring; or, perhaps more naturally, on some form of its representing spectrum. Before confronting this question, it may be useful to present a little more background on these zeta-values.

3. Symmetric and quasisymmetric functions

3.1 The formula

$$\prod_{k \geq 1} (1 + x_k z) = \sum_{k \geq 0} e_k z^k = \exp(- \sum_{k \geq 1} \frac{p_k}{k} (-z)^k) \; ,$$

where e_k is the kth elementary symmetric function, and p_k is the kth power sum, can be derived by formal manipulations very much like those in the preceding section, by expanding the logarithm of $\prod(1 + x_n z)$; such arguments go back to Newton. The specialization

$$x_k \mapsto k^{-2}$$

(cf. the second edition of MacDonald's book [43 Ch I §2 ex 21]) leads to Bernoulli numbers, but the map

$$x_k \mapsto k^{-1}$$

is trickier, because of convergence problems like those mentioned above; it defines a homomorphism from the ring of symmetric functions to the reals, sending p_k to $\zeta(k)$ when $k > 1$, while $p_1 \mapsto \gamma$ [24]. Under this homomorphism the even power sums p_{2k} take values in the field $\mathbb{Q}(\pi)$.

3.2 The Gamma-genus is thus a specialization of the formal group law with exponential

$$\mathrm{Exp}_\infty(z) = \frac{z}{e(z)} = z \prod_{k \geq 1} (1 + x_k z)^{-1} = \sum_{k \geq 0} (-1)^k h_k z^{k+1}$$

having the complete symmetric functions (up to signs) as its coefficients. This is a group law of **additive** type: its exponential, and hence its logarithm, are both defined over the ring of polynomials generated by the elements h_k. This group law is classical: it is defined by the Boardman-Hurewicz-Quillen complete Chern number homomorphism

$$MU^*(X) \to H^*(X, \mathbb{Z}[h_*])$$

defined on coefficients by the homomorphism

$$\mathbf{Lazard} \to \mathbf{Symm}$$

from Lazard's ring which classifies the universal group law of additive type. The Landweber-Novikov Hopf algebra $S_* = \mathbb{Z}[t_*]$ represents the prounipotent groupscheme $\mathbb{D}_0$ of formal diffeomorphisms

$$z \mapsto t(z) = z + \sum_{k \geq 1} t_k z^{k+1}$$

of the line, with coproduct

$$\Delta(t(z)) = (t \otimes 1)((1 \otimes t)(z)) \in (S_* \otimes S_*)[[z]] \ .$$

The universal group law $t^{-1}(t(X) + t(Y))$ of additive type is thus classified by the homomorphism

$$\mathbf{Lazard} \to \mathbf{Lazard} \otimes S_* \to \mathbb{Z} \otimes S_* \to \mathbf{Symm}$$

representing the orbit of the Thom map $\mathbf{Lazard} \to \mathbb{Z}$ (which classifies the additive group law) under the action of $\mathbb{D}_0$. This identifies the algebra S_* with the ring of symmetric functions by $t_k \mapsto (-1)^k h_k$.

3.3 The symmetric functions are a subring

$$\mathbf{Symm} \to \mathbf{QSymm}$$

of the larger ring of **quasi**symmetric functions, which is Hopf dual to the universal enveloping algebra $\mathbb{Z}\langle Z_1, \ldots \rangle$ of the free graded Lie algebra $\mathfrak{f}_*$ with one generator in each positive degree [23], given the cocommutative coproduct

$$\Delta Z_i = \sum_{i=j+k} Z_j \otimes Z_k \ .$$

Standard monomial basis elements for this dual Hopf algebra, under specializations like those discussed above [6 §2.4, 23], map to polyzeta values

$$\zeta(i_1, \ldots, i_k) = \sum_{n_1 > \cdots > n_k \geq 1} \frac{1}{n_1^{i_1} \cdots n_k^{i_k}} \in \mathbb{R} \ ;$$

note that there are convergence difficulties unless $i_1 > 1$. If we think of the Gamma-genus as taking values in the field $\mathbb{Q}(\zeta) \subset \mathbb{R}$ generated by such polyzeta values, then it is the specialization of a homomorphism

$$\mathbf{Lazard} \to \mathbf{Symm} \to \mathbf{QSymm}$$

representing a morphism from the prounipotent groupscheme $\mathfrak{F}$ with Lie algebra $\mathfrak{f}_*$ to the moduli space of one-dimensional formal group laws.

Because we are dealing with group laws of additive type, there seems to be little loss in working systematically over a field of characteristic zero, where Lie-theoretic methods are available. Over such a field any formal group is of additive type: the localization of the map from the Lazard ring to the symmetric functions is an isomorphism. Similarly, over the rationals the Landweber - Novikov algebra is dual to the enveloping algebra of the Lie algebra of vector fields

$$z_k = z^{k+1}\partial/\partial z \;, k \geq 1$$

on the line, and the embedding of the symmetric in the quasisymmetric functions sends the free generators Z_k to the Virasoro generators z_k, corresponding to a group homomorphism $\mathfrak{F} \to \mathbb{D}_0$.

3.4 It will be useful to summarize a few facts about Malcev completions and pro-unipotent groups [13 §9]. The rational group ring $\mathbb{Q}[G]$ of a discrete group G has a natural lower central series filtration; its completion $\widehat{Q[\mathbb{G}]}$ with respect to that filtration is a topological Hopf algebra, whose continuous dual represents a pro-unipotent groupscheme over $\mathbb{Q}$. Applied to a finitely-generated free group, for example, this construction yields a Magnus algebra of noncommutative formal power series.

There are many variations on this theme: in particular, an action of the multiplicative group $\mathbb{G}_m$ defines a grading on a Lie algebra. The action

$$t_k \mapsto u^k t_k \;,$$

(u a unit) on the group of formal diffeomorphisms defines an extension of its Lie algebra by a new Virasoro generator v_0, corresponding to an extension $S_*[t_0^{\pm}]$ of the Landweber-Novikov algebra. The group of formal diffeomorphisms is pro-unipotent, and this enlarged object is most naturally interpreted as a semidirect product $\mathbb{D}_0 \rtimes \mathbb{G}_m$. Grading the free Lie algebra $\mathfrak{f}$ similarly extends the homomorphism above to

$$\mathfrak{F} \rtimes \mathbb{G}_m \to \mathbb{D} \rtimes \mathbb{G}_m \;.$$

4. Motivic versions of classical K-theory and cobordism

4.1 There are now several (eg [39, 55]) good and probably equivalent constructions of a triangulated category $DM(k)$ of motives over a field k of characteristic zero. The subject is deep and fascinating, and I know at best some of its vague outlines. Since this paper is mostly inspirational, I will not try to provide an account of that category; but as it is after all modelled on spectra, it is perhaps not too much of a reach to think that some of its aspects will look familiar to topologists.

One approach to defining a motivic category $DM(k)$ starts from a category whose morphisms are elements of a group of algebraic correspondences. At some later point it becomes useful to tensor these groups with $\mathbb{Q}$, resulting in a category $DM_{\mathbb{Q}}(k)$ whose Hom-objects are rational vector spaces. The underlying concern of this paper is the relation of such motivic categories to classical topology; but stable homotopy theory over the rationals is equivalent to the theory of graded vector spaces. This has the advantage of rendering some of the conjectures below almost trivially true – and the disadvantage of making them essentially contentless. Behind these conjectures, however, lies the hope that they might say something **before** rationalization, and for that reason I have outlined here a rough theory of **integral** geometric realizations of motives:

4.2 The category $DM(k)$ contains certain canonical Tate objects $\mathbb{Z}(n)$, defined [55 §2.1, but see also 13 §2] as tensor powers of a reduced version $\mathbb{Z}(1)$ of the projective line. Grothendieck's original category of 'pure' motives, constructed from smooth projective varieties, is (in some generality [28]) semisimple, but categories of motives built from more general (non-closed) varieties admit nontrivial extensions. The (derived) category $DMT_{\mathbb{Q}}(k)$ of mixed Tate motives can be defined as the smallest tensor triangulated subcategory of $DM_{\mathbb{Q}}(k)$ containing the Tate objects. In this rationalized category, it is natural to denote the (images of) the generating objects by $\mathbb{Q}(n)$; however, I will be most interested here in the case $k = \mathbb{Q}$ and in a certain more subtle construction of a (rationalized, though I will now drop the subscript) subcategory $DMT(\mathbb{Z})$ of $DMT_{\mathbb{Q}}(\mathbb{Q})$ [14 §1.6], closely related to the motives over Spec $\mathbb{Z}$ 'with integral coefficients' in the sense of [13 §1.23, 2.1]. This is still a $\mathbb{Q}$-linear category, but its objects have stronger integrality properties than one might naively expect.

In particular: one of the foundation-stones of the theory of mixed motives is an isomorphism

$$\mathrm{Ext}^1_{MT(\mathbb{Z})}(\mathbb{Q}(0), \mathbb{Q}(n)) \cong K_{2n-1}(\mathbb{Z}) \otimes \mathbb{Q}$$

(cf. [2, 13 §8.2]. As the referee points out, one has to be careful here; the corresponding description for the category $MT(\mathbb{Q})$ involves the algebraic K-theory of $\mathbb{Q}$, which is much larger than that of $\mathbb{Z}$). The groups on the right have rank one for **odd** $n > 1$, and vanish otherwise, by work of Borel; the theory of regulators says that to some extent the zeta-values

$$\frac{(n-1)!}{(2\pi i)^n}\,\zeta(n)$$

(cf. [13 §3.7]) can be interpreted as natural generators for these groups. This is strikingly reminiscent to a homotopy-theorist of the identification (for n **even**) of the group

$$\mathrm{Ext}^1{}_{\mathrm{Adams}}(K(S^0), K(S^{2n}))$$

of extensions of modules over the Adams operations, with the cyclic subgroup of $\mathbb{Q}/\mathbb{Z}$ generated by this zeta-number. These connections between the image of the J-homomorphism and the groups $K_{4k-1}(\mathbb{Z})$, go back to the earliest days of algebraic K-theory [16, 50].

4.3 This suggests that there might be some use for a notion of geometric or homotopy-theoretic realization for motives, which manages to retain some integral information. Aside from tradition (algebraic geometers usually work with cycles over $\mathbb{Q}$, and topologists have been neglecting correspondences since Lefschetz), there seems to be no obstacle to the development of such a theory. Indeed, let E be a multiplicative (co)homology functor (ie a ring-spectrum), supplied with a natural class of E-orientable manifolds: if E were stable homotopy, for example, we could use stably parallelizeable manifolds. In the case of interest below, however, E will be complex K-theory, and the manifolds will be smooth (proper) algebraic varieties over $\mathbb{C}$.

Such manifolds $(X, Y, \dots)$ define an additive category Corr_E with $E_*(X \times Y)$ (suitably graded) as Hom-objects; composition of such morphisms can be defined using

Pontrjagin-Thom transfers [45]. It is straightforward to check that

$$X \mapsto X_+ \wedge \mathbf{E} : \mathrm{Corr}_E \to (\mathrm{Spectra})$$

is a functor: the necessary homomorphism

$$E_*(X \times Y) = [S^*, X_+ \wedge Y_+ \wedge \mathbf{E}] \to [X_+ \wedge \mathbf{E}, Y_+ \wedge \mathbf{E}]_*$$

of Hom-objects is defined by the adjoint composition

$$X_+ \wedge Y_+ \wedge \mathbf{E} \wedge X_+ \wedge \mathbf{E} \cong X_+ \wedge X_+ \wedge Y_+ \wedge \mathbf{E} \wedge \mathbf{E} \to Y_+ \wedge \mathbf{E}$$

built from the multiplication map of $\mathbf{E}$ and the composition

$$X_+ \wedge X_+ \wedge \mathbf{E} \to X_+ \wedge \mathbf{E} \to \mathbf{E}$$

of the transfer $\Delta_!$ associated to the diagonal map, with the projection of X to a point.

Following the pattern laid out by Voevodsky, we can now define a category of (topological) 'E-motives', and when $E = K$ it is a classical fact [1] that an algebraic cycle defines a nice K-theory class. This allows us to associate to an embedding of k in $\mathbb{C}$, a triangulated 'realization' functor

$$\mathbf{K}_{\mathrm{mot}} : X \mapsto X(\mathbb{C})_+ \wedge \mathbf{K} : DM(k) \to (\mathrm{Spectra}) \; .$$

[Since algebraic cycles are triangulable (by Lojasiewicz), a theorem of Sullivan allows us to define these cycles in **connective** complex K-theory.]

4.4 When k is a number field, $DMT_{\mathbb{Q}}(k)$ possesses a theory of truncations, or t-structures [14, 37], analogous to the Postnikov systems of homotopy theory; the **heart** of this structure is an abelian tensor category $MT_{\mathbb{Q}}(k)$ of mixed Tate motives. Its existence permits us to think of $DMT_{\mathbb{Q}}(k)$ as the derived category of $MT_{\mathbb{Q}}(k)$; in particular, the (co)homology of an object of the larger category becomes in a natural way [31 §2.4] an object of $MT_{\mathbb{Q}}(k)$. Similar considerations hold for the more rigid category $DMT(\mathbb{Z})$, and since we are working in a rational, stable context, I will write π_* for the homology groups of an object in this category, given this enriched structure. [For the purposes of this presentation I've reversed the logical order of construction: in fact in [14] the category $MT(\mathbb{Z})$ is constructed first.]

Now under very general conditions (involving a suitably rigid duality), an abelian tensor category with rational Hom-objects can be identified with a category of representations of a certain groupscheme of automorphisms of a suitable forgetful functor on the category; the resulting groupscheme is called a motivic (Galois) group. This theory applies to $MT(\mathbb{Z})$, and as was noted in the introduction, $\mathrm{Gal}_{\mathrm{mot}}(\overline{\mathbb{Q}}/\mathbb{Q})$ is the corresponding groupscheme [13 §8.9.5, 14 §2; cf. also 2 §5.10]. In the preceding paragraph we constructed a homological functor

$$K_{\mathrm{mot}} := \pi_* \mathbf{K}_{\mathrm{mot}}$$

from $DM_{\mathbb{Q}}$ to $\mathbb{Q}$-vector spaces, together with a preferred lift of the functor to the category of spectra. It is easy to see that

$$\pi_{\mathrm{odd}} \mathbf{K}_{\mathrm{mot}} = 0 \; , \text{ while } \pi_{2n} \mathbf{K}_{\mathrm{mot}} = \mathbb{Q}(n) \; ,$$

as representations of $\mathrm{Gal}_{\mathrm{mot}}(\overline{\mathbb{Q}}/\mathbb{Q})$, and thus that K_{mot} is represented by the mixed (Bott!)-Tate object

$$\oplus_{n \in \mathbb{Z}} \mathbb{Q}(n)[2n] = \mathbb{Q}[b^{\pm}] \; ;$$

in other words, we have constructed a lifting of the rationalized classical K-theory functor to the category of mixed Tate motives, with an action of $\mathrm{Gal}_{\mathrm{mot}}(\overline{\mathbb{Q}}/\mathbb{Q})$ which factors through the multiplicative quotient.

The possible existence of a descent spectral sequence for the automorphisms of the K-theoretic realization functor of §4.3 seems to be an interesting question, especially when restricted to some category of mixed Tate motives.

4.5 The **main conjecture** of this paper is that, similarly, a rational version of complex cobordism lifts to an object $\mathbf{MU}_{\mathrm{mot}} \in DMT(\mathbb{Z})$, with an action of $\mathrm{Gal}_{\mathrm{mot}}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\pi_*\mathbf{MU}_{\mathrm{mot}}$ defined by the obvious embedding

$$\mathfrak{F}_{\mathrm{odd}} \rtimes \mathbb{G}_m \to \mathfrak{F} \rtimes \mathbb{G}_m$$

followed by a homomorphism from the latter group to the diffeomorphisms of the formal line, cf. §2.4 above.

This is a conjecture about an object characterized by its universal properties, so it can be reformulated in terms of the structures thus classified. The theory of Chern classes is founded on Grothendieck's calculation of the cohomology of the projectification $P(V)$ of a vector bundle V over a scheme X. It follows immediately from his result that the cohomology of Atiyah's model

$$X^V := P(V \oplus 1)/P(V)$$

for a Thom space as a relative motive is free on one generator over that of X. Such a generator is a Thom class for V, but in the motivic context there seems to be no natural way to construct such a thing; this is related to the inconvenient nonexistence of abundantly many sections of vector bundles in the algebraic category.

For a systematic theory of Thom classes it is enough, according to the splitting principle, to work with line bundles L, and in this context it is relevant that the Thom complexes

$$X^L = P(L \oplus 1)/P(L) \cong P(1 \oplus L^{-1})/P(L^{-1}) = X^{L^{-1}}$$

of a line bundle and its reciprocal are isomorphic objects. The conjecture about $\mathbf{MU}_{\mathrm{mot}}$ can be thus reformulated in terms of a theory of motivic Thom and Euler classes $U_{\mathrm{mot}}(L), \mathbf{e}(L)$ for line bundles L, satisfying a **motivic Thom axiom**

$$U_{\mathrm{mot}}(L^{-1}) = -U_{\mathrm{mot}}(L) \ , \ \mathbf{e}(L^{-1}) = -\mathbf{e}(L) \ ;$$

the conjecture is then the assertion that an element $\sigma \in \mathrm{Gal}_{\mathrm{mot}}(\overline{\mathbb{Q}}/\mathbb{Q})$ sends $U_{\mathrm{mot}}(L)$ to another Thom class

$$\sigma(U_{\mathrm{mot}}(L)) = [1 + \sum_{k>0} \sigma_k \mathbf{e}(L)^k] \cdot U_{\mathrm{mot}}(L)$$

for L, with coefficients σ_k depending only on σ. Since

$$\sigma(U_{\mathrm{mot}}(L^{-1})) = -\sigma(U_{\mathrm{mot}}(L)) \ ,$$

it follows that

$$[1 + \sum_{k>0} \sigma_k(-\mathbf{e}(L))^k] \cdot (-U_{\mathrm{mot}}(L)) = -[1 + \sum_{k>0} \sigma_k \mathbf{e}(L)^k] \cdot U_{\mathrm{mot}}(L) \ ,$$

which entails that the classes $\sigma_{\rm odd} = 0$, distinguishing the Hopf subalgebra

$$S_{\rm ev} = \mathbb{Z}[t_{2k} \mid k > 0]$$

which represents the group of **odd** diffeomorphisms of the formal line [46 §3.3]. Away from the prime two, classical complex cobordism is a kind of base extension

$$\mathbf{MU}[1/2] \sim \mathrm{SO/SU} \wedge \mathbf{MSO}[1/2]$$

of oriented cobordism, and I'm suggesting the existence of a similar splitting for the hypothetical motivic lift of complex cobordism.

4.6 After this paper had been submitted for publication, I became aware of the very elegant recent work of Levine and Morel [38], where an algebraic cobordism functor is characterized as a universal cohomology theory on the category of schemes, endowed with pullback and pushforward transformations satisfying certain natural axioms pf compatibility. [Voevodsky [56] has also considered a motivic version of the cobordism spectrum; its relation with their work is discussed briefly in the introduction to their paper.] I believe their work is fundamentally compatible with the conjectures made here, given a slight difference in framework and emphasis: they suppose a Thom isomorphism (or, equivalently, a system of covariant transfers) is to be given as part of the structure of a cohomology theory on schemes, while the spectrum hypothesized here is merely a ringspectrum, with no preferred choice of orientation.

4.7 I should note that the trivial action of $\mathfrak{F}_{\rm odd}$ on $\pi_*(\mathbf{K}_{\rm mot})$, together with the usual action of $\mathbb{G}_m$ defined by the grading, is consistent with the existence of a spectral sequence

$$E_2^{*,*} = H^*_c(\mathrm{Gal}_{\rm mot}(\overline{\mathbb{Q}}/\mathbb{Q}), K^*_{\rm mot}) \Longrightarrow K^*(\mathbb{Z}) \otimes \mathbb{Q}$$

of descent type, as suggested above: using the Hochschild-Serre spectral sequence for the semidirect product decomposition of the Galois group (and confusing continuous cochain with Lie algebra cohomology) we start with

$$H^*_c(\mathfrak{F}_{\rm odd}, K^*_{\rm mot}) \cong \mathbb{Q}\langle e_{2k+1} \mid k \geq 1\rangle[b] \ ,$$

where angled brackets denote a vector space spanned by the indicated elements (with e_{2r+1} in degree $(1,0)$, and the Bott (-Tate?!) element b in degree $(0,-2)$). The $\mathbb{G}_m$-action sends b to ub, where u is a unit in whatever ring we're over; similarly,

$$e_{2k+1} \mapsto u^{-2k-1} e_{2k+1} \ .$$

Thus $e_{2k+1}b^{2k+1} \in E_2^{1,-4k-2}$ is $\mathbb{G}_m$-invariant, yielding a candidate for the standard generator in $K_{4k+1}(\mathbb{Z}) \otimes \mathbb{Q}$.

5. Quantization and asymptotic expansions

5.1 The motivic Galois group appears in Kontsevich's work through a conjectured action on deformation quantizations of Poisson structures [cf. [53]]. The framework of this paper suggests a plausibly related action on an algebra of asymptotic expansions for geometrically defined functionals on manifolds, interpreted in terms of the cobordism ring of symplectic manifolds [18, 19]. This is isomorphic to the complex cobordism (abelian Hopf) algebra $MU_*(B\mathbb{G}_m(\mathbb{C}))$ of circle bundles, and the dual (rationalized) Hopf algebra

$$MU^*_{\mathbb{Q}}(B\mathbb{G}_m(\mathbb{C})) \cong MU^*_{\mathbb{Q}}[[\hbar]] ,$$

where $\hbar = \sum_{k\geq 1} \mathbb{C}P_{k-1}\mathbf{e}^k/k$, can be interpreted [46] as an algebra generated by the coefficients of a kind of universal asymptotic expansion for geometrically defined heat kernels (or Feynman measures, via the Feynman-Kac formula [20 §3.2]), as the Chern class $\mathbf{e}$ of the circle bundle approaches infinity.

A Poisson structure on an even-dimensional manifold V is a bivector field (a section of the bundle $\Lambda^2 T_V$) satisfying a Jacobi identity modelled on that satisfied by the inverse of a symplectic structure. A symplectic manifold is thus Poisson, and although I am aware of no useful notion of Poisson cobordism (but cf. [7]) one expects a natural restriction map from asymptotic invariants Poisson manifolds to the corresponding ring for symplectic manifolds. If the conjectures above are correct, then one might further hope that (some motivic version of) such a restriction map would be equivariant, with respect to some motivic Galois action.

5.2 Working in the opposite (local to global) direction, Connes and Kreimer have recently developed a systematic program for understanding classical quantum field theoretic renormalization in terms of its symmetries. The standard methods (eg dimensional regularization) for dealing with the singular integrals which appear in classical perturbation theory replaces them with certain meromorphic functions, and through work of Broadhurst, Kreimer, and others it has become more and more clear that the polar coefficients of these meromorphic functions are frequently elements of the polyzeta algebra. [Kontsevich has suggested that this is always so, but his program of proof fails, by the arguments of [4], and at present the question seems to be open.] Connes and Kreimer[12, 21, 26, 35 §12] have developed a systematic approach to the theory of Feynman integrals through certain Hopf algebras related to automorphism groups [42] of operads defined by graphs of various sorts.

There are deep connections between the Grothendieck - Teichmüller group and the Lie algebras of these automorphism groups [29, 54], and it seems likely that they (and the theory of quasisymmetric functions, via free Lie algebras) will eventually be understood to be intimately related; the appearance of polyzeta values in the theory of quantum knot invariants (cf. eg [36]) is another source of recent interest in this subject. Perhaps the deepest (and most precise) approach to the relations between these topics may be the work of Goncharov, who associates to a field F a certain Hopf algebra $\mathcal{T}_\bullet(F)$ of F-decorated planar trivalent trees and a closely related Hopf algebra $\mathcal{I}_\bullet(F)$ of motivic iterated integrals. According to the correspondence principle of [21 §7], the renormalization Hopf algebra corresponding to certain types of Feynman integrals should be closely related to a precisely defined subgroup of the motivic Tate Galois group of F. A slight strengthening of this correspondence principle would settle the question of the role of polyzeta values in perturbative expansions of Feynman integrals.

5.3 In exemplary cases Connes and Kreimer construct a very interesting representation of the prounipotent groupscheme underlying their renormalization algebra in the group of **odd** formal diffeomorphisms of the line [10 §1 eq. 20, §4 eq. 2].

It also seems quite possible that the action of their groupscheme on asymptotic expansions defined by Feynman measures associated to suitable Lagrangians [30] factor through an action of the motivic Galois group on cobordism, along the lines suggested in §4.5 above.

appendix: motivic models for the little disk operad

1 In 1984 Grothendieck suggested the study of the action of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ as automorphisms of the moduli of algebraic curves, understood as a collection of stacks linked by morphisms representing various geometrically natural fusion operations. There are remarkable analogies between his ideas and contemporary work in physics on conformal field theories, and in 1990 Drinfel'd [15] unified at least some of these lines of thought by constructing a pronilpotent group GT of automorphisms of certain braided tensor categories, together with a faithful representation of the absolute Galois group in that group.

This program has been enormously productive; the LMS notes [42, 51] are one possible introduction to this area of research, but evolution has been extremely rapid. In the late 90's Kontsevich [33, 34] recognized connections between these ideas, Deligne's question on Hochschild homology, deformation quantization, and other topics, while physicists [9-12] interested in the algebra of renormalization were developing sophisticated Hopf-algebraic techniques, which are now believed [5 §8,9] to be closely related to the Hopf-algebraic constructions of Drinfel'd.

The point of this appendix is to draw attention to a central conjecture in this circle of ideas: that the Lie algebra of GT, which acts as automorphisms of the system of Malcev completions of the braid groups, is a **free** graded Lie algebra, with one generator in each odd degree greater than one. The braid groups in fact form an operad, and I want to propose the related problem of identifying the automorphisms of the operad of Lie algebras defined by the braid groups (cf. [8]), in hope that this will shed some light on this question, and the closely related conjecture that Deligne's motivic group acts **faithfully** on the unipotent motivic fundamental group [13] of $\mathbb{P})_1 - \{0, 1, \infty\}$ (with nice tangential base point).

2 For the record, an operad (in some reasonable category) is a collection of objects $\{\mathcal{O}_n,\ n \geq 2\}$ together with composition morphisms

$$c_I : \mathcal{O}_{r(I)} \times \prod_{i \in I} \mathcal{O}_i \to \mathcal{O}_{|I|}$$

where $I = i_1, \dots, i_r$ is an ordered partition of $|I| = \sum i_k$ with $r(I)$ parts. These compositions are subject to a generalized associativity axiom, which I won't try to write out here; moreover, the operads in this note will be **permutative**, which entails the existence of an action of the symmetric group Σ_n on $\mathcal{O}_n$, also subject to unspecified axioms. Not all of the operads below will be unital, so I haven't assumed the existence of an object $\mathcal{O}_1$; but in that case, and under some mild assumptions, an operad can be described as a monoid in a category of objects with symmetric group action, with respect to a somewhat unintuitive product [cf. eg. [17]].

The moduli $\{\overline{\mathcal{M}}_{0,n+1}\}$ of stable genus zero algebraic curves marked with $n+1$ ordered smooth points form such an operad: if the final marked point is placed at infinity, then composition morphisms are defined by gluing the points at infinity of a set of r marked curves to the marked points **away** from infinity on some curve marked with $r+1$ points. This is an operad in the category of algebraic stacks defined over $\mathbb{Z}$, so the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on many of its topological and cohomological invariants.

By definition, a stable algebraic curve possesses at worst ordinary double point singularities. The moduli of **smooth** genus zero curves thus define a system $\mathcal{M}_{0,*} \subset \overline{\mathcal{M}}_{0,*}$ of subvarieties, but not a suboperad: the composition maps glue curves together at smooth points, creating new curves with nodes from curves without them. However, the spaces $\mathcal{M}_{0,n+1}(\mathbb{C})$ of complex points of these moduli objects have the homotopy types of the spaces of configurations of n distinct points on the complex line, which are homotopy-equivalent to the spaces of the little disks operad $\mathcal{C}_2$.

Behind the conjectures of Deligne, Drinfel'd, and Kontsevich lies an apparently unarticulated question: **is the little disks operad defined over $\mathbb{Q}$?**; or, more precisely: is there a version of the little disks operad in which the morphisms, as well as the objects, lie in the category of algebraic varieties defined over the rationals? Thus in [33] (end of §4.4) we have

> The group GT maps to the group of automorphisms in the homotopy sense of the operad $\mathrm{Chains}(\mathcal{C}_2)$. Moreover, it seems to coincide with $\mathrm{Aut}(\mathrm{Chains}(\mathcal{C}_2))$ when this operad is considered as an operad not of complexes but of differential graded cocommutative coassociative coalgebras ...

and in [34] (end of §3)

> There is a natural action of the Grothendieck-Teichmüller groups on the rational homotopy type of the [Fulton-MacPherson version of the] little disks operad ...

although the construction in [34 §7] is not (apparently) defined by algebraic varieties. It may be that the question above is naive [cf. [53]], but a positive answer would imply the existence of a system of homotopy types with action of the Galois groups, whose algebras of chains would have the properties claimed above; moreover, the system of fundamental groups of these homotopy types would yield an action of the Galois group on the system of braid groups, suitably completed.

3 This is probably a good place to note that $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts by automorphisms of the etalé homotopy type of a variety defined over $\mathbb{Q}$, which is not at all the same as a continuous action on the space of complex points of the variety. In fact one expects to recover classical invariants of a variety (the cohomology, or fundamental group, of its complex points, for example) only up to some kind of completion. Various kinds of invariants [etalé, motivic, Hodge-deRham, ...] each have their own associated completions, some of which are still quite mysterious; the fundamental group, in particular, comes in profinite [49] and prounipotent versions.

Drinfel'd works with the latter, which corresponds to the Malcev completion used in rational homotopy theory. A free group on n generators corresponds to the graded Lie algebra defined by n noncommuting polynomial generators, and the Lie algebra $\mathfrak{p}_n$ defined by the pure braid group P_n on n strands [the fundamental group of the space of ordered configurations of n points in the plane] is generated by elements $x_{ik}, 1 \leq i < k \leq n$ subject to the relations

$$[x_{ik}, x_{st}] = 0$$

if i, k, s, t are all distinct, and

$$[x_{ik}, x_{is}] = -[x_{ik}, x_{ks}]$$

if i, k, s are all distinct [32].

The fundamental groups of the little disks operad define a (unital) operad $\{P_n\}$ (the symmetric group action requires some care [47 §3]); in particular, cabling

$$c_I : P_{r(I)} \times \prod_{i \in I} P_i \to P_{|I|}$$

of pure braids defines composition operations which extend to homomorphisms

$$\mathfrak{c}_I : \mathfrak{p}_{r(I)} \times \prod_{i \in I} \mathfrak{p}_i \to \mathfrak{p}_{|I|}$$

of Lie algebras, defining an operad $\{\mathfrak{p}_n\}$ in that category as well. The natural product of Lie algebras is the direct sum of underlying vector spaces, so $\mathfrak{c}_I$ is a sum of two terms, the second defined by the juxtaposition operation

$$\mathfrak{p}_{i_1} \oplus \cdots \oplus \mathfrak{p}_{i_r} \to \mathfrak{p}_{i_1 + \cdots + i_r} \ .$$

The remaining information is contained in a less familiar homomorphism

$$\mathfrak{c}_I^0 : \mathfrak{p}_{r(I)} \to \mathfrak{p}_{|I|}$$

defined on the first component by any partition I of $|I|$ in r parts. It is not hard to see that

$$\mathfrak{c}_I^0(x_{st}) \equiv \sum_{p \in i_s, q \in i_t} x_{pq} + \dots \ ;$$

it would be very useful to know more about this expansion ...

4 Groups act on themselves by conjugation, and thus in general to have lots of (inner) automorphisms; Lie algebras act similarly on themselves, by their adjoint representations. It would also be useful to understand something about the relations between the automorphisms of an operad in groups or Lie algebras (as monoids in a category of objects $\{\mathcal{O}_*\}$ with $\{\Sigma_*\}$-action, as in §2 above), and systems of inner automorphisms of the objects $\mathcal{O}_n$: thus the adjoint action of a system of elements $\phi_n \in \mathfrak{p}_n$ defines an operad endomorphism if $\mathfrak{c}_I^0(\phi_*) = \phi_{|I|}$ for all partitions I. From some perspective, the classification of such endomorphisms is really part of the theory of symmetric functions.

This may be relevant, because the action GT on the completed braid groups is relatively close to inner. Drinfel'd [15 §4] describes elements of GT as pairs (λ, f), where λ is a scalar (ie, an element of the field of definition for the kind of Lie algebras we're working with: in our case, $\mathbb{Q}$, for simplicity), and f lies in the commutator subgroup of the Malcev completion of the free group on two elements. The pairs

(λ, f) are subject to certain restrictions which I'll omit here, on the grounds that the corresponding conditions on the Lie algebra are spelled out below. It is useful to regard elements $f = f(a,b)$ of the free group as noncommutative functions of two parameters a, b; if σ_i is a standard generator of the braid group, and

$$y_i = \sigma_{i-1} \cdots \sigma_2 \sigma_1^2 \sigma_2 \ldots \sigma_{i-1} \in P_n$$

for $2 \leq i \leq n$ (with $y_1 = 1$), then the action of GT on the braid group is defined by

$$(\lambda, f)(\sigma_i) = f(\sigma_i^2, y_i) \sigma_i^\lambda f(\sigma_i^2, y_i)^{-1} \ ;$$

the omitted conditions imply that when $i = 1$, this reduces to an exponential automorphism $\sigma_1 \mapsto \sigma_1^\lambda$ defined in the Malcev completion.

5 Here are some technical details about the Lie algebra of GT, reproduced from [15 §5]. Drinfel'd observes [remark before Prop. 5.5] that the scalar term λ can be used to define a filtration on this Lie algebra, and he describes the associated graded object $\mathfrak{grt}$. The following formalism is useful: $\mathfrak{fr}$ is the free formal $\mathbb{Q}$-Lie algebra defined by power series in two noncommuting generators A, B: it is naturally filtered by total polynomial degree, with the free graded Lie algebra on two generators as associated graded object.

The algebra $\mathfrak{grt}$ consists of series $\psi = \psi(A,B) \in \mathfrak{fr}$ which are antisymmetric $[\psi(A,B) = -\psi(B,A)]$ and in addition satisfy the relations

$$\psi(C,A) + \psi(B,C) + \psi(A,B) = 0$$

and

$$[B, \psi(A,B)] + [C, \psi(A,C)] = 0$$

when $A + B + C = 0$, as well as a third relation asserting that

$$\psi(x_{12}, x_{23} + x_{24}) + \psi(x_{13} + x_{23}, x_{34}) - \psi(x_{12} + x_{13}, x_{24} + x_{34})$$

equals

$$\psi(x_{23}, x_{34}) + \psi(x_{12}, x_{23}) \ ,$$

assuming that the x_{ik} satisfy the relations defining $\mathfrak{p}$. The bracket in $\mathfrak{grt}$ is

$$\langle \psi_1, \psi_2 \rangle = [\psi_1, \psi_2] + \partial_{\psi_2}(\psi_1) - \partial_{\psi_1}(\psi_2) \ ,$$

where ∂_ψ is the derivation of $\mathfrak{fr}$ given by $\partial_\psi(A) = [\psi, A],\ \partial_\psi(B) = 0$.

Drinfel'd shows [15 §5.6] that this Lie algebra is in fact isomorphic to the Lie algebra of GT [omitting the subalgebra corresponding to the scalars, used in this description to define the grading], but the isomorphism is defined inductively, so describing its action on the braid groups is not immediate. Nevertheless, Ihara (cf. [15 §6.3]) has shown that for each odd $n > 1$ there are elements $\psi_n \in \mathfrak{grt}$ such that

$$\psi_n(A,B) \equiv \sum_{1 \leq m \leq n-1} \binom{n}{m} (\mathrm{ad} A)^{m-1} (\mathrm{ad} B)^{n-m-1} [A, B]$$

modulo $[\mathfrak{fr}', \mathfrak{fr}']$ (where $\mathfrak{fr}'$ is the derived Lie algebra of $\mathfrak{fr}$].

It is conjectured that $\mathfrak{grt}$ is free on these generators. Aside from [41], the cabling described in §3 doesn't seem to have been considered very closely, in this context.

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