Stacks and homotopy theory

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Torsors

\( S = \) “decent” scheme, ie. noetherian, locally of finite type, ...

\( G = \) group-scheme defined over \( S \), eg. \( \text{Gl}_n \), etc.

\( G \) represents a sheaf of groups \( G = \text{hom}(, G) \) for the standard geometric topologies (eg. Zariski, flat, étale, Nisnevich) on \( \text{Sch}|_S = \) schemes locally of fin. type/\( S \).

\( G \)-torsor: sheaf \( X \) with free \( G \)-action such that \( X/G \to * \) is isomorphism, \( * = \) terminal sheaf.

ie. \( G \) acts freely on \( X \), and there is sheaf epi \( U \to * \) and map \( \sigma: U \to X \) s.t. following dia. is a pullback:

\[
\begin{array}{ccc}
G \times U & \xrightarrow{\sigma_*} & X \\
\downarrow pr & & \downarrow \\
U & \xrightarrow{\sigma} & * \\
\end{array}
\]

\( \sigma_*(g, u) = g\sigma(u) \)
Cocycles

\[ \sigma(u_2) = c(u_1, u_2)\sigma(u_1) \] for uniquely determined \( c(u_1, u_2) \in G \) for all sections \( u_1, u_2 \) of \( U \).

\((u_1, u_2) \mapsto c(u_1, u_2)\) defines a cocycle \( c \), from which the torsor \( X \) can be reassembled up to iso. from a \( G \)-equivariant coequalizer (twist one projection by \( c \))

\[
G \times U \times U \rightrightarrows G \times U \to X
\]

The cocycle is a map of sheaves of groupoids \( c : U_\bullet \to G \) where \( U_\bullet \) = the trivial groupoid arising from the sheaf epi \( U \to * \). It is also a map of simplicial sheaves

\[
c : U_\bullet \to BG
\]

where \( U_\bullet \) = Čech resolution for \( U \to * \).
Classifying objects

\( BC \) denotes the nerve of a category \( C \).

\( BC_n = \) strings \( a_0 \to a_1 \to \cdots \to a_n \) with faces and degeneracies defined by composition, insertion of identities resp.

eg: \( G = \) group is a category with one object \( * \), then \( n \)-simplices of \( BG \) are strings

\[
* \xrightarrow{g_1} * \xrightarrow{g_2} \cdots \xrightarrow{g_n} *
\]

or elements of \( G^{\times n} \).

The construction \( C \mapsto BC \) is functorial and preserves the sheaf condition, so there is a simplicial sheaf \( BG \) associated to a sheaf of groups (or groupoids) \( G \).
Homotopy theory

Simplicial homotopy corr. to isomorphism of $G$-torsors, so we have bijection

$$\pi(U_\bullet, BG) \cong \{\text{iso. classes of } G\text{-torsors, trivial over } U\}$$

Joyal (mid 80s): A map of simp. sheaves $X \to Y$ is a weak equiv. iff it induces a weak equiv. $X_x \to Y_x$ of simp. sets in all stalks. Cofibrations are monomorphisms. There is a model structure, with ass. htpy. category $\text{Ho}(s \text{Shv})$.

$$\left[*, BG\right] \cong \lim_{\text{hypercovers } V \to *} \pi(V, BG)$$

$$\cong \lim_{\text{epi } U \to *} \pi(U_\bullet, BG)$$

$$\cong \{\text{iso. classes of } G\text{-torsors}\} = H^1(S, G)$$
Examples

For the étale topology/\( S \):

1) \([*, BGl_n] \cong \{\text{iso. classes of rank } n \text{ vector bundles}\} \)
2) \([*, BO_n] \cong \{\text{iso. classes of rank } n \text{ sym. bil. forms}\} \)
3) \([*, BPGL_n] \cong \{\text{iso. classes of rank } n^2 \text{ Azumaya algebras}\} \)
\([*, BPGL_n] \to Br(S) = \text{similarity classes.}\)

General fact: \([*, K(A, n)] \cong H^n(S, A)\) for all abelian sheaves \( A \).

\( X = \text{simplicial scheme or sheaf}: \)

\[
H^n(X, A) := [X, K(A, n)]
\]

(cup products, Steenrod operations ...)
Brauer group

There is a fibre seq. $BG_m \to BGl_n \to BPGL_n$

$BG_m$ acts freely on $BGl_n$ with quotient $BPGL_n$, so there is a fibre sequence

$$BGl_n \to EBG_m \times_{BG_m} BGl_n \to BBG_m$$

$BBG_m \simeq K(^G_m, 2)$, Borel const. $\simeq BPGL_n$, so there is an induced map

$$[* , BPGL_n] \to [* , K (^G_m, 2)] (n\text{-torsion})$$

Assembling these gives a monomorphism

$$Br(S) \to H^2(S, ^G_m)_{ tors }.$$ 

NB: no mention of non-abelian $H^2$. 
Effective descent

A stack is a sheaf of groupoids which satisfies the effective descent condition.

**Effective descent datum** \( \{x_\phi\} \) in \( G \) for cov. sieve \( R \subset \text{hom}(U) \)

- objects \( x_\phi \in G(V) \), one for each \( \phi : V \to U \) in \( R \)
- isomorphisms \( c_\psi : \psi^* x_\phi \to x_{\phi \psi} \) for each composable pair \( W \xrightarrow{\psi} V \xrightarrow{\phi} U \)

s.t. \( c_1 = 1 \) and following diagrams commute

\[
\begin{array}{ccc}
\omega^* \psi^* x_\phi & \xrightarrow{\omega^* c_\psi} & \omega^* x_{\phi \psi} \\
\downarrow & & \downarrow c_\omega \\
(\psi \omega)^* x_\phi & \xrightarrow{c_{\psi \omega}} & X_{\phi \psi \omega}
\end{array}
\]

\( W' \xrightarrow{\omega} W \xrightarrow{\psi} V \xrightarrow{\phi} U \)
The effective descent data for $R$ in $G$ are members of a category $\text{hom}(E_R, G)$ with morphisms $\{x_\phi\} \rightarrow \{y_\phi\}$ given by families $x_\phi \rightarrow y_\phi$ in $G(V)$ which are compatible with structure.

There is a functor

$$G(U) \rightarrow \text{hom}(E_R, G)$$

defined by $x \mapsto \{\phi^*x\}$.

A sheaf of groupoids $G$ is a stack if all functors $G(U) \rightarrow \text{hom}(E_R, G)$ are equivalences of groupoids for all covering sieves $R$.
\( \text{hom}(E_R, G) \)

\( R \subseteq \text{hom}(\ , U) \) is a subfunctor, also defines a category \( R \) with objects \( \phi : V \to U \) in \( R \) and comm. triangles for morphisms. There is a functor \( R \to \text{Shv} \) defined by sending \( \phi : V \to U \) to \( V = \text{hom}(\ , V) \).

\( E_R \) is “translation category” defined by this functor in the sheaf category.

\[
E_R : \begin{array}{c}
\begin{array}{c}
W \\
\downarrow \\
W \to V \\
\phi \\
V \\
\downarrow \\
V \phi \\
U
\end{array}
\end{array} \Rightarrow \begin{array}{c}
\begin{array}{c}
W \\
\downarrow \\
W \to V \\
\phi \\
V \\
\downarrow \\
V \phi \\
U
\end{array}
\end{array}
\]

Effective descent datum is functor \( E_R \to G \) (simplicial sheaf map \( BE_R \to BG \)), morphism is natural transformation (homotopy) of such functors.
Stack completion

A sheaf of groupoids $G$ is a stack if and only if all induced maps

$$BG(U) \to \text{hom}(BE_R, BG) = B \text{hom}(E_R, G)$$

are weak equivalences of simplicial sets.

Stack completion:

$$St^p(G)(U) = \lim_{R \in \text{hom}(,U)} \text{hom}(E_R, G)$$

defines a presheaf of groupoids $St^p G$.

The stack completion $St(G)$ of $G$ is the associated sheaf for $St^p(G)$.
More homotopy theory

Joyal-Tierney (mid 80s): There is a homotopy theory for $\text{Shv}(Gpd) = \text{sheaves of groupoids}$, for which a map $f : G \to H$ is a local weak equivalence (resp. fibration) if the induced map $f : BG \to BH$ is a local weak equivalence (resp. fibration) of simplicial sheaves.

Every sheaf of groupoids $G$ has **fibrant model** $j : G \to G^\wedge$ (ie. weak equiv., $BG^\wedge$ fibrant simplicial sheaf).
Some results

**Fact:** Every fibrant groupoid $H$ is a stack.
Proof: $BE_R \to U$ is a local weak equivalence, so $\hom(U, BH) \to \hom(BE_R, BH)$ is a weak equivalence of simplicial sets.

**Fact:** If $G$ is a stack and $G \to G^\wedge$ is a fibrant model, then all $G(U) \to G^\wedge(U)$ are equivalences of groupoids, ie $BG \to BG^\wedge$ is sectionwise weak equivalence.
Proof: There is isomorphism $\pi_0 BG(U) \cong [U, BG]$.

**Fact:** $G \to St(G)$ is a weak equivalence of sheaves of groupoids.
Proof: Descent data lift to $G$ locally.

**Corollary:** The stack completion $St(G)$ and fibrant model $G^\wedge$ coincide up to natural sectionwise equivalence.
Stacks are fibrant groupoids

**Fact:** (old) There is a homotopy theory for $sPre = \text{simplicial presheaves}$ for which weak equivs. are stalkwise weak equivs. and cofibrations are monomorphisms. The ass. sheaf map $X \to \tilde{X}$ is a weak equiv. $Ho(sPre) \simeq Ho(sShv)$.

**Fact:** (Hollander) There is a homotopy theory for $Pre(Gpd) = \text{presheaves of groupoids}$ for which weak equivs. (resp. fibrations) are maps $G \to H$ such that $BG \to BH$ are local weak equivs. (resp. fibrations). The assoc. sheaf map $G \to \tilde{G}$ is a weak equiv. $Ho(Pre(Gpd)) \simeq Ho(Shv(Gpd))$.

**Observation:** Every fibrant sheaf of groupoids is a fibrant presheaf of groupoids.

Stacks are homotopy types of presheaves of groupoids.
Torsors, revisited

$G = \text{sheaf of groups}$:

$$St^p(G)(U) \cong \{G\text{-cocycles over } U\}$$

$$\cong \{G|_U \text{ torsors over } U\}$$

**Fact:** $St^p(H)(U) \rightarrow St(H)(U)$ is an equivalence for all sheaves of groupoids $H$, in all sections.

**Consequence:** $St(G)(U) \cong \{G|_U \text{ torsors over } U\}$ for all $U$.

**NB:** $St(G)$ is a sheaf or presheaf of groupoids rather than a fibre functor.
Quotient stacks

$G \times N \to N$ is action by sheaf of groups on a sheaf $N$.

The quotient stack $G - \text{Tors}/N$, in sections, has objects all $G$-equivariant maps $P \to N$ where $P$ is a $G$-torsor, and has morphisms all $G$-equivariant diagrams

$$
\begin{array}{c}
P \\
\theta \\
P'
\end{array} \xymatrix{\ar[r]^p & \ar[d]^{\cong} \ar[r]_p & N \\
\theta \ar[r]_{\cong} & N \\
\theta \ar[r]_{p} & N
\end{array}
$$

There is a groupoid of translation categories $E_GN$ arising from the $G$-action, and $B(E_GN) \cong EG \times_G N$.

Fact: $[\ast, E_G \times_G N] \cong \pi_0(G - \text{Tors}/N)(S)$. $St(E_GN)$ is sectionwise equivalent to $G - \text{Tors}/N$. 


\[\text{p.16/22}\]
Alternative description of torsors

\( G = \) sheaf of groups: a \( G \)-torsor is a sheaf \( X \) with \( G \)-action such that the simplicial sheaf map \( EG \times_G X \rightarrow \ast \) is a weak equivalence.

\( EG \times_G X = \operatorname{holim}_G X \) for the diagram on groupoid \( G \) defined by \( G \)-action.

\( H = \) sheaf of groupoids: an \( H \)-torsor is a functor \( X : H \rightarrow \operatorname{Shv} \) (internally defined) such that the map \( \operatorname{holim}_H X \rightarrow \ast \) is a weak equiv. \([\ast, BH] \cong \pi_0(H - \operatorname{Tors})\).
Gerbes

A **gerbe** $H$ is a locally connected stack (ie. $\tilde{\pi}_0(H) \cong \ast$).

$G$ sheaf of groups: a $G$-gerbe is a gerbe locally equivalent to $G - \text{Tors}$.

$\text{Aut}(G)$ is two groupoid of automorphisms of $G$ and homotopies.

Luo, J. (2004): $[\ast, B(\text{Aut}(G)^{o})] \cong \pi_0(G - \text{gerbe})$
Algebraic stacks

Specialize to the étale topology for $Sch|_S$:

Laumon, Moret-Bailly: Algebraic stacks occur as stack completions of (ie. htpy types represented by) special sheaves of groupoids $X_1 \rightrightarrows X_0$, where

- $X_1, X_0$ are algebraic spaces (ie. étale quotients of schemes)
- the source and target maps $X_1 \to X_0$ are smooth
- the map $X_1 \to X_0 \times_S X_0$ is separated and quasi-compact.

If the source and target maps are étale, the stack is a Deligne-Mumford stack.

Example: $\mathcal{M}_{g,n} =$ moduli stack of proper smooth curves/$S$ of genus $g$ with $n$ ordered points.
Stack cohomology

Perspective: $X = \text{algebraic stack}$.

“Naive” stack cohomology $H^*(X, A)$ is defined by the étale site of a simplicial algebraic space, which is $BX$.

Claim:

$$H^n(X, A) \cong [BX, K(A, n)]$$

for abelian sheaves $A$ defined on the big site $\text{Sch}|_S$.

This is a generalization of a corresponding result for simplicial schemes.
Some details

The site $et|_{BX}$ has objects all étale maps $\phi : U \to BX_n$ (algebraic spaces) and morphisms

$$
\begin{aligned}
V & \longrightarrow U \\
\psi & \downarrow \\
BX_m & \theta^* \longrightarrow BX_n
\end{aligned}
\quad \theta : n \to m \text{ in } \Delta
$$

Coverings are generated by étale coverings $U_i \to U$. $F = \text{sheaf on } (Sch|_S)_{et}: F|_{BX}(\phi) = \hom(U, F)$. $F|_{BX}$ is sheaf on $et|_{BX}$ and $F \mapsto F|_{BX}$ is exact, so restriction preserves weak equivs.

Idea: show that there is a bijection $[BX, Y] \cong [* , Y|_{BX}]$. 

May as well assume that $Y$ is fibrant.

Choose fibrant model $Y|_{BX} \to Z$ on $et|_{BX}$. (*) Show that this map is a weak equivalence in all sections by showing that this is true for all restrictions to $et|_{BX_n}$ for all $n$.

$1_{BX}$ is the simplicial sheaf on $et|_{BX}$ rep. by $1_{BX_n}$, $n \geq 0$.

\[
\begin{array}{ccc}
V & \longrightarrow & BX_n \\
\psi \downarrow & & \downarrow 1 \\
BX_m & \xrightarrow{\theta^*} & BX_n
\end{array}
\]

det. by $\theta$. $1_{BX}(\psi) = \Delta^m \simeq \ast$, so $1_{BX} \to \ast$ is weak equiv.

$\text{hom}(1_{BX}, Y|_{BX}) \to \text{hom}(1_{BX}, Z) \simeq Z(\ast)$ is weak equiv. by (*).

$\text{hom}(1_{BX}, Y|_{BX}) \cong \text{hom}(BX, Y)$. Compare $\pi_0$. 