Lectures at AIM – April 23-26, 2004

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Lecture I

The Grothendieck Teichmüller Group

Let $G_{\mathbb{Q}}$ be the absolute Galois group of $\mathbb{Q}$, i.e. the (topological) group of automorphisms of the separable closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ which act trivially on $\mathbb{Q}$.

Central Theme: Study $G_{\mathbb{Q}}$ via its *geometric* actions, i.e. its actions on fundamental groups of geometric objects (varieties, schemes, stacks...)

Use this to characterize properties of the elements of $G_{\mathbb{Q}}$.

Note: Apart from complex conjugation, it is impossible to 'write down' an element of $G_{\mathbb{Q}}$.

The purpose of the theory developed by Grothendieck in §2 of *Esquisse d'un Programme* is:

1) to identify each element $\sigma \in G_{\mathbb{Q}}$ with a pair

$$(\chi(\sigma), f_\sigma) \in \widehat{\mathbb{Z}}^* \times \widehat{F}_2'.$$

Here $\chi : G_{\mathbb{Q}} \to \widehat{\mathbb{Z}}^*$ is just the cyclotomic character giving the action of $G_{\mathbb{Q}}$ on roots of unity; we have the exact sequence

$$1 \to G_{\mathbb{Q}^{\mathrm{ab}}} \to G_{\mathbb{Q}} \to \widehat{\mathbb{Z}}^* \to 1.$$

Grothendieck indicated how to do this, and it was completed by Drinfel'd and Ihara.

The cyclotomic character is well-understood, so the deep part is the element f_σ in the (derived subgroup of the) free profinite group on two generators. In fact, one can restrict to the subgroup $G_{\mathbb{Q}^{\mathrm{ab}}}$ and associate an element $f_\sigma \in \widehat{F}_2'$ to each $\sigma \in G_{\mathbb{Q}^{\mathrm{ab}}}$.

The free group is obviously a *much* simpler group than $G_{\mathbb{Q}}$! But which elements of $\widehat{F}_2'$ come from $G_{\mathbb{Q}}$? The second part of Grothendieck's program is:

2) Find necessary and sufficient conditions on $f \in \widehat{F}_2'$ for it to come from a $\sigma \in G_{\mathbb{Q}}$.

Various necessary conditions have been found, coming from geometry of the moduli spaces. But it is not known whether they are or are not sufficient.

§1. Galois groups and fundamental groups

Grothendieck's suggestion for approaching $G_{\mathbb{Q}}$ is by *geometric Galois actions*, i.e. considering actions of $G_{\mathbb{Q}}$ on objects which are geometric/topological rather than directly on the algebraic numbers.

Here we discuss actions of $G_{\mathbb{Q}}$ on two kinds of topological objects:

- dessins d'enfants; these are graphs embedded into topological surfaces, whose faces are all cells.

- diffeomorphisms (well, actually pro-diffeomorphisms) of topological surfaces.

Recall that the *profinite completion* of a group is given by the inverse limit of the system of all its finite quotients:

$$\widehat{G} = \lim_{\leftarrow} G/N$$

where N runs through the normal subgroups of finite index of G.

When I wrote "pro-diffeomorphisms" above, I meant elements of the profinite completion of the group of diffeomorphisms of a topological surface.

The two kinds of actions I want to talk about both actually stem from one main type of action.

Namely, if X is any algebraic variety defined over $\mathbb{Q}$, let $\pi_1(X)$ denote its topological fundamental group and $\widehat{\pi}_1(X)$ its algebraic fundamental group, which is the profinite completion of the topological one. Then there is a **canonical outer action**

$$G_{\mathbb{Q}} \to \mathrm{Out}\big(\widehat{\pi}_1(X)\big). \tag{1}$$

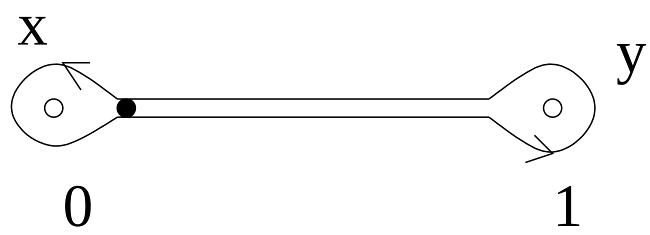
Moreover this outer action *preserves conjugacy classes of inertia groups.*

Here is where that outer Galois action comes from: the left-hand column shows a finite cover Y of X, sitting under the universal cover $\widetilde{X}$ of X, with Galois group the topological π_1, the middle column shows the function field situation over $\mathbb{C}$, where the top field is the compositum of all the function fields of the finite covers Y and therefore the Galois group is the profinite completion of the topological π_1, and the right-hand column uses the Lefschetz theorem to descend from $\mathbb{C}$ to the algebraically closed subfield $\overline{\mathbb{Q}}$ without changing the Galois group, so that the natural inclusion of the field $\mathbb{Q}(X)$ into $\overline{\mathbb{Q}}(X)$, with Galois group $G_{\mathbb{Q}}$, gives a canonical outer action of $G_{\mathbb{Q}}$ on $\widehat{\pi}_1(X)$.

$$
\begin{array}{cccccc}
 & \widetilde{X} & \widetilde{\mathbb{C}(X)} & & \widetilde{\overline{\mathbb{Q}}(X)} & \\
 & | & | & & | & \\
\pi_1(X) & Y & \mathbb{C}(Y) & \widehat{\pi}_1(X) & \overline{\mathbb{Q}}(Y) & \widehat{\pi}_1(X) \\
 & | & | & & | & \\
 & X & \mathbb{C}(X) & & \overline{\mathbb{Q}}(X) & \\
 & & & & | & G_{\mathbb{Q}} \\
 & & & & \mathbb{Q}(X) &
\end{array}
$$

§2. The case $\mathbb{P}^1 - \{0,1,\infty\}$

Let $X = \mathbb{P}^1 - \{0,1,\infty\}$, so that the topological π_1 is F_2, the free group on two generators, which we write $\langle x, y, z \mid xyz = 1\rangle$, identifying x, y and z with loops around 0, 1 and ∞ respectively.



We saw in §1 that we have a canonical homomorphism

$$G_{\mathbb{Q}} \to \mathrm{Out}\big(\widehat{\pi}_1(\mathbb{P}^1 - \{0,1,\infty\}\big)$$

i.e.

$$G_{\mathbb{Q}} \to \mathrm{Out}(\widehat{F}_2).$$

The inertia groups are $\langle x\rangle$, $\langle y\rangle$ and $\langle z\rangle$, so we know that for each $\sigma \in G_{\mathbb{Q}}$, there exist $\alpha, \beta, \lambda \in \widehat{\mathbb{Z}}^*$ and $f,\, g \in \widehat{F}_2$ such that

$$\begin{cases} \sigma(x) = x^\alpha \\ \sigma(y) = g^{-1} y^\beta g \\ \sigma(z) = h^{-1} z^\lambda h \end{cases}$$

lifts the canonical outer action of σ on $\widehat{F}_2$.

In $\widehat{F}_2^{\mathrm{ab}} = \widehat{\mathbb{Z}} \times \widehat{\mathbb{Z}}$, this means that $x^\alpha y^\beta z^\lambda = 1$, which is only possible if $\alpha = \beta = \lambda$. Suppose $g \equiv x^\delta y^\epsilon$ in $\widehat{F}_2^{\mathrm{ab}}$, and set $f = y^{-\epsilon} g x^\delta$. Then

$$\begin{cases} \sigma(x) = x^\alpha \\ \sigma(y) = f^{-1} y^\beta f \end{cases}$$

is the unique lifting of the outer action of σ such that $f \in \widehat{F}_2'$.

We have obtained a map

$$G_{\mathbb{Q}} \to \widehat{\mathbb{Z}}^* \times \widehat{F}_2'.$$

This map is NOT a group homomorphism. It corresponds to associating to $\sigma \in G_{\mathbb{Q}}$ the automorphism $F_\sigma \in \mathrm{Aut}(\widehat{F}_2)$ associated to the pair $(\lambda_\sigma, f_\sigma)$ such that

$$\begin{cases} x \mapsto x^{\lambda_\sigma} \\ y \mapsto f_\sigma^{-1} y^{\lambda_\sigma} f_\sigma. \end{cases}$$

If $\sigma, \tau \in G_{\mathbb{Q}}$, the product $\sigma \cdot \tau$ corresponds to applying first the automorphism τ, then σ, so we get

$$x \overset{\tau}{\mapsto} x^{\lambda_\tau} \overset{\sigma}{\mapsto} x^{\lambda_\sigma \lambda_\tau}$$

$$y \overset{\tau}{\mapsto} f_\tau^{-1} y^{\lambda_\tau} f_\tau \overset{\sigma}{\mapsto} F_\sigma(f_\tau)^{-1} f_\sigma^{-1} y^{\lambda_\sigma \lambda_\tau} f_\sigma F_\sigma(f_\tau).$$

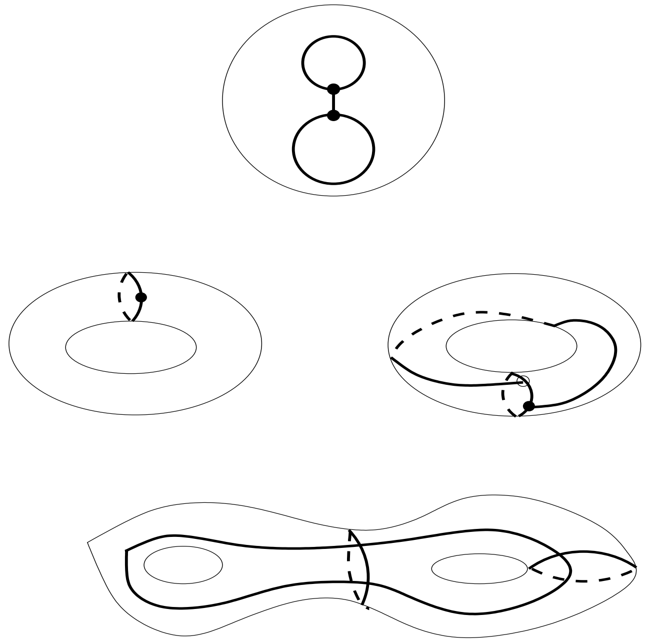
In other words, the pair corresponding to $\sigma \cdot \tau$ is

$$\big(\lambda_\sigma \lambda_\tau, f_\sigma F_\sigma(f_\tau)\big).$$

§3. Dessins d'enfants

Definition. A *dessin d'enfant* is a triple $X_0 \subset X_1 \subset X_2$ where X_0 is a finite set of points on a compact topological surface X_2 of genus g, and X_1 is a subset of X_2 such that $X_2 \setminus X_1$ is a disjoint union of open cells (simply connected regions) of X_2. The set X_1 consists of edges connecting the vertices.

The dessin is only defined up to isotopy on the surface, and we also require it to be *bicolorable*, i.e. we want to be able to color the vertices in two colors, black and white, in such a way that all neighbors of every vertex of a given color are of the opposite color.



WHICH ONES ARE DESSINS?

We have bijections between the following sets

- $\{$dessins d'enfant$\}$

- $\{$finite covers of $\mathbb{P}^1$ unramified outside $\{0, 1, \infty\}$ $\}$,

  known as *Belyi covers*

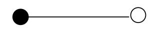
- $\{$finite etale covers of $\mathbb{P}^1 - \{0, 1, \infty\}$ $\}$

- $\{$conj. classes of subgroups of finite index of $\widehat{F_2}\}$

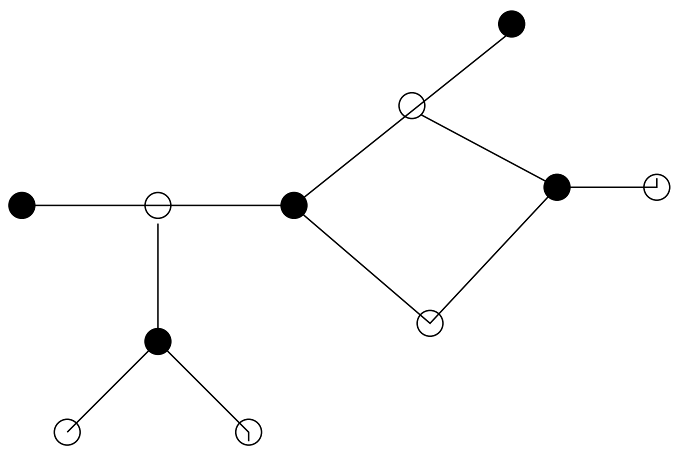
The first bijection is given by associating to a Belyi cover

$$\beta : X \to \mathbb{P}^1$$

the preimage $\beta^{-1}([0,1])$ of the segment $[0,1]$ in $\mathbb{P}^1$ (automatically bicolorable). The second and third bijections are basic facts about Riemann surfaces and topological covers.

The degree of the cover is equal to the number of edges  of the dessin.

The points over 0 correspond to black vertices of the dessin, the points over 1 to white vertices.



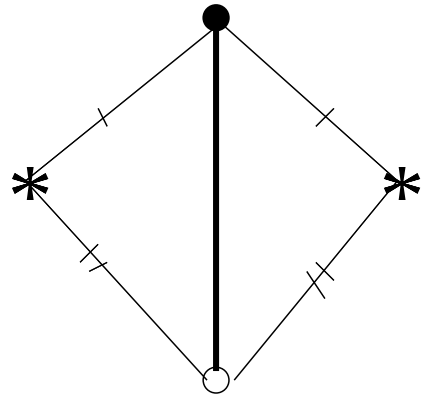
Example. Genus=0, Degree = 11

5 preimages of 0, 6 preimages of 1

2 preimages of ∞

You can visualize the cover topologically by triangulating the dessin (adding a vertex marked $\star$ in each face, and adding edges joining it up to the black and white vertices).

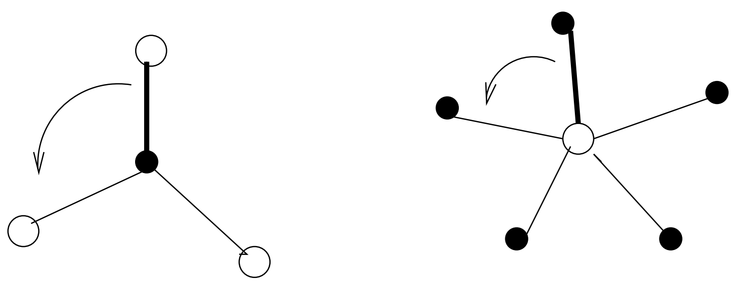
This paves the dessin surface with diamonds



each of which contains exactly one edge of the actual dessin.

The cover identifies the marked pairs of edges, so the quotient is a sphere with three branch points.

The group F_2 acts on the set of edges of the dessin D as follows:



Pick any edge e of the dessin and let $N = \mathrm{Stab}(e)$; then N is a finite-index subgroup of $\widehat{F}_2$. The stabilizers of the different flags from a conjugacy class of finite-index subgroups in $\widehat{F}_2$, and this conjugacy class corresponds to a finite cover of $\mathbb{P}^1$, namely exactly the Belyi cover $\beta : X \to \mathbb{P}^1$.

The degree of the cover is the number of edges e, and the set of edges is in bijection with the coset space $\widehat{F}_2/N$; furthermore the action of $\widehat{F}_2$ on the edges is exactly the action on $\widehat{F}_2/N$ by right multiplication. Obviously, $\widehat{F}_2$ acts via a finite quotient, called the *monodromy group* of the dessin or the cover.

You can reconstruct the whole dessin just by knowing N (up to conjugacy):

• Edges are in bijection with $\widehat{F}_2/N$;

• orbits of $\widehat{F}_2/N$ under x are sets of stars centered around black vertices (edges attached to same black vertex);

• similarly, orbits of $\widehat{F}_2/N$ under y are sets of stars centered around white vertices.

Galois action on dessins

The action of $G_{\mathbb{Q}}$ on $\widehat{F}_2$ sends N to N^σ, so it sends the dessin D to a dessin D^σ. The field

$$K_D = \text{fixed field of } \{\sigma \in G_{\mathbb{Q}} \mid N^\sigma = N, \text{ i.e. } D^\sigma = D\}$$

is called *the moduli field of D.*

Thus, each dessin is naturally defined over a number field, and the set of dessins is naturally equipped with a Galois action.

Now, what we would like is to give a list of **combinatorial Galois invariants** of dessins, the dream being to give a list sufficient to determine Galois orbits of dessins. To start with, there are some obvious Galois invariants:

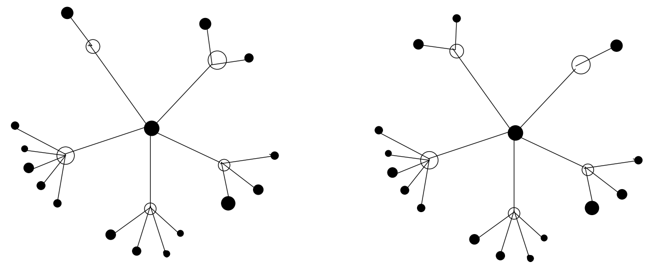
• number of edges, faces, black, white vertices

• ramification indices, i.e. valencies of black and white vertices;

• monodromy group...

All these are *geometric*, i.e. they have to do with the ramification information of the associated Belyi cover.

Example:



Every one of the preceding, geometric invariants of these two dessins is equal. There are 24 dessins having the same valency lists. However, it is actually possible to EXPLICITLY COMPUTE the associated number fields and see that these two dessins are NOT Galois conjugates.

The valencies at the black vertices are $(5, 1, \cdots, 1)$ and at the white vertices $(2, 3, 4, 5, 6)$. If you take dessins with the same black valencies and various 5-tuples of white valencies, you sometimes get a Galois orbit of 24 and sometimes two Galois orbits of 12, as here.

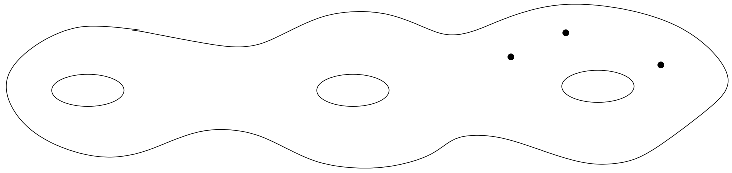
Y. Kochetkov computed many examples and noticed that the Galois orbit appeared to split exactly when the white valencies are (a, b, c, d, e) such that

$$abcde(a+b+c+d+e) \text{ is a square.}$$

This conjecture was generalized and proved by Leonardo Zapponi (1997), who actually came up with a NEW GALOIS INVARIANT – arithmetic, not geometric – for a large family of dessins.

§4. Diffeomorphisms of topological surfaces

Now, let S be a topological surface of genus g, with n distinct ordered marked points $(x_1, \ldots, x_n)$.



Let $M_{g,n}$ denote the *moduli space of Riemann surfaces of type* (g, n). The points of $M_{g,n}$ are isomorphism classes of these Riemann surfaces; it can also be considered as the space of analytic structures on S up to isomorphism.

In the case of *genus zero*, we are working with Riemann spheres marked with n distinct ordered points $(x_1, \dots, x_n)$, up to isomorphism. The isomorphisms are $\mathrm{PSL}_2(\mathbb{C})$, which is triply transitive; this means that we can always find a unique representative

$$(0, 1, \infty, y_1, \dots, y_{n-3})$$

for each isomorphism class (=point of $M_{0,n}$), or in other words, a unique element $\gamma \in \mathrm{PSL}_2(\mathbb{C})$ such that $\gamma(x_1) = 0$, $\gamma(x_2) = 1$, $\gamma(x_3) = \infty$.

Thus, the space $M_{0,n}$ is isomorphic to

$$(\mathbb{P}^1 - \{0, 1, \infty\})^{n-3} - \Delta,$$

where Δ denotes the union of the lines $x_i = x_j$.

PATHS on moduli space are thus continuous parametrized deformations of the analytic structure of the starting point x (a given Riemann surface).

In particular, LOOPS (up to homotopy) are exactly (orientation preserving) diffeomorphisms of x (up to those homotopic to the identity).

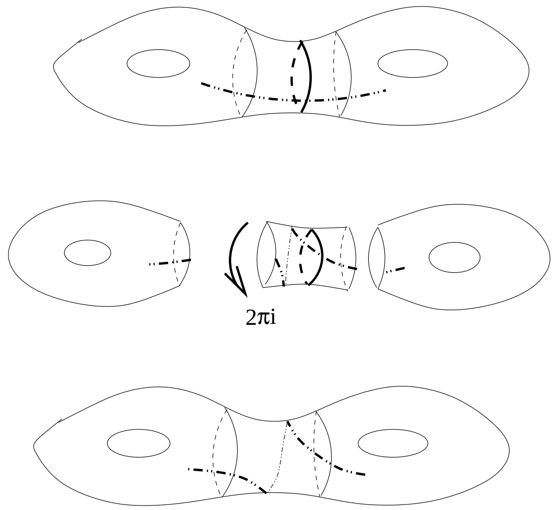
This means that if we define the *mapping class group* to be

$$\Gamma_{g,n} = \mathrm{Diff}^+(S)/\mathrm{Diff}^0(S)$$

and fix a base point $x \in M_{g,n}$, we have an isomorphism

$$\Gamma_{g,n} \simeq \pi_1(M_{g,n}, x) \simeq \mathrm{Diff}^+(S)/\mathrm{Diff}^0(S).$$

The group $\Gamma_{g,n}$ is generated by a certain set of diffeomorphisms called **Dehn twists** along simple closed loops.



Dehn twists correspond to certain particularly well-understood loops in the fundamental group, corresponding to classical inertia generators (see next talk).

§5. The Grothendieck-Teichmüller group

Recall we have an injective set map

$$G_{\mathbb{Q}} \to \widehat{\mathbb{Z}}^* \times \widehat{F}_2'$$

$$\sigma \mapsto (\chi(\sigma), f)$$

(put back pages 9-10). We can view this with moduli spaces now.

As we saw (page 26), the moduli space $M_{0,4}$ is isomorphic to $\mathbb{P}^1 - \{0, 1, \infty\}$, since is the moduli space of Riemann spheres with 4 ordered marked points, and each isomorphism class of such spheres has a unique representative with marked points

$$(x_1, x_2, x_3, x_4) = (0, 1, \infty, x).$$

There are three basic loops, one around x_1 and x_2, one around x_2 and x_3 and one around x_1 and x_3.

The fundamental group $\pi_1(M_{0,4})$ is just F_2, the free group on two generators. The three Dehn twists along the three loops above are the generators x, y, z with $xyz = 1$.

Notation: For any group homomorphism

$$\widehat{F}_2 \to G$$

$$x, y \mapsto a, b$$

we write $f(a, b)$ for the image of $f \in \widehat{F}_2$.

For example:

- under $\mathrm{id} : \widehat{F}_2 \to \widehat{F}_2$, we have $f = f(x, y)$;
- under the map $\widehat{F}_2 \to \widehat{F}_2$ exchanging the generators x and y, we have

$$f = f(x, y) \mapsto f(y, x).$$

Definition. The *Grothendieck-Teichmüller group* $\widehat{GT}$ is the group of pairs $(\lambda, f) \in \widehat{\mathbb{Z}}^* \times \widehat{F}_2'$ such that $x \mapsto x^\lambda$ and $y \mapsto f^{-1} y^\lambda f$ induces an automorphism of $\widehat{F}_2$, and such that

(I) $f(x, y)f(y, x) = 1$,

(II) $f(x, y)x^m f(z, x)z^m f(y, z)y^m = 1$ where $xyz = 1$ and $m = (\lambda - 1)/2$,

(III) (5-cycle relation) $f(x_{34}, x_{45})f(x_{51}, x_{12})f(x_{23}, x_{34})f(x_{45}, x_{51})f(x_{12}, x_{23}) = 1$ in $\widehat{\Gamma}_{0,5}$, where x_{ij} is the Dehn twist along a loop (on a sphere with 5 numbered marked points) surrounding points i and j.

This definition clearly shows that $G_{\mathbb{Q}} \to \widehat{GT}$, and injectivity is easy.

The defining relations (I), (II) and (III) are exactly what is needed in order to ensure that we have homomorphisms

$$\widehat{GT} \to \mathrm{Out}(\widehat{\Gamma}_{0,n})$$

for $n = 4, 5$ which extend the homomorphisms of $G_{\mathbb{Q}}$. But we have more.

Theorem. (D, I-M) For all $n \geq 4$, there is a homomorphism $\widehat{GT} \to \mathrm{Out}(\widehat{\Gamma}_{0,n})$ extending the action of $G_{\mathbb{Q}}$ on these fundamental groups.

The Teichmüller tower

One can go further by identifying $\widehat{GT}$ with a specific automorphism group of mapping class group structures in genus zero.

Consider two specific types of natural morphisms between moduli spaces coming from topological operations on surfaces: (i) erasing marked points, (ii) subsurface inclusion (cutting out subsurfaces by disjoint simple closed loops). These give two natural types of homomorphisms between the corresponding mapping class groups.

First,

$$\eta : \Gamma_{0,n} \to \Gamma_{0,n-1}$$

corresponds to erasing one point (pulling one strand out of a braid). Then, if a sphere with n marked points (or boundary components) is cut out of a sphere with n' marked points (or boundary components), then we have the homomorphism

$$\eta : \Gamma_{0,n} \to \Gamma_{0,n'}$$

where every Dehn twist along a simple closed loop of S is mapped to the Dehn twist along the same simple closed loop of S'.

Definition. The genus zero *Teichmüller tower* is the collection of the profinite genus zero mapping class groups $\widehat{\Gamma}_{0,n}$ linked by all the above homomorphisms.

Theorem. *$\widehat{GT}$ is the inertia-preserving automorphism group of the Teichmüller tower. Namely, if Φ is a tuple $(\phi_n)_{n\geq 4}$ where each $\phi_n \in \mathrm{Out}(\widehat{\Gamma}_{0,n})$ preserves conjugacy classes of*

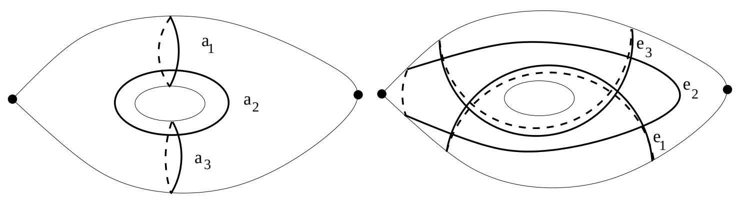
Dehn twists (inertia) and the diagrams

$$\begin{tikzcd} \widehat{\Gamma}_{0,n} \arrow[r, "\eta"] \arrow[d, "\phi_n"'] & \widehat{\Gamma}_{0,m} \arrow[d, "\phi_m"] \\ \widehat{\Gamma}_{0,n} \arrow[r, "\eta"] & \widehat{\Gamma}_{0,m}, \end{tikzcd}$$

commute (up to inners) for all homomorphisms η of the tower, then Φ is an element of $\widehat{GT}$.

§6. Higher genus and the two-level principle

The above theorem shows that it is enough to ensure that $\widehat{GT}$ acts as automorphisms of $\widehat{\Gamma}_{0,4}$ and $\widehat{\Gamma}_{0,5}$ in order to get an action on all of the $\widehat{\Gamma}_{0,n}$. This corresponds to the "two-level" principle stated by Grothendieck, that the action on all mapping class groups of higher dimension ($= 3g - 3 + n$, dimension of the moduli space) should be completely determined by the action on those of dimension 1 and 2, namely $\widehat{\Gamma}_{0,4}$, $\widehat{\Gamma}_{0,5}$, $\widehat{\Gamma}_{1,1}$, $\widehat{\Gamma}_{1,2}$.



We define a subgroup Λ of $\widehat{GT}$ by adding one relation; we assume here for simplicity that $\lambda = 1$, and require f to satisfy

$$f(e_3, a_1)f(a_2^2, a_3^2)f(e_2, e_3)f(e_1, e_2)f(a_1^2, a_2^2)f(a_3, e_1) = 1$$

in $\widehat{\Gamma}_{1,2}$, where a_i and e_i are twists along the loops in the figure above.

Λ has the property that there is a homomorphism $\Lambda \to \mathrm{Out}(\widehat{\Gamma}_{1,2})$ extending the canonical homomorphism of $G_{\mathbb{Q}^{\mathrm{ab}}}$ (can also do $G_{\mathbb{Q}}$ with a more complicated relation).

Grothendieck's two-level principle turns out to be right! Namely, we also obtain:

Theorem. *There is a homomorphism*

$$\Lambda \to \mathrm{Out}(\widehat{\Gamma}_{g,n})$$

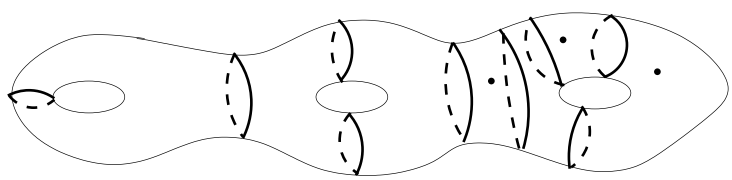
extending the canonical Galois ($G_{\mathbb{Q}^{\mathrm{ab}}}$) homomorphism for all (g,n).

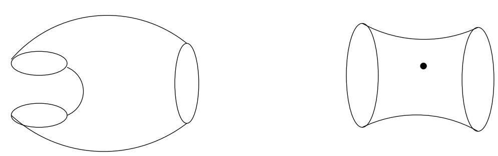
§7. The lego

Grothendieck justified the two-level principle by saying that the expression of the action of an element of $G_{\mathbb{Q}}$ (or $\widehat{GT}$) on any Dehn twist in any $\widehat{\Gamma}_{g,n}$ should be given by a 'game of lego', fitting together the action on the Dehn twists in dimension 1, i.e. in the groups $\widehat{\Gamma}_{0,4}$ and $\widehat{\Gamma}_{1,1}$.

He was right! The game of lego is now fully understood and works as follows.

Let a **pants decomposition** on S be a maximal set of $3g-3+n$ disjoint simple closed loops; they cut S into "pants".





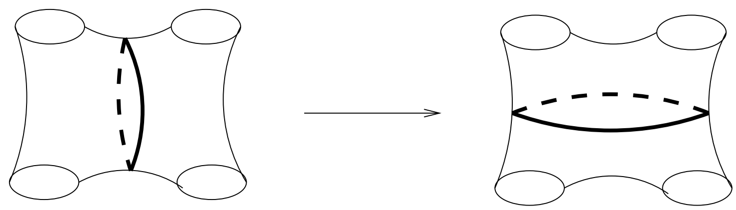
If we erase any one of these loops, then the pants decomposition becomes a decomposition into many pairs of pants and one larger piece, which is always

• either a genus zero piece with four boundary components

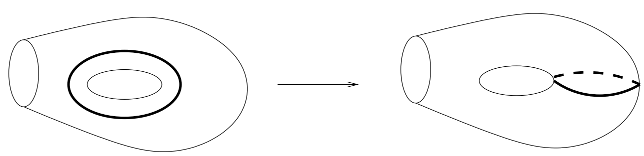
• or a genus one piece with one boundary component.

We call this piece the *neighborhood of the loop in the pants decomposition.*

• An A-move on a pants decomposition P is a new pants decomposition obtained from P by erasing one loop and replacing it by another one which intersects the first one in 2 points.



• An S-move on a pants decomposition P is a new pants decomposition obtained from P by erasing one loop and replacing it by another one which intersects the first one in 1 point.



Theorem. *Let S be a topological surface of type (g,n) and let P be a pants decomposition on S. Then there exists an injective homomorphism*

$$G_{\mathbb{Q}} \to \mathrm{Aut}_P(\widehat{\Gamma}_{g,n})$$

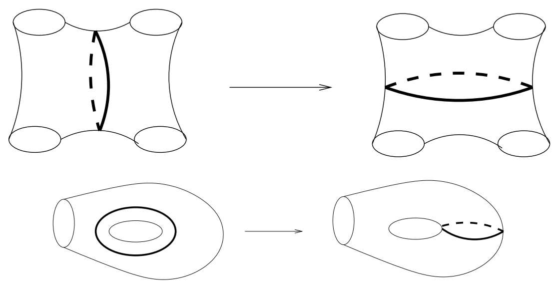
lifting the canonical homomorphism $G_{\mathbb{Q}} \to \mathrm{Out}(\widehat{\Gamma}_{g,n})$, such that:

(i) $\sigma(\tau_a) = \tau_a^{\lambda}$ if $a \in P$;

(ii) $\sigma(\tau_b) = f(\tau_a, \tau_b)^{-1}\tau_b^{\lambda} f(\tau_a, \tau_b)$ *if* $a \to b$ *is an A-move on* P*;*

(iii) $\sigma(\tau_c) = f(\tau_a^2, \tau_c^2)^{-1}\tau_c^{\lambda} f(\tau_a^2, \tau_c^2)$ *if* $a \to c$ *is an S-move on* P*. This homomorphism (with* $\lambda = 1$*) extends to a homomorphism* $\Lambda \to \mathrm{Out}(\widehat{\Gamma}_{g,n})$*.*

This means that in acting on a Dehn twist τ_a along a loop a, Galois not only conjugates it (we knew that – it's inertia!), but it conjugates it by a *local* element of $\widehat{\Gamma}_{g,n}$, i.e. a profinite product of Dehn twists living right on the neighborhood of the loop a!



We say the Galois action on Dehn twists is *local.* This is what Grothendieck called a game of "Lego-Teichmüller" (see next lecture).