Motivic spaces and the motivic stable category

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Nisnevich topology

$Sm|_S =$ schemes which are smooth and of finite type over a scheme S of finite dimension.

A **Nisnevich covering** (originally "cd" covering) for an S-scheme U is a finite family of étale maps $V_i \to U$ which is an étale cover for U, s.t. every map $Sp(K) \to U$ lifts to some V_i, for all fields K.

étale $\geq$ Nisnevich $\geq$ Zariski

eg: $K =$ field. F is a sheaf for the Nisnevich topology on $et|_K$ iff F is additive: $F(\sqcup_i Sp(L_i)) \cong \prod_i F(Sp(L_i))$.

Stalks are computed on henselizations $\mathcal{O}^h_{x,U}$ of local rings, because residue fields must lift to covers.

Nisnevich cohomology

Nisnevich cohomology and étale cohomology are generally quite different, eg. fields are acyclic (but not points) for the Nisnevich topology.

Nisnevich cohomological dimension behaves similarly to Zariski cohomological dimension:

Theorem: (Kato-Saito) $X =$ Noetherian scheme of finite Krull dimension d. Then X has cohomological dimension d wrt. all abelian Nisnevich sheaves.

Simplicial sheaves, presheaves

$s\,\mathrm{Shv}$ = simplicial sheaves, $s\,\mathrm{Pre}$ = simplicial presheaves on $(Sm|_S)_{Nis}$.

Homotopy theory: $f : X \to Y$ is a weak equivalence if all induced maps $X_x \to Y_x$ on stalks are weak equivs. of simplicial sets. Cofibrations are monomorphisms. Ass. sheaf map $X \to \tilde{X}$ is weak equiv. and $\mathrm{Ho}(s\,\mathrm{Shv}) \simeq \mathrm{Ho}(s\,\mathrm{Pre})$.

Fibrant model: weak equivalence $Y \to Y_f$ such that Y_f is fibrant for the homotopy theory.

Fibrant simplicial presheaves are determined by a right lifting property with respect to trivial cofibrations. Behave like injective resolutions, homotopy groups in all sections determined by cohomology with coeffs. in sheaves of homotopy groups.

Nisnevich descent

Distinguished square:

$$\begin{array}{ccc} \phi^{-1}(U) & \longrightarrow & V \\ \downarrow & & \downarrow \phi \\ U & \underset{j}{\longrightarrow} & X \end{array}$$

j open immersion, ϕ étale, $\phi^{-1}Z \cong Z$ where $Z = X - U$.

$\{j, \phi\}$ is Nisnevich cover of X.

Def'n: simplicial presheaf Y has the **Nisnevich descent property** (aka. B.G. property) if Y is presheaf of Kan complexes and Y takes all dist. squares to htpy. cartesian squares.

Nisnevich descent theorem: If Y has the Nisnevich descent property and $Y \to Y_f$ is a fibrant model for Y, then $Y(U) \simeq Y_f(U)$ in all sections.

Consequences of Nisnevich descent

Simplicial presheaves or sheaves having the Nisnevich descent property approximate fibrant objects so well that there is no difference in practice.

Class of simplicial presheaves having Nisnevich descent is closed under filtered colimits.

This is special to the Nisnevich topology: not true in general that filtered colimits of fibrant objects are well behaved.

Motivic homotopy theory

$\mathbb{A}^1 = S \times \mathbb{A}^1_{\mathbb{Z}}$ is the affine line over S.

Motivic homotopy theory is constructed from the htpy. theory for simp. sheaves and presheaves on $(Sm|_S)_{Nis}$ by formally contracting the affine line to a point: want $\mathbb{A}^1 \to *$ to be a weak equivalence.

How to do it: ("Bousfield localization")

- Z is **injective** if it is fibrant for Nis. topology and has RLP wrt. all maps

$$(\mathbb{A}^1 \times A) \cup_A B \to \mathbb{A}^1 \times B$$

  ass. to all inclusions $A \subset B$
- $Y \to Y'$ is **motivic weak equiv.** if $\pi(Y', Z) \to \pi(Y, Z)$ is bijection for all injective Z.

Motivic homotopy theorems

Every weak equivalence for the Nisnevich topology is a motivic weak equivalence.

Cofibrations are monomorphisms. **Motivic fibrations** are defined by a RLP wrt. all motivic trivial cofibrations.

Theorem: There is a **proper** closed simplicial model structure on $s\,\mathrm{Pre}(Sm|_S)$ with cofibrations, motivic weak equivalences and motivic fibrations defined as above.

Fact: Injective objects $=$ motivic fibrant objects.

Fact: Z is motivic fibrant iff Z if fibrant for the Nisnevich topology and $Z(U) \to Z(U \times \mathbb{A}^1)$ a weak equivalence of simp. sets for all U/S.

Fact: Every Y has motivic fibrant model, ie. motivic weak equiv. $Y \to Y_f$ s.t. Y_f is motivic fibrant.

Motivic spaces

Other models for motivic homotopy theory:

- simplicial sheaves on $(Sm|)_{Nis}$
- sheaves on $(Sm|)_{Nis}$
- presheaves on $(Sm|)_{Nis}$

Cosimplicial scheme $\mathbb{A}^n = Sp(\mathbb{Z}[t_0, \ldots, t_n]/(\sum t_i - 1)$: $d_i(t_i) = 0$. $s_j(t_j) = t_j + t_{j+1}$

Singular functor $S : \mathrm{Pre} \to s\,\mathrm{Pre}$: $S(X)_n = X(U \times \mathbb{A}^n)$. has right adjoint $|\ | =$ realization.

Cofibrations are monomorphisms, $W \to W'$ is weak equiv. iff $K(W, 0) \to K(W', 0)$ is motivic weak equiv.

Quillen equivalence: $S : \mathrm{Pre} \leftrightarrows s\,\mathrm{Pre}$.

Old-timey concept: motivic space $=$ Nisnevich sheaf

Examples

Pushout square:

$$\begin{array}{ccc} \mathbb{G}_m & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1 \end{array}$$

All maps are inclusions, hence cofibrations and $\mathbb{A}^1 \simeq *$.

- $\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$, where $S^1 = \Delta^1/\partial\Delta^1$
- $\mathbb{P}^1 \simeq \mathbb{A}^1/\mathbb{G}_m = \mathbb{A}^1/(\mathbb{A}^1 - 0) =: T$.
- $T \wedge T \wedge T \cong \mathbb{A}^3/(\mathbb{A}^3 - 0)$, $c =$ cyclic perm. of order 3 acts on $T^{\wedge 3}$ via action $Gl_3 \times T^{\wedge 3} \to T^{\wedge 3}$. c is prod. of $E_{i,j}(a)$ in $Gl_3(\mathbb{Z})$, and there is path $\mathbb{A}^1 \to Gl_3$ def. by $t \mapsto E_{i,j}(ta)$. There is homotopy $T^{\wedge 3} \times \mathbb{A}^1 \to T^{\wedge 3}$ from c to identity.

Purity

Theorem: $i : Z \to X$ closed embedding of smooth schemes/S. Then there is a natural equivalence of pointed motivic homotopy types

$$X/(X - i(Z)) \simeq Th(N_{X,Z})$$

$N_{X,Z} =$ normal bundle for the imbedding i, and Thom space $Th(N_{X,Z})$ is defined by

$$Th(N_{X,Z}) = N_{X,Z}/(N_{X,Z} - i_0(Z)),$$

where i_0 is zero section.

The point of this: X has an open cover $\{U\}$ for which the normal bundle trivializes, and then

$$U/(U - (U \cap Z)) \simeq (U \cap Z)_+ \wedge (\mathbb{A}^c/(\mathbb{A}^c - 0))$$

Stable homotopy theory

Homotopy theory of spectra:

- **Def'n**: A **spectrum** X consists of pointed spaces X^n and pointed maps $S^1 \wedge X^n \to X^{n+1}$, $n \geq 0$.
- A map of spectra $f : X \to Y$ consists of pointed maps $X^n \to Y^n$ which respect structure.
- A spectrum X has stable homotopy groups: the maps $X^n \to \Omega X^{n+1}$ induce a system

$$\pi_{n+k} X^n \to \pi_{n+k+1} X^{n+k+1} \to \dots$$

  and $\pi_k X$ is the colimit of this system, $k \in \mathbb{Z}$.
- A map $X \to Y$ of spectra is a *stable equivalence* if it induces isomorphisms in all stable homotopy groups.

More stable homotopy theory

The stable equivalences are the weak equivalences for a homotopy theory (model structure) on $\mathbf{Spt}$ = category of spectra. $\mathrm{Ho}(\mathbf{Spt})$ is the (traditional) stable homotopy category. $S^1 \wedge X \simeq X[1]$.

Can do similar things for "ordinary" chain complexes (suspension $=$ shift by -1, spectrum objects are unbounded chain complexes), presheaves of spectra, presheaves of chain complexes, spectrum objects in simplicial abelian groups, ...

There are many stable homotopy categories, depending on underlying topologies, choice of suspension object.

T-spectra

Idea: Formally invert $X \mapsto T \wedge X$ in the setting of motivic homotopy theory for good objects T (eg. $T = \mathbb{P}^1$, $T = S^1$, $T = \mathbb{G}_m$).

- **Def'n**: A T-spectrum X consists of pointed simplicial presheaves X^n, pointed maps $T \wedge X^n \to X^{n+1}$, $n \geq 0$.
- A map of T-spectra $f : X \to Y$ consists of pointed maps $X^n \to Y^n$ which respect structure.

Motivic stable equivalence

Every T-spectrum X has a natural levelwise motivic fibrant model $X \to X_f$, ie. the maps $X^n \to X_f^n$ are motivic fibrant models in all levels.

There is T-spectrum QX_f with QX_f^n def. by filtered colimit

$$X_f^n \to \Omega_T X_f^{n+1} \to \Omega_T^2 X_f^{n+2} \to \dots$$

There are natural maps $X \to X_f \to QX_f$.

$f : X \to Y$ is **motivic stable equivalence** if all maps $QX_f^n(U) \to QY_f^n(U)$ are weak equivalences of pointed simplicial sets.

This makes sense because each QX_f^n is sectionwise equivalent to its motivic fibrant model, by Nisnevich descent.

Motivic stable categories

Def'n: A **cofibration** $i : A \to B$ is a map s.t. $A^0 \to B^0$ is a monomorphism, and all induced maps

$$(T \wedge B^n) \cup_{(T \wedge A^n)} A^{n+1} \to B^{n+1}$$

are monomorphisms (by analogy with ordinary stable homotopy theory).

Theorem: The motivic stable equivalences and cofibrations determine a homotopy theory (model structure) for T-spectra.

$\mathrm{Ho}(\mathbf{Spt}_{\mathbb{P}^1})_S$ is the standard motivic stable category of Morel and Voevodsky.

$\mathrm{Ho}(\mathbf{Spt}_{S^1})_S$ is the motivic stable category of S^1-spectra.

Examples, Facts

- S^0 : $S^0, T, T \wedge T, T \wedge T \wedge T, \ldots$ is motivic sphere spectrum.
- suspension spectrum $\Sigma_T^\infty A$: $A, T \wedge A, T \wedge T \wedge A, \ldots$.
- shift functor $X[k]$: $X[k]^n = X^{n+k}$ defined for all $k \in \mathbb{Z}$. $T \wedge X \simeq X[1]$

Fact: Y is stably fibrant if and only if Y^n is motivic fibrant and $Y^n \to \Omega_T Y^{n+1}$ is a motivic weak equiv. for all $n \geq 0$.

Fact: Cofibre sequences $=$ fibre sequences.

Fact: $X \vee Y \to X \times Y$ is motivic stable equivalence.

Fact: There is a good theory of smash products, via $\mathbf{Spt}_T^\Sigma =$ symmetric T-spectra.

Eilenberg-Mac Lane spectrum

$S = Sp(k)$, k a field. $Y =$ smooth scheme of finite type/k.

$LY(U) =$ free $\mathbb{Z}$-module gen. by closed irred. subschemes $Z \subset U \times Y$ such that composite $Z \subset U \times Y \to Y$ is finite and surjective. The graph determines a presheaf morphism $Y \longrightarrow L(Y)$.

Suslin-Voevodsky: $LY(U) \cong \mathbb{Z}_{qfh}Y(U)$. LY is a universal "presheaf with transfers".

$H(\mathbb{Z})^n = K(\mathbb{Z}(n), 2n) = L(\mathbb{A}^n)/L(\mathbb{A}^n - 0) \simeq L((\mathbb{P}^1)^{\wedge n}) \simeq L(\mathbb{G}_m^{\wedge n})[-n]$.

The graph map and pairings make $H(\mathbb{Z})$ into a $\mathbb{P}^1$-spectrum.

Motivic cohomology

X/S smooth, $T = \mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$, $p, q \in \mathbb{Z}$

$$H^{p,q}(X, \mathbb{Z}) = [\Sigma^\infty_T X_+, H(\mathbb{Z})(q)[p]]$$
$$= [\Sigma^\infty_T X_+, H(\mathbb{Z}) \wedge (S^1)^{p-q} \wedge \mathbb{G}_m^q]$$

More generally, if E is a T-spectrum

$$H^{p,q}(X, E) = [\Sigma^\infty X_+, E \wedge (S^1)^{p-q} \wedge \mathbb{G}_m^q].$$

$\mathbb{G}_m^q = \mathbb{G}_m^{\wedge q}$ if $q \geq 0$.

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Fact: X motivic fibrant pointed simplicial presheaf U/S smooth. $T = \mathbb{P}^1$. Then $\pi_r X(U) \cong [S^r, X|_U]$.

$Y = T$-spectrum, all Y^n motivic fibrant

$$Q_T Y^n = \varinjlim (Y^n \to \Omega_T Y^{n+1} \to \dots)$$

$\pi_r Q_T Y^n(U)$ is colimit

$$[S^r, Y^n|_U] \to [S^r \wedge S^1 \wedge \mathbb{G}_m, Y^{n+1}|_U] \to [S^r \wedge S^2 \wedge \mathbb{G}_m^{\wedge 2}, Y^{n+2}|_U] \to \dots$$

$$\begin{aligned}\pi_{t,s} Y(U) &= \varinjlim ([S^{t+n} \wedge \mathbb{G}^{s+n}, Y^n|_U] \to [S^{t+n+1} \wedge \mathbb{G}^{s+n+1}, Y^{n+1}|_U]) \\ &\cong [S^0 \wedge S^t \wedge \mathbb{G}_m^s, Y|_U]\end{aligned}$$

Motivic stable homotopy groups

$U \mapsto \pi_{t,s} Y(U)$ is the presheaf of motivic stable homotopy groups.

Fact: $E \to F$ is a motivic stable equivalence iff all $\pi_{t,s}E \to \pi_{t,s}F$ are isomorphisms of presheaves.

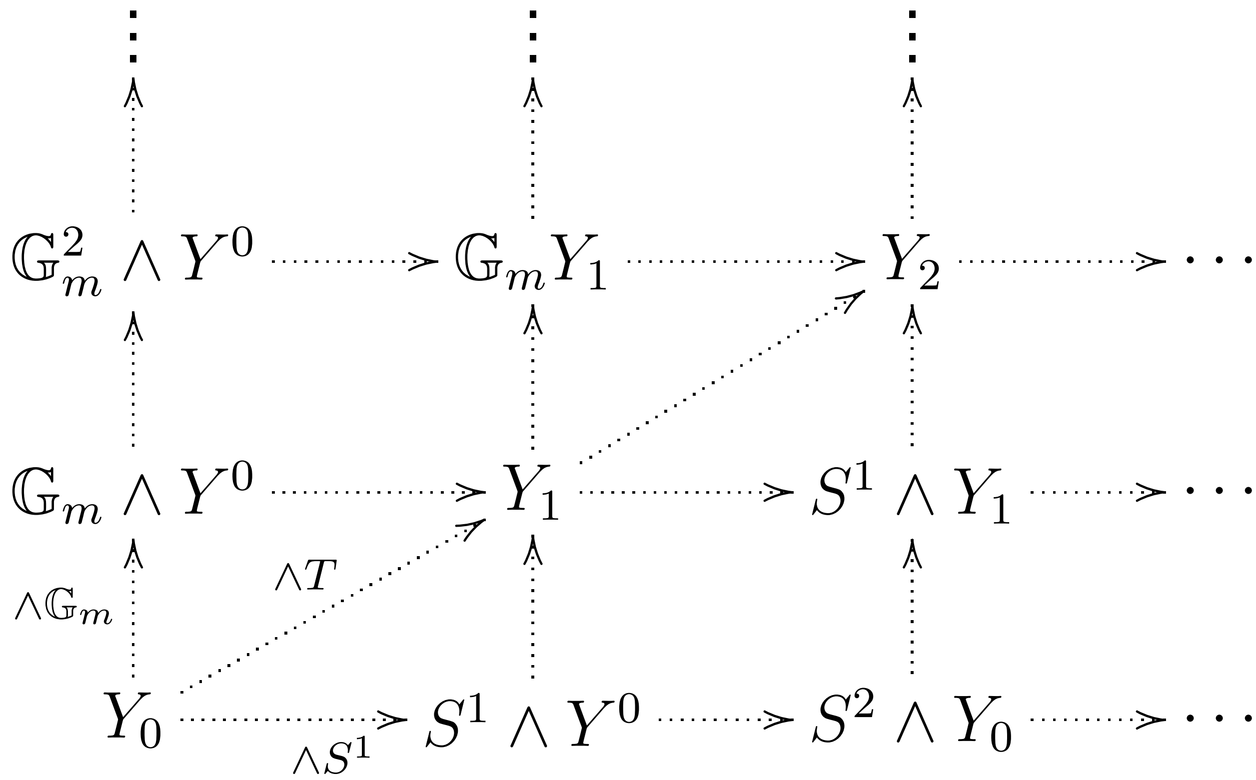
$$\begin{aligned} H^{p,q}(S,E) &= [S^0, E \wedge S^{p-q} \wedge \mathbb{G}_m^p] \\ &\cong [S^0 \wedge S^{q-p} \wedge \mathbb{G}_m^{-p}, E] \\ &\cong \pi_{q-p,-p} E(*). \end{aligned}$$

Fact: Cofibre sequence $A \to B \to B/A$ ind. long ex. seq.

$$\cdots \to \pi_{t+1,s} B/A \to \pi_{t,s} A \to \pi_{t,s} B \to \pi_{t,s} B/A \to \dots$$

Asymmetric bispectra

$Y = \mathbb{P}^1$-spectrum. $\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$.



$\mathbb{G}_m$-spectrum in S^1-spectra, or S^1-spectrum in $\mathbb{G}_m$-spectra