Theory of motives, homotopy theory of varieties, and dessins d'enfants

The American Institute of Mathematics

Table of Contents

A. Participant contributions 3

1. Barwick, Clark
2. Dhillon, Ajneet
3. Dugger, Dan
4. Ellenberg, Jordan
5. Esnault, Hélène
6. Isaksen, Dan
7. Jardine, Rick
8. Joseph, Ayoub
9. Kahn, Bruno
10. Khadjavi, Lily
11. Levine, Mark
12. Morava, Jack
13. Schneps, Leila

Chapter A: Participant contributions

A.1 Barwick, Clark

Toën and Vezzosi's homotopical algebraic geometry (HAG) is a very general theory in which one can do algebraic geometry over any reasonable "category of affines with homotopy theory." Such a category may be a closed model category, or a more general type of object, called a Segal category. It is ordinarily taken to be the opposite of a "category of rings with homotopy theory." The dream is that the homotopical study of schemes can be done "from the ground up," that is, that the homotopy theory of rings gives rise to a homotopy theory of sheaves (or stacks) over them, and thus in particular on schemes themselves.

Of course, Morel and Voevodsky have already constructed a beautiful homotopy theory for (i.e., a closed model category structure on) the category of sheaves with the affine line inverted. Why, then, would we be interested in changing our affines to allow for even more homotopy theory ? The answer is that changing or enlarging our category of rings means changing or enlarging our subcategory of representables. This gives us the opportunity to construct fascinating new geometric objects (from gluing together these representables) which are in some ways better suited for studying certain moduli problems.

Brave new algebraic geometry (BNAG) is the newest instantiation of HAG, in which our category of rings with homotopy theory is taken to be the closed model category of brave new affines, i.e., the opposite closed model category of a closed model category of structured ring spectra. There are many constructions of such closed model categories of ring spectra (which are all basically equivalent), including commutative E_∞–ring spectra, commutative $\mathbf{S}$–algebras, commutative symmetric ring spectra, and others. The Eilenberg–Mac Lane functor H provides a way to embed the old–fashioned category of rings into the (homotopy) category of structured ring spectra. If topologies are chosen well, H turns out to be a continuous morphism, and $H_!$ gives an embedding of the stack theory over ordinary rings into the stack theory over structured ring spectra. Now it turns out that, if X is any ordinary affine scheme, the closed model category of modules over $H_!X$ is Quillen equivalent to the closed model category of (unbounded) complexes of modules over X. The closed model category of stacks over the closed model category of brave new affines therefore contains the classical objects of algebraic geometry, along with the homotopy theory of the complexes of modules over them. The task of BNAG is to study stacks over these kinds of objects (for well–behaved topologies).

The idea of hidden smoothness (due, I think, variously to Deligne, Drinfel'd, Kapranov, and Kontsevich) is that the singularity of certain moduli spaces arises from certain truncations that we implicitly, but unnecessarily, have performed when we compute the parameter space of a moduli problem. (Recall that the tangent complex of the moduli space of vector bundles of fixed rank on a smooth surface at E is a truncation of the shifted Zariski cohomology complex $C^\bullet(X, \underline{\mathrm{End}}(E))[1]$.) Thus we should be seeking *derived* moduli spaces, whose cotangent complexes have not been truncated. This indicates that we might do well to work from the ground up with the derived category of modules over a scheme, in lieu of the category of modules. But observe that this is precisely what enlarging our category of rings to the category of structured ring spectra does for us.

BNAG can enter into the Morel–Voevodsky $\mathbf{A}^1$–homotopy theory of schemes at two different levels. First, following a suggestion I think due to Manin, one can study algebraic geometry over the category of structured ring spectra (with $\mathbf{P}^1$ as the circle) in the $\mathbf{A}^1$ closed model category. There is a lot of great work being done on the study of the representables in this case, i.e., on the category of structured ring spectra in the $\mathbf{A}^1$ closed model category. These have been shown to be fascinating and useful coefficients for Bloch–Ogus cohomology theories.

The focus of some of my work in the past few months has been a somewhat different role for BNAG in the Morel–Voevodsky theory. I am interested in constructing and using a "brave new Morel–Voevodsky" category over any brave new affine R. This is the closed model category MV/R of pointed prestacks on the category of brave new affines over R with the brave new affine line inverted. One can stabilize this closed model category (following Hovey) with respect to a brave new Tate circle and a simplicial circle to yield the stable brave new Morel–Voevodsky category SMV/R. This gives rise to a closed model category of cohomology theories on brave new stacks. That is, spectra (in fact, symmetric spectra) $F_\bullet$ in this category will have the property that for any brave new stack X, the homotopy groups of the stacks $\mathrm{R}\underline{\mathrm{Hom}}(\Sigma^\infty X, F_\bullet)$ will give the values of cohomology theories on X.

The study of these cohomology theories will lead to interesting results. In particular, any reasonable cohomology theory $E^{p.q}$ of schemes should have a brave new coefficient spectrum $F_\bullet$ of this type, so that

$$E^{p,q}X = \pi_{q-p}\mathrm{R}\underline{\mathrm{Hom}}(H_!X, S_T^q \wedge F_\bullet) \tag{1}$$

for any scheme X. As an example, Weibel's (nonconnective) homotopy K–theory (which agree with Thomason–Trobaugh's K–theory for regular schemes) has such a brave new classifying spectrum $\Lambda_\bullet$, and it is for the moment a conjecture that the brave new stack $\mathrm{R}\underline{\mathrm{Hom}}(H_!X, S_T^q \wedge \Lambda_\bullet)$ is representable, and the representing structured ring spectrum is the homotopy K–theory ring spectrum $KH(X)$. It would be fascinating to study the brave new classifying spectra of other cohomology theories such as algebraic cobordism, motivic cohomology, Deligne cohomology, and étale cohomology. I conjecture that for any Bloch–Ogus cohomology theory, the Chern characters from K–theory will be induced by morphisms of brave new stacks.

The assignment $R \mapsto MV/R$ gives rise to a (Segal) 1–stack on the brave new site, whose pullback along the Eilenberg–Mac Lane functor is equivalent (in a certain precise sense) to the prestack given by assigning to any affine scheme A the traditional Morel–Voevodsky category. Thus this brave new Morel–Voevodsky category is a true generalization of the original Morel–Voevodsky category.

A deeper statement, which I still must leave as a conjecture, is that the deep results of Morel–Voevodsky on the universality of the Mayer–Vietoris, Gysin, and blow–up squares, valid in the classical Morel-Voevodsky category only on a smooth site, have true analogues in the brave new Morel–Voevodsky category on the site of all schemes, using the yoga of hidden smoothness.

Deeper still (and yet more conjecturally), through some process of linearization, it may be possible to construct, following Voevodsky, a triangulated tensor category of mixed motives. The difficulties presented by the need for resolution of singularities in Voevodsky, Friedlander, and Suslin's construction of the category of correspondences may be avoidable by using descent arguments and hidden smoothness. In a later section, I will discuss this type of argument a little further. In particular, it would be especially nice if the use of brave new cohomology theories of the type described here permitted a construction of the

Ω–spectrum $K(\mathbf{Z}(n), 2n)$ for motivic cohomology that no longer depends on the existence of a strong resolution of singularities.

Much of this work was (notwithstanding its rather abstract façade) inspired more or less directly by arithmetic and positive–characteristic considerations. Over the algebraic closure of $\mathbf{Q}$, where no such considerations exist, one studies the moduli space of branched covers of the thrice–punctured projective line by using Belyi's theorem to reduce to the study of curves over $\mathbf{C}$, where one employs homotopy theory. Likewise, many of the great successes of Hodge theory and geometric Langlands follow from the availability of homotopy–theoretic techniques. Morally, many of the advances in the zero–characteristic, algebraically closed milieu involve bringing the algebraic geometry down to homotopy theory. The main goal of HAG and BNAG is to find ways of bringing the homotopy theory up to the algebraic geometry.

An especially nice display of these sort of techniques in arithmetic has come in the form of Hain and Matsumoto's proof of the Deligne–Ihara conjecture (sans freeness) using Levine's theory of Tate motives. A robust homotopy theory of motives is clearly the key to the arithmetic study of covers of the thrice–punctured projective line.

A.2 Dhillon, Ajneet

Currently I am studying Tamagawa numbers of algebraic groups in positive characteristic. Specifically, I have been trying to understand some conjectures of Harder and Weil on these numbers. These conjectures can be reduced to some statements about the ℓ-adic cohomology of the moduli stack of G-torsors over a smooth projective. Over the complex numbers, these cohomology groups can be studied by homotopy theoretic methods. My method is to try to generalize these methods to the positive characteristic case via the machinery of A^1 homotopy theory.

A.3 Dugger, Dan

I am interested in applications of motivic homotopy to concrete problems in algebra and geometry. I'd like to come away from the workshop with new ideas about where these might lie. I'm especially interested in Morel's various approaches to the second Milnor conjecture on quadratic forms.

Here are some questions related to the more abstract side of motivic homotopy theory:

1) Does the slice filtration, applied to stable motivic sphere spectrum, give the motivic cohomology spectrum in characteristic p? How might one tackle this?

2) Does one expect a motivic spectral sequence converging to Hermitian K-theory analagous to the Atiyah-Hirzebruch spectral sequence for topological KO theory?

A.4 Ellenberg, Jordan

I'm especially interested in the relation between Galois actions on fundamental groups and rational points on varieties. If X/K is a variety, then every point P of $X(K)$ induces a section of the surjection in the basic exact sequence

$$1 \to \pi_1^{geom}(X) \to \pi_1(X) \to Gal(K) \to 1$$

and much is conjectured about the relationship between $X(K)$ and the set of conjugacy classes of such sections. If $\pi_1(X)$ is replaced by a characteristic quotient (for instance, the quotient by the commutator of $\pi_1^{geom}(X)$ or some other term of the lower central or derived series) one gets a different set of sections which can perhaps be viewed as an "approximation" to the set $X(K)$. For example, if Π is the maximal pro-l quotient of the quotient of $\pi_1(X)$ by

$$[\pi_1^{geom}(X), [\pi_1^{geom}(X), \pi_1^{geom(X)}]$$

what can one say about the sections of $\Pi \to Gal(K)$?

I would also like to think about the various novel obstructions to existence and adelic density of rational points (e.g. those developed by Skorobogatov, Harari, etc...) To what extent are these obstructions to the existence of sections of fundamental groups, and to what extent can this viewpoint yield new and interesting obstructions? We started discussing some of these questions at a previous AIM workshop ("Rational and integral points on higher dimensional varieties) and I hope to continue the discussion here.

A.5 Esnault, Hélène

Several subjects of interest not far from the main topic of the conference.

1) Rational points on varieties defined over finite fields. Motivic ideas were used in our article (Varieties over a finite field with trivial Chow group of 0-cycles have a rational point, Invent. math. **151** (2003), 187-191) in order to prove the Lang-Manin conjecture asserting that a Fano variety over a finite field has a rational point. In the singular hypersurface case, we could prove a motivic version of the Ax-Katz theorem for level 1 ((with S. Bloch and M. Levine) Decomposition of the diagonal and eigenvalues of Frobenius for Fano hypersurfaces, preprint 2003, 16 pages, appears in the Am. J. of Mathematics) and also prove that the divisibilities of eigenvalues predicted by the Ax-Katz theorem holds true for all levels ((with N. Katz) Cohomological divisibility and point count divisibility, preprint 2003, 10 pages, appears in Compositio Mathematica.). This together with recent work of Fakhruddin and Rajan (math.NT/0402230) treating Fanos which are the degeneration of smooth ones, gives some hope to understand what is true in the singular case.

2) Additive Chow groups, as developed in ((with S. Bloch) An additive version of higher Chow groups, Annales Scientifiques de l'École Normale Supérieure, 4-o série, **36** (2003), 463-477), together with ((with S. Bloch) The additive dilogarithm, Documenta Math. Extra Volume: Kazuya Kato's Fiftieth Birthday (2003), 131-155) give the first step to describe the de Rham-Witt complex of fields by a complex of cycles. At the same time, understood as motives over $k[\epsilon], \epsilon^2 = 0$, this has been further developed as a generalization of Bloch's complex by Goncharov (math.AG/0401354). This gives some material to discuss some of the questions raised in Goncharov's manuscript.

A.6 Isaksen, Dan

I am interested in various kinds of concrete computations that can be made in motivic homotopy theory, especially with the purpose of making connections to fields of mathematics outside of algebraic K-theory and algebraic cycles. A good example is Morel's computation of the motivic stable homotopy group $\pi_{0,0}$.

One possible source of questions is from algebraic problems that have been addressed with topological methods. I would be very interested to learn of examples of this situation.

Off the top of my head, here are a few more examples. I'm not sure that these are necessarily interesting or doable.

1. Express the motivic cohomology of a toric variety in terms of the combinatorics of the associated fan.

2. Use framed bordism to produce elements of the stable motivic homotopy groups (suggested to me by Daniel Biss).

3. If q is any quadratic form, compute the motivic cohomology of the projective variety defined by $q = 0$.

A.7 Jardine, Rick

It would be interesting to see new calculational techniques arise for the motivic cohomology or motivic homotopy types of certain moduli spaces (or stacks). There are only isolated examples of such calculations at the moment: one result that we know, due to Morel and Voevodsky, is that the etale stack associated to a finite etale group-scheme is a motivic stack in the sense that its classifying space is motivic fibrant. The calculations in this area are so far either based on results of this type, or on finding concrete geometric models for the motivic homotopy types of classifying spaces of algebraic groups.

There is a motivic homotopy theory of groupoids with a corresponding notion of motivic stacks that one can write down, but what it means calculationally is still unclear. One feature of the theory is that classifying spaces of groupoids can have non-trivial Nisnevich sheaves of higher homotopy groups: in particular, when one puts the general linear group into the machine one gets the higher algebraic K-theory sheaves back. One might expect that the motivic stack associated to the fundamental groupoid of a simplicial sheaf is equivalent to the fundamental groupoid of the associated motivic space, but this has not been proved yet. The question is strongly related to a difficult question about the connectivity of motivic spaces which has only recently been solved in the stable case by Morel.

A.8 Joseph, Ayoub

For a field k of caracteristic zero, let $MGal(k)$ be the conjectural motivic Galois group. Is the following sequence of pro-algebraic groups a split sequence? (like the case for the Galois groups):

$$1 \to RMGal(k(X)/k) \to MGal(k(X)) \to MGal(k) \to 1$$

Where $RMGal(k(X)/k)$ is the relative motivic Galois group wich can be defined to be the kernel of the second map above. Is there a way to define directly the relative motivic Galois group (without supposing the existance of the absolute one)? Is the relative motivic Galois group something easier to study than the absolute one?

A.9 Kahn, Bruno

The title of the programme has three parts: theory of motives, homotopy theory of varieties, and dessins d'enfants. The presentation of the workshop by the organisers only involves the last two. I must say that my work has been mostly related to the first, and I only have a "honest man" knowledge of the two others. So I will come mostly to learn, my current interest being firmly in arithmetic geometry.

Certainly I would like to learn more on moduli spaces (of curves of others) from the viewpoint of the workshop. Concerning suggestions, I don't have anything very original: I am aware of the work of Madsen and Tillmann on stable moduli spaces of curves, which I think predates [MW]. Certainly it would be interesting to put more algebraic geometry in this, and the obvious way to do this would be via the homotopy theory of schemes. Where would this lead? I hope to learn it during the workshop.

A.10 Khadjavi, Lily

By Belyi's Theorem only algebraic curves (i.e., defined over $\overline{\mathbf{Q}}$) admit maps to the projective line ramified over at most three points. Can one define a Belyi height, say the minimal degree of such a map for a given curve, which would establish an invariant characterizing algebraic curves? (The degree of a Belyi map gives the number of edges in the associated *dessin*.)

Could one construct a sort of moduli space for curves in terms of this Belyi height? If so, how does the Belyi height behave with respect to other invariants of curves, such as the genus or, for elliptic curves, the j-invariant? Also, can one find effective methods for computing a Belyi height? If not, could one at least give a lower bound on the degree?

A.11 Levine, Mark

I am interested in learning about relations between the motivic Galois group of $\mathbb{Q}$ and the Grothendieck-Teichmüller group, and also motives that arise from configuration spaces. I'd like to learn about motivic aspects of topological operads, like the little disks operad.

A.12 Morava, Jack

See separate document.

A.13 Schneps, Leila

At the present time I am starting to work on two aspects of Grothendieck-Teichmüller theory which have a pronounced motivic flavour. One of my main goals in attending the AIM conference is to increase my understanding of the role of motive theory in providing answers and proofs in this domain.

The first aspect of Grothendieck-Teichmüller theory in question is the *linear* aspect, concerning the comparison and conjectural isomorphism between (i) the Grothendieck-Teichmüller Lie algebra, also known as Ihara's stable derivation Lie algebra, (ii) the double shuffle Lie algebra, also known as the formal multizeta Lie algebra, (iii) the Drinfeld-Ihara Lie algebra coming from considering representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on the quotients of $\widehat{F}_2$ by its descending central series; (iv) the Lie algebra associated to the fundamental group of the category of mixed Tate motives over $\mathbf{Z}$. The second aspect (joint work with P. Lochak) comes from considering periods of genus zero moduli spaces. These are real numbers of which some are obviously equal to multizeta values; Goncharov has shown that all these periods can be viewed as periods of framed mixed Tate motives, and has conjectured that the latter are always multizeta values. Whether or not they are all classical multizeta values, the periods of genus zero moduli spaces (and in some more complicated sense, the periods of higher genus moduli spaces) satisfy a shuffle relation generalising that of classical multizeta values. We conjecture that they also satisfy fundamental 2, 3 and 5 cycle relations corresponding to those satisfied by the classical multizeta values (i.e. by the Drinfeld associator), and that this is a consequence of the geometric role of the 2, 3 and 5 cycle relations in the moduli spaces. These questions and their motivic aspects are what I am interested in understanding more deeply.