Lecture III

Five Lie Algebras

Introduction

The aim of this talk is to connect five different Lie algebras which arise naturally in different theories. Various conjectures, put together, state that they are all isomorphic. Given their different sources, this seems very deep and surprising.

• A free Lie algebra (Lie algebra of the fundamental group of the category of mixed Tate motives);

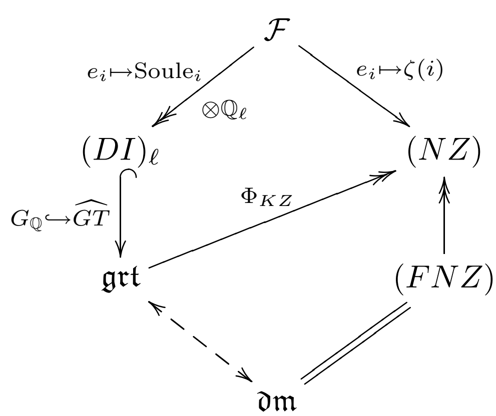
• the Deligne-Ihara Lie algebra defined from the action of $G_{\mathbb{Q}}$ on $\widehat{F}_2$;

• the Lie algebra of new multizeta values

• the double shuffle Lie algebra (formal multizeta values);

• the Grothendieck-Teichmüller Lie algebra

Let $\mathcal{F}$ be the free graded Lie algebra over $\mathbb{Q}$ generated by e_i, $i \geq 3$ odd, graded by giving weight i to e_i. Then the Lie algebras listed above are connected by the following known maps:



The definitions

The Deligne-Ihara Lie algebra. Let $F_2^{(\ell)}(i)$ denote the i-th group in the descending central series of the pro-ℓ completion $F_2^{(\ell)}$ of F_2, defined by

$$F_2^{(\ell)}(1) = F_2^{(\ell)}, \quad F_2^{(\ell)}(i+1) = [F_2^{(\ell)}, F_2^{(\ell)}(i)].$$

Filter the absolute Galois group $G_{\mathbb{Q}}$ by setting

$$G_{\mathbb{Q}}^{(i)} = \mathrm{Ker}\big(G_{\mathbb{Q}} \to \mathrm{Out}(F_2^{(\ell)}/F_2^{(\ell)}(i+1))\big).$$

Consider the associated graded object

$$(DI)_\ell = \oplus_{i\geq 1} G_{\mathbb{Q}}^{(i)}/G_{\mathbb{Q}}^{(i+1)}$$

Then $(DI)_\ell$ is a $\mathbb{Z}_\ell$-Lie algebra (the Lie bracket is induced by the commutators in $G_{\mathbb{Q}}$).

The new zeta Lie algebra. For all tuples of positive integers $(k_1, \ldots, k_r)$, define the *multizeta value*

$$\zeta(k_1, \ldots, k_r) = \sum_{n_1 \rangle \cdots n_r \rangle 0} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}.$$

This converges if $k_1 \rangle 1$. It is easy to see that a product of two multizeta values can be written as a linear combination of multizeta values. Thus, these real numbers generate an algebra over $\mathbb{Q}$. The quotient of this algebra by $\zeta(2)$ and all products of multizeta values is the *vector space of new multizeta values.* The algebra itself is naturally graded by $k_1 + \cdots + k_r$, as is the quotient vector space. It has conjectured to have a natural Lie coalgebra structure.

The formal new multizeta Lie algebra. The multizeta values are known to satisfy two sets of "fundamental quadratic relations" coming from:

(i) multiplying two multizeta values and re-expressing the product as a sum of multizeta values using the series definition

$$\zeta(k_1, \ldots, k_r) = \sum_{n_1 \rangle \cdots n_r \rangle 0} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}$$

as above.

(ii) using the integral identities $\zeta(k_1, \ldots, k_r) =$

$$\int_0^1 \int_0^{t_n} \cdots \int_0^{t_2} \frac{dt_n}{t_n} \wedge \frac{dt_{n-1}}{t_{n-1} - \epsilon_{n-1}} \wedge \cdots \wedge \frac{dt_2}{t_2 - \epsilon_2} \wedge \frac{dt_1}{t_1 - 1}$$

where $n = k_1 + \cdots + k_r$, all the ϵ_i are equal to 0 or 1, and

$$(0, \epsilon_{n-1}, \ldots, \epsilon_2, 1) = (0, \ldots, 0, 1, 0, \ldots, 0, 1, \ldots, 0, \ldots, 0, 1)$$

$$k_1 - 1 \qquad k_2 - 1 \qquad\qquad k_r - 1$$

All other algebraic relations between multizeta values are conjectured to arise from these. Therefore, it is natural to define the *formal multizeta algebra* FZ to be generated by symbols $Z(k_1, \ldots, k_r)$ subject only to the fundamental quadratic relations above. Like the multizeta algebra itself, FZ can be quotiented by products, yielding a formal multizeta Lie algebra FNZ. This is known to be a Lie (co)algebra. Goncharov has given the Lie coproduct explicitly.

The double shuffle algebra. An element P of the non-commutative polynomial ring $\mathbb{Q}\langle x, y\rangle$ belongs to the free Lie algebra $Lie[x, y]$ if and only if it is primitive for the standard coproduct $\Delta : \mathbb{Q}\langle x, y\rangle \to \mathbb{Q}\langle x, y\rangle \otimes \mathbb{Q}\langle x, y\rangle$ is the standard coproduct defined by $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\Delta(y) = y \otimes 1 + 1 \otimes y$, i.e. if and only if

$$\Delta(P) = P \otimes 1 + 1 \otimes P.$$

This holds if and only if the coefficients $(P|w)$ of words w in P satisfy the *shuffle relations*:

$$\sum_{w \in \mathrm{sh}(w_1, w_2)} (P|w) = 0.$$

Let $\pi_y(P)$ be the projection of P onto the words ending in y, and let

$$\tilde{P} = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} (P|x^{n-1}y) y^n + \pi_y(P).$$

This polynomial can be written in the variables $y_i = x^{i-1}y$. Let $\Delta_* : \mathbb{Q}\langle y_1, y_2, \ldots\rangle \to \mathbb{Q}\langle y_1, y_2, \ldots\rangle \otimes \mathbb{Q}\langle y_1, y_2, \ldots\rangle$ be the coproduct defined by $\Delta_*(y_i) = \sum_{k+l=i} y_k \otimes y_l$. Then $\tilde{P}$ is primitive for Δ_* if and only if P satisfies the *stuffle relations*:

$$\sum_{v \in \mathrm{st}(v_1, v_2)} (\tilde{P}|v) = 0,$$

where v_1 and v_2 (and the v) are words in the variables y_i.

Another formulation of the stuffle relations: Let $P \in L[x, y]$ (so P satisfies the shuffle relations), and let $\tilde{P}$ be defined as above.

Set

$$u_n = \sum_{j=1}^{n} \frac{(-1)^{j-1}}{j} \sum_{i_1+\cdots+i_j=n} y_{i_1} \ldots y_{i_j},$$

and let $\tilde{P}_u$ denote the same polynomial again, but written in terms of the u_i.

Then $P \in L[x, y]$ is said to satisfy the *stuffle relations* if $\tilde{P}_u \in Lie[u_1, u_2, \ldots]$.

Let $\mathfrak{dm}$ denote the set of polynomials P satisfying the shuffle and stuffle relations.

Goncharov and more recently Racinet have shown that the Poisson bracket makes this set into a Lie algebra, the *double shuffle Lie algebra* $\mathfrak{dm}$. The Poisson bracket is defined by

$$f, g = [f, g] + D_f(g) - D_g(f)$$

where $[f, g]$ is the Lie bracket in $Lie[x, y]$ and D_f is the derivation of $Lie[x, y]$ associated to any $f \in Lie[x, y]$ by $D_f(x) = 0$, $D_f(y) = [y, f]$.

The Grothendieck-Teichmüller Lie algebra. This Lie algebra, defined by Ihara and also called the stable derivation algebra, is defined to be the set of elements $f \in Lie[x, y]$ satisfying

(I) $f(x, y) + f(y, x) = 0$

(II) $f(x, y) + f(z, x) + f(y, z) = 0$ where $x + y + z = 0$

(III) $\sum_{i \in \mathbb{Z}/5\mathbb{Z}} f(x_{i,i+1}, x_{i+1,i+2}) = 0$

where x_{12}, x_{23}, x_{34}, x_{45}, x_{51} are the usual generators of the Lie algebra associated to $\Gamma_{0,5}$.

Ihara showed that this set forms a Lie algebra under the Poisson bracket.

The known maps between these Lie algebras are shown in the diagram preceding the definitions, coming from various different theories. Not one of them is yet known to be an isomorphism, yet all of them are conjectured to be.