Lecture II:

Curve Complexes,

Tensor Categories,

Fundamental Groupoids

Goal

The aim of this talk is to relate the concepts of:

- divisor at infinity on the moduli space of curves
- fundamental group(oid) of the moduli space
- tensor category
- complexes of curves (built from simple closed loops)
- Grothendieck-Teichmüller theory, Dehn twists

Outline

§1. Infinity of the moduli space of curves

§2. The braided tree tensor category ...

§3. ... and the fundamental groupoid of the genus 0 moduli space

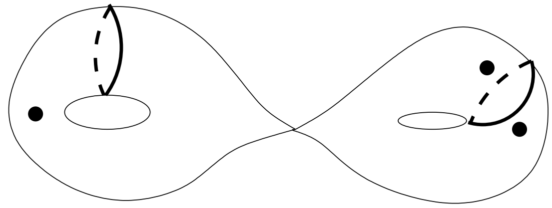
§4. Higher genus: the Hatcher-Thurston curve complex

§5. The role of HT in Grothendieck-Teichmüller theory

§1. Infinity of the moduli space of curves

Pinching simple closed loops

The moduli space $M_{g,n}$ of Riemann surfaces of genus g with n ordered marked points can be compactified by adding all *stable curves* (curves with nodes); these can be considered as Riemann surfaces of type (g,n) equipped with disjoint geodesic simple closed loops of length 0.



The divisor at infinity of $M_{g,n}$

The "infinite part" or divisor at infinity $\overline{M}_{g,n} - M_{g,n}$ consists of strata of decreasing dimension; each simple closed loop on the topological surface (up to action of the mapping class group) corresponds to a stratum of codimension 1, each pair of disjoint loops to an intersection of two of these, which is a stratum of codimension 2, and so on. A pants decomposition corresponds to a point on the divisor at infinity.

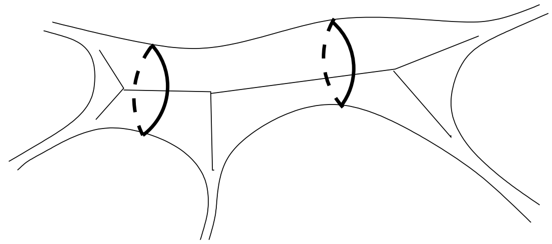
Dehn twists

The mapping class group $\Gamma_{g,n}$ can be identified with the fundamental group of $M_{g,n}$. A *Dehn twist* along a simple closed loop c, in this fundamental group, corresponds to a loop around the corresponding codimension 1 strata at infinity.

This shows that Dehn twists are *inertia elements* in the fundamental group, which explains the result that $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the conjugacy class of a Dehn twist by raising it to the $\chi(\sigma)$-th power.

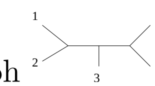
Tangential base points

Equipping a topological surface of type (g, n) with disjoint simple closed loops is equivalent to giving a graph "inside it" with an inner edge corresponding to each loop and a tail to each marked point.



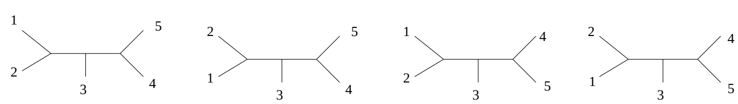
Trivalent trees correspond to pants decompositions.

Consider, in genus zero, the real part of the moduli space $M_{0,n}(\mathbb{R})$ of Riemann spheres with n marked points in $\mathbb{R}$. Then it is a useful fact that if P denotes a point of maximal degeneration in $\overline{M}_{0,n}$ and V_P denotes a neighborhood of this point, the intersection $V_P \cap M_{0,n}$ falls naturally into 2^{n-3} disjoint simply connected regions corresponding to the different possible cyclic orders of the real marked points. Being simply connected, these regions can be used as base points for a fundamental groupoid; they are called *tangential base points of maximal degeneration.*

Example: $n = 5$, point of maximal degeneration corresponding to the graph 

, i.e. to the pants decomposition with a loop around 1 and 2, and a loop around 4 and 5. The possible cyclic orders corresponding to this graph are $(1,2,3,4,5)$, $(2,1,3,4,5)$, $(1,2,3,5,4)$, $(2,1,3,5,4)$.

In other words, *the neighborhood in $M_{0,n}$ of a point of maximal degeneration consists of 2^{n-3} simply connected regions corresponding to planar embeddings of the trivalent graph (up to turning them over in the plane).*



§2. The braided tree tensor category and the fundamental groupoid of $M_{0,n}$

A *tensor category* is a category of objects V and morphisms $f : V \to W$, equipped with a tensor product $V \otimes W$ satisfying suitable functorial conditions with respect to the morphisms, and a set of morphisms called *associativity constraints*

$$a_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$$

satisfying Mac Lane's pentagon relation:

$$\begin{array}{ccc}
((U \otimes V) \otimes W) \otimes X & \xrightarrow{a_{U,V,W}} & (U \otimes (V \otimes W)) \otimes X \\
\Big\downarrow {\scriptstyle a_{U \otimes V,W,X}} & & \Big\downarrow {\scriptstyle a_{U,V \otimes W,X}} \\
(U \otimes V) \otimes (W \otimes X) & & U \otimes ((V \otimes W) \otimes X) \\
\Big\downarrow {\scriptstyle a_{U,V,W \otimes X}} & \swarrow {\scriptstyle a_{V,W,X}} & \\
U \otimes (V \otimes (W \otimes X)) & &
\end{array}$$

A *braided tensor category* is a tensor category further equipped with morphisms called *commutativity constraints*

$$c_{U,V} : U \otimes V \to V \otimes V$$

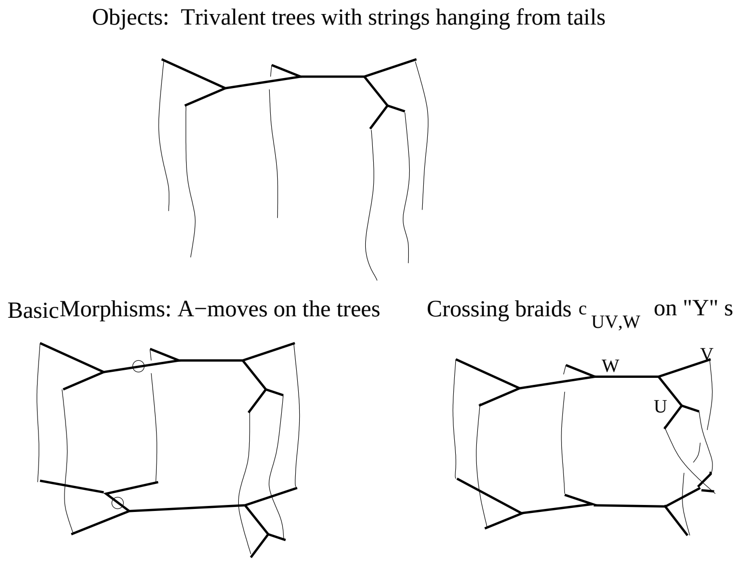
satisfying Mac Lane's two hexagons:

$$\begin{array}{ccccc}
(U \otimes V) \otimes W & \xrightarrow{a_{U,V,W}} & U \otimes (V \otimes W) & \xrightarrow{c_{U,V \otimes W}} & (V \otimes W) \otimes U \\
\Big\downarrow {\scriptstyle c_{U,V}} & & & & \Big\downarrow {\scriptstyle a_{V,W,U}} \\
(V \otimes U) \otimes W & \xrightarrow{a_{V,U,W}} & V \otimes (U \otimes W) & \xrightarrow{c_{U,W}} & V \otimes (W \otimes U)
\end{array}$$

and

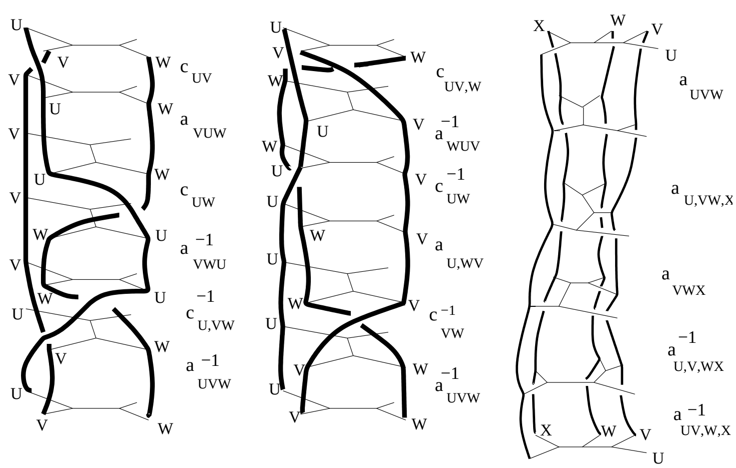
$$\begin{array}{ccccc}
(U \otimes V) \otimes W & \xrightarrow{c_{U \otimes V,W}} & W \otimes (U \otimes V) & \xrightarrow{a^{-1}_{W,U,V}} & (W \otimes U) \otimes V \\
\Big\uparrow {\scriptstyle a^{-1}_{U,V,W}} & & & & \Big\uparrow {\scriptstyle c_{U,W}} \\
U \otimes (V \otimes W) & \xrightarrow{c_{V,W}} & U \otimes (W \otimes V) & \xrightarrow{a^{-1}_{U,W,V}} & (U \otimes W) \otimes V
\end{array}$$

Objects and morphisms of the braided tree category:



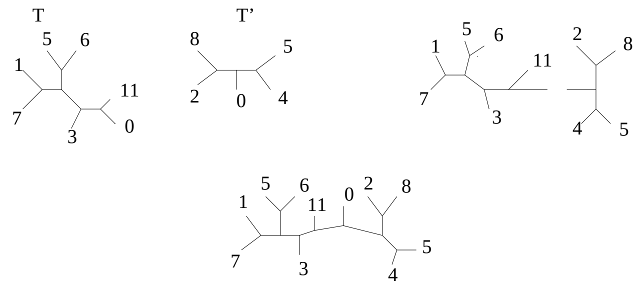
The trivalent trees are numbered as follows: every tail is labeled with a positive integer except for one distinguished tail labeled 0.

Faces: Mac Lane's two hexagons and pentagon drawn as series of morphisms in the braided tree category:



The tensor product

Tensor product: Two trees T and T' give a new tree $T \otimes T'$ by joining the tails numbered 0 and adding a new tail numbered 0.



This category has two basic functions:

1) For each $n \geq 4$, the full subcategory of objects (trees) which are n-tailed trees labeled with the integers $0, \ldots, n-1$ is naturally isomorphic to the *fundamental groupoid of the moduli space $M_{0,n}$ based at tangential base points at infinity.*

2) The braided tree category is a free tensor category whose automorphism group is $\widehat{GT}$. This was proved by Drinfel'd in a similar situation: he showed that for a field k, the k-pro-unipotent group $GT(k)$ defined by the same relations as $\widehat{GT}$ was the fundamental group of a free Tannakian category of vector spaces.

Here **free** means that the only relations between morphisms are those coming from Mac Lane's hexagons and pentagon.

§**3. ... and the fundamental groupoid of** $M_{0,n}$

Let us now explain how the n-tailed part of the braided tree category describes the fundamental groupoid of the genus zero moduli space.

• **Objects:** Identify a trivalent tree graph numbered from 0 to $n-1$ with a point of maximal degeneration on $\overline{M}_{0,n}$, i.e. a genus zero Riemann surface with n marked points numbered 1 to n (renumber 0 to n) with $n-3$ disjoint simple geodesic loops pinched to length 0 (see page 7).

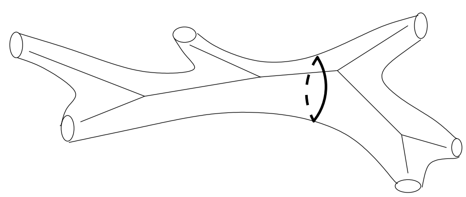
Next, identify each *planar embedding* of the numbered trivalent tree with a tangential base point (see pages 8-9).

• **A-move:** Every point on $M_{0,n}(\mathbb{R})$ has a natural *cyclic order* since the marked points lie on $\mathbb{R}$. Since a tangential base point corresponds to a numbered planar tree, it has a natural cyclic order.

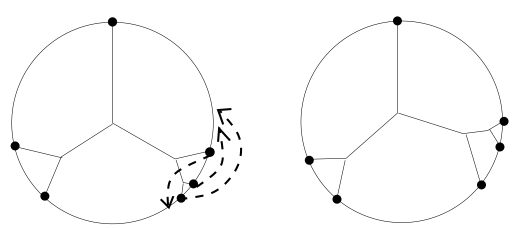
The region described in moduli space by a *fixed cyclic order* is clearly simply connected (all ways of sliding the points along the real axis without letting them cross are homotopic).

An A-move takes a planar tree T to another T' with the same cyclic order. Therefore, the tangential base points B_T and $B_{T'}$ associated to these trees both lie in the simply connected region associated to their common cyclic order. We interpret the A-move from T to T' as the unique (up to homotopy) path on $M_{0,n}$ from B_T to $B_{T'}$ lying in this region.

• **Crossing-braid:** We interpret a crossing braid as a path on the moduli space given by a 1/2 Dehn twist, as follows.



Another way to see this is as a movement of points on the sphere, parametrizing a path (up to homotopy) on the moduli space $M_{0,n}$:



The pentagon and hexagon relations interpreted on the moduli space mean that certain loops formed by sequences of commutativity and A (associativity) paths are homotopic to the identity.

This is clear by the figure on page 13. Indeed, these sequences of paths form braids, thus elements of the fundamental group of $M_{0,n}$ (which are basically braid groups), and by inspection the braids in question are trivial braids.

The original definition of $\widehat{GT}$

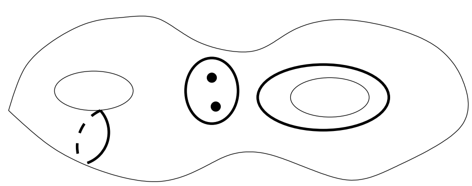
Drinfel'd defined a pro-algebraic (k-pro-unipotent) group $GT(k)$ by taking a braided tensor category defined above, but whose objects were vector spaces over a field k, and asking for all ways in which one can modify the associativity and commutativity constraints while preserving all objects and diagrams. The answer is as follows: one can change

$$\begin{cases} c_{U,V} & \text{to } c_{U,V}\cdot(c_{V,U}c_{U,V})^m \\ a_{U,V,W} & \text{to } a_{U,V,W}\cdot f(c_{V,U}c_{U,V}, c_{W,V}c_{V,W}) \end{cases}$$

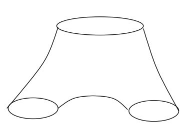
for a certain profinite number m and $f\in\widehat{F}_2'$. Then making sure that these new associativity and commutativity constraints still satisfy the pentagon and hexagon relations yields exactly (I), (II) and (III). One can do this just as well, directly in the profinite situation.

§4. Higher genus: the Hatcher-Thurston curve complex

Curve complexes: basic notions



Simple closed loops on a topological surface

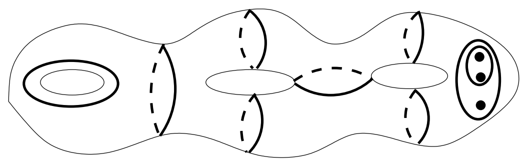


Pants

There are at most 3g−3+n disjoint simple closed loc
on a surface of genus g with n marked points.
They cut the surface into 2g−2+n pairs of pants

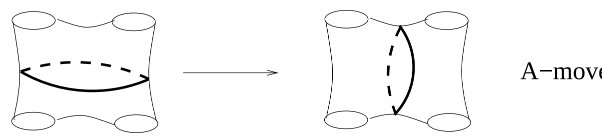
The 2-dimensional Hatcher-Thurston complex is defined as follows.

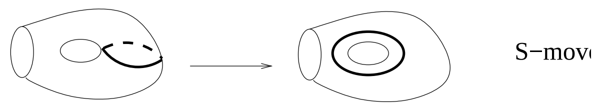
Vertices:



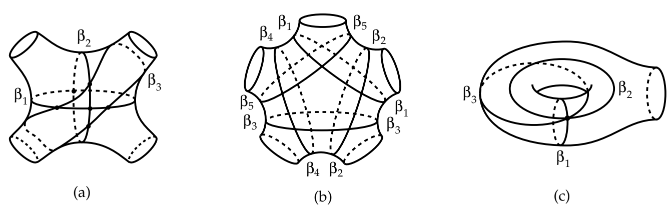
Pants decompositions (here of type (3,3))

Edges:





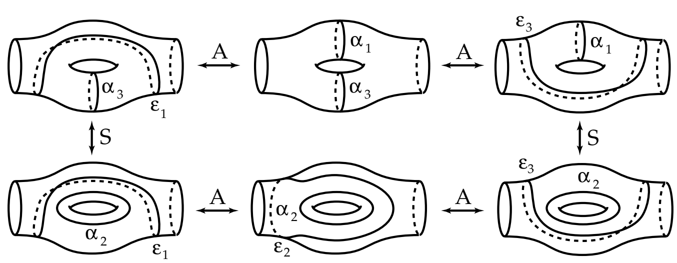
Faces:



(a) A sequence of 3 A-moves forms a triangle

(b) A sequence of 5 A-moves forms a pentagon

(c) A sequence of 3 S-moves forms a triangle



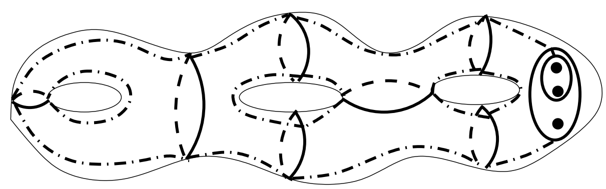
(d) A sequence of 6 mixed A and S-moves forms a hexagon

Theorem. *The Hatcher-Thurston complex of curves is simply connected.*

We will explain the role of this complex in Grothendieck-Teichmüller theory in §5. First let us explain the relation with the fundamental groupoid of $M_{g,n}$. For this, we introduce the *seamed Hatcher-Thurston complex (SHT).*

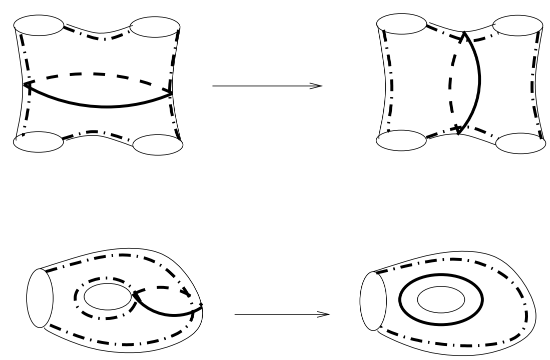
One adds structure to the pants decompositions by adding *seams* as in the following figure. It remains simply connected. There is a natural action of the mapping class group $\Gamma_{g,n}$ on the complexes HT and SHT. Quotienting SHT by $\Gamma_{g,n}$ yields a complex which exactly describes the fundamental groupoid of the moduli space $M_{g,n}$ based at tangential base points at infinity.

Vertices:



Seamed pants decompositions

Edges:



Genus zero: To see this, consider the genus zero situation, and compare SHT with the braided tree complex. By forgetting the seams, an object gives a pants decomposition of a topological sphere with n numbered points, and this pants decomposition is (as we saw) dual to a numbered trivalent graph. Adding the seams (modulo action of $\Gamma_{0,n}$) corresponds exactly to determining an embedding of this tree in the plane, so the seamed pants decompositions correspond to tangential base points. An A-move on a seamed pants decomposition corresponds to an A-move on the tree, and a 1/2-twist corresponds to a crossing braid. So we have the same groupoid as before, which is isomorphic to the fundamental groupoid of $M_{0,n}$.

• The formulation in terms of seamed pants decompositions is better for passing to higher genus, where the mapping class group is no longer a braid group.

• One loses the connection with classically defined braided tensor categories, which are deeply connected to genus zero. A tensor product has not been defined on the seamed pants decompositions; this would give an interesting generalization of braided tensor categories.

• The non-seamed complex HT is obtained from the seamed complex by forgetting the seams. It is enough for the proof of the $\widehat{GT}$ theorem.

§5.The role of HT in Grothendieck-Teichmüller theory

Recall that $\widehat{GT}$ was defined by modifying the associativity and commutativity constraints of a braided tensor category via

$$\begin{cases} c_{U,V} \mapsto c_{U,V} \cdot (c_{V,U} c_{U,V})^m \\ a_{U,V,W} \mapsto a \cdot f(c_{V,U} c_{U,V}, C_{W,V} c_{V,W}), \end{cases}$$

and that its defining relations (I), (II) and (III) came from requiring the new morphisms (i.e. m and f) to respect the pentagon and hexagon diagrams of the braided tensor category.

Recall from last time that we defined a subgroup Λ of $\widehat{GT}$ (with $\lambda = 1$) by four relations:

(I) $f(x,y)f(y,x) = 1$,

(II) $f(x,y)f(z,x)f(y,z) = 1$ where $xyz = 1$ (for simplicity we are restricting here to $\lambda = 1$),

(III) (5-cycle relation) in $\widehat{\Gamma}_{0,5}$

$$f(x_{34},x_{45})f(x_{51},x_{12})f(x_{23},x_{34})f(x_{45},x_{51})f(x_{12},x_{23})=1$$

(IV) (new 6-cycle relation) in $\widehat{\Gamma}_{1,2}$

$$f(e_3,a_1)f(a_2^2,a_3^2)f(e_2,e_3)f(e_1,e_2)f(a_1^2,a_2^2)f(a_3,e_1)=1$$

These four relations come from modifying the morphisms of the HT complex in a manner which generalizes the formulae for $\widehat{GT}$ above. Explicitly, if a denotes a simple closed loop in a pants decomposition P, and $A_{a,b}$ is an A-move on P (resp. $S_{a,b}$ is an S-move on P), we have the formulae

$$\begin{cases} A_{a,b} \mapsto A_{a,b} \cdot f(\tau_a,\tau_b) \\ S_{a,b} \mapsto S_{a,b} \cdot f(\tau_a^2,\tau_b^2) \end{cases}$$

for an element $f \in \widehat{F}_2'$, and requiring the image A and S-moves to satisfy the same four types of relations (faces) is equivalent to requiring f to satisfy relations (I), (II), (III), (IV).

Note: One can also work in SHT and modify the 1/2-twist morphism by

$$c_a \mapsto c_a\tau_a^m,$$

completing the analogy with $\widehat{GT}$.

Setting $\lambda = 2m+1$ and requiring (λ, f) to satisfy the defining relations of SHT then yields somewhat more complicated relations (II) and (IV) defining a subgroup of $\widehat{GT}$ which contains all of $G_{\mathbb{Q}}$.

We stick with HT here for simplicity, in order to concentrate on the basic idea of simple connectedness and avoid all the little factors that show up in the general case.

We saw the following "Lego" or "local action" theorem last time:

Theorem. *Let S be a topological surface of type (g,n) and let P be a pants decomposition on S. Then there exists an injective homomorphism*

$$\eta_P : \Lambda \to \mathrm{Aut}_P(\widehat{\Gamma}_{g,n})$$

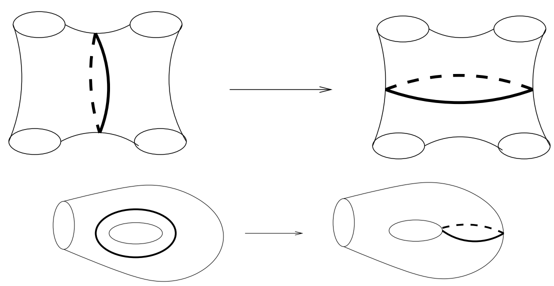
extending and lifting the canonical homomorphism $G_{\mathbb{Q}} \to \mathrm{Out}(\widehat{\Gamma}_{g,n})$*, such that:*

(i) $\sigma(\tau_a) = \tau_a$ *if* $a \in P$*;*

(ii) $\sigma(\tau_b) = f(\tau_a, \tau_b)^{-1}\tau_b f(\tau_a, \tau_b)$ *if* $a \to b$ *is an A-move on* P*;*

(iii) $\sigma(\tau_c) = f(\tau_a^2, \tau_c^2)^{-1}\tau_c f(\tau_a^2, \tau_c^2)$ *if* $a \to c$ *is an S-move on* P*.*

If a is a loop in P and we make an A-move on P taking a to a loop a', then a' cannot 'bump into' any other loops of P and thus it lives on a neighborhood of a of type equal to either $(0,4)$ or $(1,1)$.



Let us show how the Hatcher-Thurston complex is essential in proving the theorem. Fix a pants decomposition P. We will define the Λ-action on any Dehn twist, starting from the fact that it acts trivially on the loops of P. So take any simple closed loop c, and let τ_c be the Dehn twist along it.

The theorem states that if P' is a pants decomposition differing from P by a single A-move, then the homomorphisms

$$\eta_P, \eta_{P'} : \Lambda \to \mathrm{Out}(\widehat{\Gamma}_{g,n})$$

are related by

$$\eta_{P'} = \mathrm{inn}(f(\tau_a, \tau_b)) \circ \eta_P,$$

and if P' is obtained from P by a single S-move, they are related by

$$\eta_{P'} = \mathrm{inn}(f(\tau_a^2, \tau_b^2)) \circ \eta_P.$$

Thus, for a fixed P, to compute the action of Λ on any Dehn twist τ_c along a loop c, it suffices to fit c into a pants decomposition P' and to go from P to P' by a sequence $M_1 \dots M_n$ of A-moves and S-moves. If M_i takes the loop a_i to b_i, the action of $f \in \Lambda$ on τ_c is then given by

$$\Big(\prod_{i=1}^{n} f(\tau_{a_i}^{\epsilon_i}, \tau_{b_i}^{\epsilon_i})\Big)^{-1}(\tau_c)\Big(\prod_{i=1}^{n} f(\tau_{a_i}^{\epsilon_i}, \tau_{b_i}^{\epsilon_i})\Big) \qquad (*)$$

where $\epsilon_i = 1$ if M_i is an A-move, 2 if M_i is an S-move.

So the only question is whether this action of Λ on all Dehn twists is well-defined, i.e. whether (*) depends on the choice of the sequence of moves $M_1 \cdots M_n$.

The answer is that it does not depend on this choice, precisely thanks to the simple-connectedness of the Hatcher-Thurston complex.

It is enough to consider the case where $M_1 \cdots M_n$ is a sequence taking P to itself with each loop ending up in its original place, i.e. in the case where the sequence $M_1 \cdots M_n$ is homotopic to the identity in the complex. We have to show that for every loop c of P and every $f \in \Lambda$, we have

$$\Big(\prod_{i=1}^{n} f(\tau_{a_i}^{\epsilon_i}, \tau_{b_i}^{\epsilon_i})\Big)^{-1}(\tau_c)\Big(\prod_{i=1}^{n} f(\tau_{a_i}^{\epsilon_i}, \tau_{b_i}^{\epsilon_i})\Big) = \tau_c.$$

The definition of the complex by its four types of faces and the fact that it is simply connected shows that (up to inserting trivial pairs of moves AA^{-1} or SS^{-1}) the sequence $M_1 \cdots M_n$ can be broken up into r subsequences *each of which is one of the four types of basic subsequences homotopic to the identity.*

This means that the product

$$\prod_{i=1}^{n} f(\tau_{a_i}^{\epsilon_i}, \tau_{b_i}^{\epsilon_i})$$

can be broken up into corresponding subproducts.

But each of those subproducts is trivial, because they are exactly the defining relations of Λ! (see page 29)

This proves the theorem.