The Evans Function: an operator-theoretical perspective

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"Dealing with any mathematical phenomena, search for a [symmetric] operator sitting behind the curtains"

[Mark Krein, private communication]

My favorite operator

Given a linear differential equation

$$\frac{dy}{dx} = A(x)y(x), \quad x \in \mathbb{R},$$

on a (generally, infinite dimensional) space $\mathcal{Y}$ with matrix-(or, generally, operator-) coefficients $A(x)$

Consider on $L^2(\mathbb{R};\mathcal{Y})$ the operator

$$(G_A u)(x) = -\frac{du}{dx} + A(x)u(x)$$

More general: given an infinite dimensional propagator

$$\Phi(x, x') - \text{strongly continuous exp bounded}$$

consider (strongly continuous) **evolution semigroup**

$$(E^t u)(x) = \Phi(x, x-t)u(x-t), \quad x \in \mathbb{R},\ t \geq 0$$

Theorem. *Infinitesimal generator* $\mathbf{G}$ *of the semigroup* $\{E^t\}_{t\geq 0}$ *is the closure of* G_A

[G_A is closed for very general "parabolic"; diGiorgio/Lunardi/Schnaubelt]

My favorite operator: perturbed

Given

$$\begin{aligned}\frac{dy}{dx} =& A(x)y(x), \quad x \in \mathbb{R}, \\ \frac{dy}{dx} =& [A(x) + R(x)]y(x)\end{aligned}$$

Assume $G_A = -\frac{d}{dx} + A(\cdot)$ is invertible

Then $(G_{A+R}u)(x) = -\frac{du}{dx} + A(x)u(x) + R(x)u(x)$ could be written as

$$G_{A+R} = G_A + R = G_A(I + G_A^{-1}R)$$

Factorize $R(x) = R_\ell(x)R_r(x)$. Then

$$G_{A+R} = G_A(I + G_A^{-1}R_\ell \cdot R_r)$$

Observation ("Birman-Schwinger principle"):

Operator $I + T_1 \cdot T_2$ is invertible if and only if operator $I + T_2 \cdot T_1$ is invertible

Conclusion. *G_{A+R} is invertible (Fredholm) if and only if $I + R_r G_A^{-1} R_\ell$ is invertible (Fredholm)*

Definition:

$$H = R_r G_A^{-1} R_\ell$$

is called the sandwiched resolvent

Main Results

Given **finite dimensional** equations

$$\begin{aligned}\frac{dy}{dx} =& A(x)y(x),\ x\in\mathbb{R},\ A(\cdot)=A(z,\cdot)\\ \frac{dy}{dx} =& [A(x)+R(x)]y(x),\ x\in\mathbb{R},\ R(\cdot)=R(z,\cdot)\\ z\in&\mathbb{C}\text{ spectral parameter}\end{aligned}$$

Theorem (Evans=Fredholm). *The Evans function $E(z)$ is equal to the (modified) Fredholm determinant of the identity plus the sandwiched resolvent $H(z)=R_r(d/dx+A(\cdot))^{-1}R_\ell$.*

For the Scrödinger equation

Theorem (Evans=Jost). *The Evans function $E(z)$ is equal to the classical Jost function*

Open Question: What if $A(x)$ and $R(x)$ are infinite dimensional operators?
What is the Evans function?
Are there interesting examples?

Based on

1. F. Gesztesy, K. Makarov and Y. Latushkin
   The Evans function and Fredholm determinants, in preparation

2. F. Gesztesy, K. Makarov and Y. Latushkin
   Evans functions and modified Fredholm determinants, Oberwolfach Reports, 2005 (to appear)

3. D. Cramer and Y. Latushkin
   Fredholm determinants and the Evans function for difference equations, preprint

4. T. Kapitula and B. Sandstede
   Eigenvalues and resonances, DCDS (2004)
   Edge bifurcations SIAM J. Math. Anal.(2002)
   (connections Evans=Fredholm for Schrödinger)

My favorite operator: bended

$$G_A = -\frac{d}{dx} + A(x) \text{ on } L^2(\mathbb{R};\mathcal{Y})$$

View $\frac{d}{dx}$ as the generator of the flow (translation on $\mathbb{R}$)

$$\varphi^x : x' \to x' + x,\ x' \in \mathbb{R}$$

"Bend" the line $\mathbb{R}$

Consider

$$\begin{gathered} X \text{ a manifold} \\ u \text{ a vector field} \\ \varphi^t \text{ a flow generated by the vector field} \end{gathered}$$

Replace $\frac{d}{dx}$ by $\langle u, \nabla\rangle$ – derivative along the flow

Then G_A becomes

$$G_A = -\langle u, \nabla\rangle + \text{operator}$$

Kinematic dynamo

Example (Lie bracket or kinematic dynamo)

$$G_A v = -\langle u, \nabla \rangle v + \langle v, \nabla \rangle u$$

v vector field (say L^2-divergence free)

Comes from eqns for induction $\mathbf{H}$

$$\frac{d}{dt}\mathbf{H} = \nabla \times (u \times \mathbf{H}) + \frac{1}{\text{Reynolds}}\Delta \mathbf{H},\ \operatorname{div} \mathbf{H} = 0$$

for Reynolds$= \infty$ this equation becomes

$$\frac{d}{dt}\mathbf{H} = -\langle u, \nabla \rangle \mathbf{H} + \langle \mathbf{H}, \nabla \rangle u.$$

3-min Course in Linear Hydrodynamics

Euler equations for an ideal incompressible fluid on $\mathbb{T}^d$, $d = 2$ or $d = 3$

u – velocity vector field, $u = u(t, x)$, $x \in \mathbb{T}^d$

P – (a scalar) pressure

$$\partial_t u + < u, \nabla > u + \nabla P = 0, \operatorname{div} u = 0$$

Fix a C^∞-steady state $u = u(x)$

$$< u, \nabla > u + \nabla P = 0$$

Linearized Euler

Linearize Euler equations about a steady-state u.
Linearized operator is

$$Lv = - < u, \nabla > v - < v, \nabla > u - \nabla P, \quad \operatorname{div} v = 0$$

$$L = L_{\text{vel}} \ , \quad d = 2 \text{ or } 3$$

Vorticity $w = \operatorname{curl} v$ $(= w\vec{k}$ in 2D, w scalar$)$

$$L = L_{\text{vort}} \ , \quad d = 2$$

$$Lw = - < u, \nabla > w - < \operatorname{curl}^{-1} w, \nabla > \operatorname{curl} u$$

Here: $\operatorname{curl} u = -\partial_2 u_1 + \partial_1 u_2$;

$v = \operatorname{curl}^{-1} w$ solves equation $\operatorname{curl} v = w, \operatorname{div} v = 0$

Problem: describe $\sigma(L)$ and $\sigma(e^{tL})$

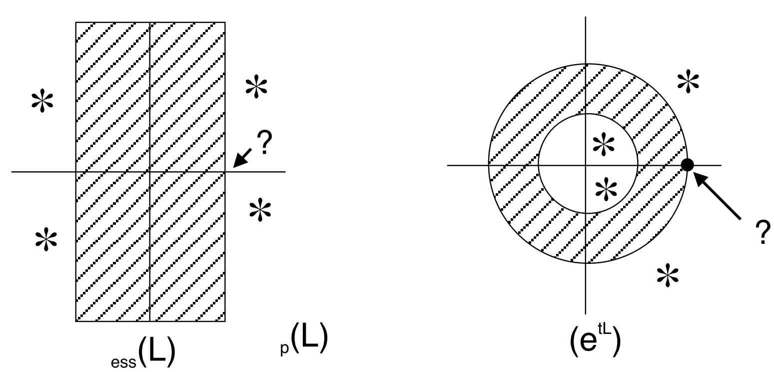
on $L_2^s(\mathbb{T}^d)$ or $H_m^s(\mathbb{T}^d)$ for L_{vel} (s - means divergence-free)
on $H_m^0(\mathbb{T}^2)$ for L_{vort} (0 means $\int w\,dx = 0$)

Point spectrum: classical

(L. Howard, Drazin/Reid, C.C. Lin)

Essential Spectrum:

S. Friedlander, M. Vishik, W. Strauss, ...



- Find boundaries
- Is $\sigma(e^{tL}) = e^{t\sigma(L)}$?
- Describe the geometry of $\sigma_{\text{ess}}(L)$ and $\sigma_{\text{ess}}(e^{tL})$

Results for 2D Euler

$$\begin{aligned} u & \quad C^\infty\text{-steady state}, \quad \langle u, \nabla\rangle u + \nabla P = 0, \quad \operatorname{div} u = 0 \\ Lw &= -\langle u, \nabla\rangle w - \langle \operatorname{curl}^{-1} w, \nabla\rangle \operatorname{curl} u \\ & \quad \text{linearized Euler operator on } H^0_m(\mathbb{T}^2), \quad m \in \mathbb{Z}. \\ \Lambda &= \lim_{t\to\infty} \frac{1}{t} \log \max_{x\in\mathbb{T}^2} \|D\varphi_t(x)\| \\ & \quad \text{maximal Lyapunov-Oseledets exponent} \end{aligned}$$

Theorem (Shvidkoy/Latushkin)

1. If $\Lambda > 0$ then

$$\begin{aligned} \sigma_{\text{ess}}(L) &= \{z \in \mathbb{C} : -|m|\Lambda \le \operatorname{Re} z \le |m|\Lambda\} \\ \sigma_{\text{ess}}(e^{tL}) &= \{z \in \mathbb{C} : e^{-t|m|\Lambda} \le |z| \le e^{t|m|\Lambda}\} \end{aligned}$$

2. If $\{\varphi_t\}$ has arbitrarily long trajectories and $\Lambda = 0$ then $\sigma_{\text{ess}}(L) = i\mathbb{R}$
3. $\frac{1}{t}\log$ radius of $\sigma_{\text{ess}}(e^{tL}) = |m|\Lambda$ (implies linear instability)
4. $e^{t\sigma(L)} = \sigma(e^{tL})$.

Formulation by pictures

u steady state for 2D-Euler, C^∞

φ^t corresponding flow on $\mathbb{T}^2$, $\frac{d}{dt}\varphi^t(x) = u(\varphi^t x)$

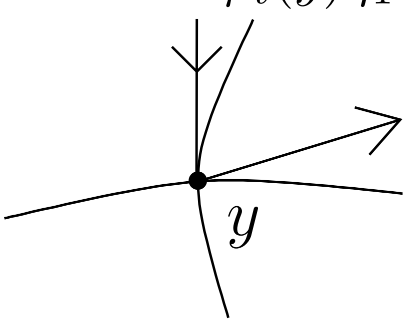
y stagnation point $u(y) = 0$

May be hyperbolic: $\sigma(Du(y)) = \{-\lambda, \lambda\}$

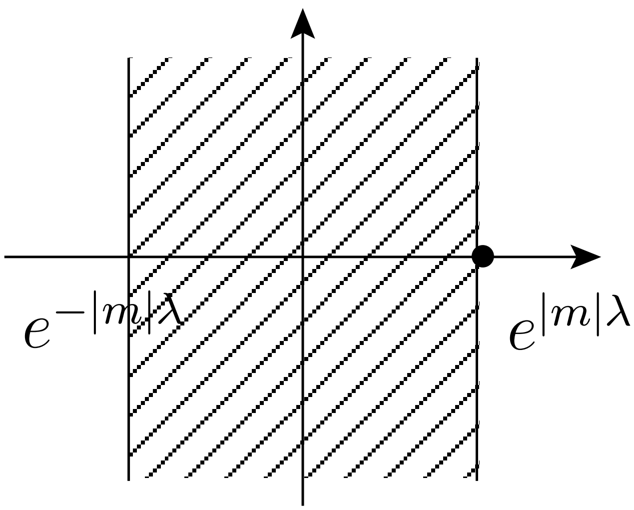
$Du(y)$ Jacoby matrix

$$D\varphi_t(y)\eta_1 = e^{-\lambda^t}\eta_1$$

$$D\varphi_t(y)\eta_2 = e^{\lambda t}\eta_2$$



$$\Lambda = \sup\{\lambda : \text{stable eigenvalues at hyperbolic stagnation points}\}$$
$$= \sup\{\Sigma = \text{Lyapunov-Oseledets exponents}\}$$



$\sigma_{\text{ess}}(L)$

$\sigma_{\text{ess}}(e^{(tL)})$

Linearized Euler $Lw = -\langle u, \nabla \rangle w - \langle \operatorname{curl}^{-1} w, \nabla \rangle \operatorname{curl} u$

Idea of the proof

Recall

$$\begin{aligned} L &= -< u, \nabla > w - < \operatorname{curl}^{-1} w, \nabla > \operatorname{curl} u \\ &= -A + K \end{aligned}$$

$Aw =< u, \nabla > w$ generates $E^t w = w \circ \varphi_t$

K compact operator on all H^0_m, $m = 0, 1, 2, \dots$

Main Lemma

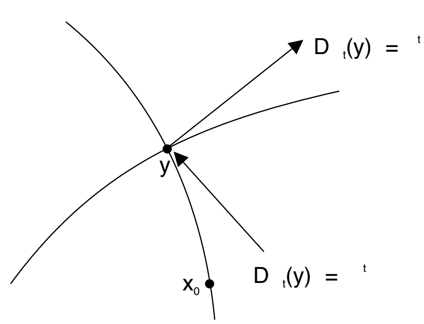
If $\alpha = \lambda + i\xi$, $\lambda \in \Sigma$, $\lambda \neq 0$, $\xi \in \mathbb{R}$, then there is

$$\{f_n\} : \|f_n\|_{H^0_m} = 1, \quad \text{weak} - \lim_{n \to \infty} f_n = 0, \|(\alpha - A) f_n\| \to 0.$$

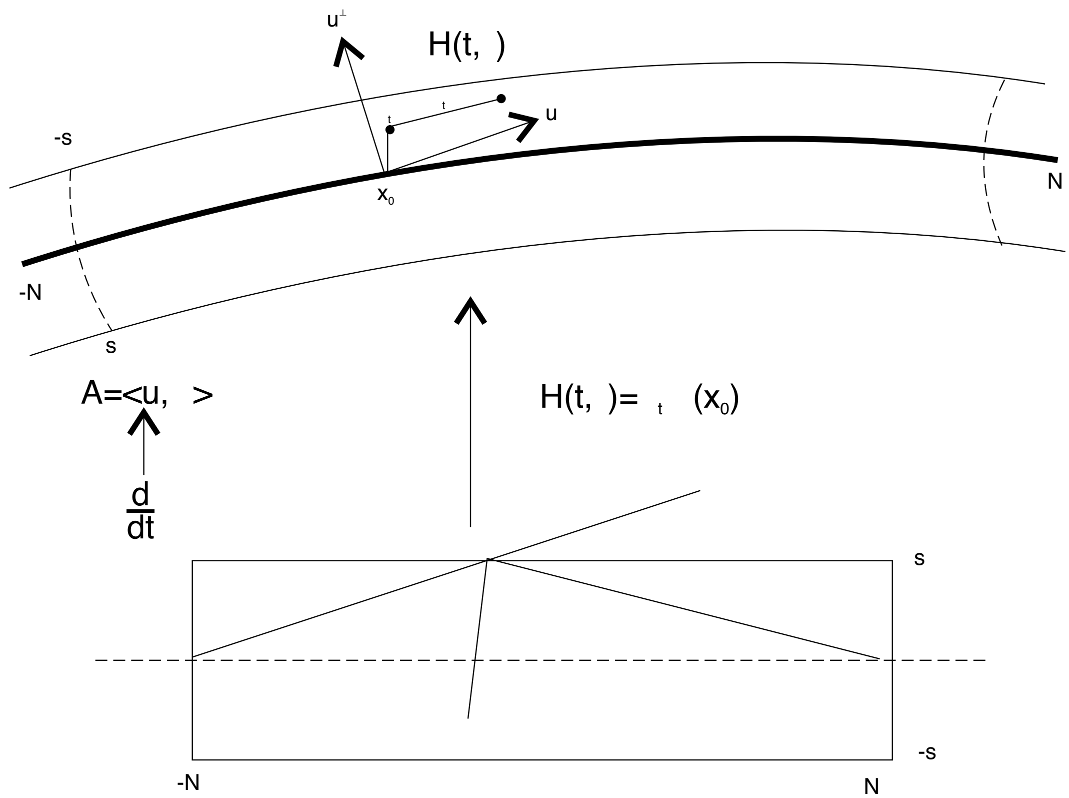
Conclusion: $-\alpha$ is an approximate eigenvalue for L

If $\lambda \in \Sigma$, $\lambda \neq 0$, then there is a stagnation point y, $u(y) = 0$, such that

$$\sigma(Du(y)) = \{-\lambda, \lambda\}, \quad D\varphi_t(y) = e^{tDu(y)}$$



For $m = 1$:



$$F(t,\tau) = e^{\alpha t}\gamma(t)\delta(\tau), \quad \delta(\tau) = (s - |\tau|)\mathbb{I}_{[-s,s]}$$

$$f = F \circ H^{-1}, \quad \operatorname{supp}\gamma(t) \subset [-N, N]$$

$$Af - \alpha f \mid_{H(t,\tau)} = e^{\alpha t}\gamma'(t)\delta(\tau) =: \tilde{F}(t,\tau), \quad \alpha = \lambda + i\xi$$

we show

$$\|L + \alpha\|^2_{\bullet, H^0_1} \leq \frac{\int_{-s}^{s}\int_{-N}^{N} |\nabla \tilde{F}|^2}{\int_{-s}^{s}\int_{-N}^{N} |\nabla F|^2} \to 0 \text{ as } \begin{array}{c} s \to 0 \\ N \to \infty \end{array}$$

Open: Evans vs. Euler

The Evans function detects the point spectrum of

$$G_{A+R} = -\frac{d}{dx} + A(x) + R(x)$$

The Euler operator

$$L = -\langle u, \nabla \rangle + K$$

Is there an Evans function which detects the point spectrum of L?

Big literature on the point spectrum (a classical problem)

Friedlander, Yudovich, Vishik (continuous fractions)

Spectral Mapping theorems

We know for Euler: $\sigma(e^{tL}) = e^{t\sigma(L)}$

We know for matrices: $\sigma(e^{tA}) = e^{t\sigma(A)}$

Given infinite dimensional equation

$$\dot{v} = Av,\ v(t) = e^{tA}v(0)$$

A generator of a strongly continuous semigroup

Will the spectral mapping theorem hold?

$$\sigma(e^{tA})\backslash 0 = e^{t\sigma(A)},\ t > 0.$$

Important: given nonlinear equation

$$\begin{aligned}\frac{d}{dt}u =& F(u) \text{ linearized around fixed point}\\ \frac{d}{dt}v =& Av\end{aligned}$$

we know $\sigma(A)$ but often need $\sigma(e^{tA})$ for stability/instability/invariant manifolds

An abstract nonlinear instability result

$\frac{dv}{dt} = Av + F(v)$, $v \in \mathcal{Y}$ (a Banach space), $F(0) = 0$

Assume

$$\sigma(A) \cap \{\operatorname{Re} z > 0\} \neq \emptyset$$

Then $e^{t\sigma(A)} \subset \sigma(e^{tA})$ implies

$$\frac{d}{dt}v = Av \text{ is linearly unstable : } \quad \sigma(e^{tA}) \cap \{|z| > 1\} \neq \emptyset$$

Theorem (Shatah/Strauss). *If $F : \mathcal{Y} \to \mathcal{Y}$ is continuous and $\|F(v)\| \leq c\|v\|^{1+\eta}$ for $\eta > 0$ and small $\|v\|$ then zero is nonlinearly unstable*

stable means:
$\forall \epsilon \exists \delta \forall \|v(0)\| < \delta \exists ! v \in C([0,\infty];X) : \sup_{0 \leq t < \infty} \|v(t)\| < \epsilon$

Question: Does $\sigma(A) \subset \{\operatorname{Re} z < 0\}$ implies stability?
Even linear stability?

Troubles that one wants to avoid

A generator of a strongly continuous semigroup

$$\dot{v} = Av, \quad v(t) = e^{tA}v(0)$$

$$\begin{aligned} s(A) &= \sup\{\operatorname{Re}\alpha : \alpha \in \sigma(A)\} \text{ spectral bound} \\ \omega(A) &= \lim_{t\to\infty} \frac{1}{t} \log \|e^{tA}\| \text{ growth bound} \end{aligned}$$

$$\dot{x} = Ax \text{ is uniformly stable } \Leftrightarrow \omega(A) < 0$$

Question: is $s(A) = \omega(A)$ [Lyapunov Theorem]

Answer: NO. Only $\overline{\exp t\sigma(A)} \subset \sigma(e^{tA})$ NOT $=$

Why: $s_{\text{ess}}(A) < \omega_{\text{ess}}(A)$

$$\begin{aligned} \omega_{\text{ess}}(A) &= \frac{1}{t_0} \log \text{radius of } \sigma_{\text{ess}}(e^{t_0 A}) \\ \omega(A) &= \frac{1}{t_0} \log \text{radius of } \sigma(e^{t_0 A}) \end{aligned}$$

Example: M. Renardy, $u_{tt} = u_{xx} + u_{yy} + e^{iy} u_x$

$u(t,x,y)$ 2π-periodic in x, y

Rewrite $\dot{v} = Av$ A is 2×2

$$s(A) = 0 < \frac{1}{2} = \omega(A)$$

Conclusion: Spectral mapping theorems hold for *parabolic* equations (analytic semigroups)

Generally, does NOT hold for hyperbolic equations

Gearhart-Prüss vs Lyapunov

$\mathcal{Y}$ is a Hilbert space, A is C_0-semigroup generator

Assume $s(A) < 0$: $\sigma(A)$ is to the left of $i\mathbb{R}$

What should we add to have $\omega(A) < 0$
that is $\sigma(e^{tA})$ is inside of $\{|z| < 1\}$?

Answer: $\sup\{\|(z - A)^{-1}\| : \operatorname{Re} z \geq 0\} < \infty$

Theorem (Gearhart-Greiner-Herbst-Prüss). $\mathcal{Y}$ ***Hilbert***

$\sigma(e^{tA})\backslash\{0\} = \{e^{\lambda t}\ |$*either* $\mu_k = \lambda + \frac{2\pi i k}{t} \in \sigma(A)$ *for some* $k \in \mathbb{Z}$ *or the sequence* $\{\|(\mu_k - A)^{-1}\|\}_{k\in\mathbb{Z}}$*is unbounded*$\}$

Theorem (Latushkin, Montgomery-Smith,...). $\mathcal{Y}$ ***Banach***

$\sigma(e^{tA})\backslash\{0\} = \{e^{\lambda t}\ |$ *either* $\mu_k = \lambda + \frac{2\pi i k}{t} \in \sigma(A)$ *for some* $k \in \mathbb{Z}$ *or the sequence* $(\mu_k - A)^{-1}$ *is not an* $L^p([0, 2\pi]; \mathcal{Y})$*-Fourier multiplier*$\}$

Operator sequence $\{M_k\}_{k\in\mathbb{Z}}$ is a Fourier multiplier if

$$\sum_{|k|\leq N} y_k e^{ik(\cdot)} \to \sum_{|k|\leq N} M_k y_k e^{ik(\cdot)}, \quad y_k \in \mathcal{Y},$$

extends to a bounded operator on $L^p([0, 2\pi]; \mathcal{Y})$, $p > 1$.

Gearhart-Prüss vs Evans

Gearhart-Prüss theorem has been used by many nice people to treat stability of traveling waves

Kapitula/Sandstede	Physica D	1998
Miller/Weinstein	CPAM	1996
Pego/Weinstein	CMP	1994

See a collection in Cramer/YL "...a survey"

[KdV,BBM,Boussinesq, Green-Naghdi, NLS...]

General strategy:
linearize a PDE $\partial_t u = L(\partial_x)u + \mathcal{N}(u)$ about a traveling wave $Q_c = Q_c(x - ct)$

$$\mathcal{L}u = L(\partial_x)u + c\partial_x u + DN(Q_c)u$$

1. Use the Evans function to detect the isolated eigenvalues of $\mathcal{L}$
2. Use resolvent estimates to apply Gearhart-Prüss for $A = \mathcal{L}$

Gearhart-Prüss vs linearized nonlinear Schrödinger

Using spectral mapping theorems to find invariant manifolds for nonlinear Schrödinger [Kapitula/Sandstede, Gesztesy/Jones/Stanislavova/YL, Weinstein]

$$\begin{aligned} u_t =& \Delta u + f(x, |u|^2)u + \beta u \\ u =& u(x,t) \in \mathbb{C},\ x \in \mathbb{R}^n, t \geq 0,\ \beta < 0 \end{aligned}$$

Separate real and imaginary parts $u = v + iw$

$$\begin{aligned} v_t =& \Delta w + f(x, v^2 + w^2)w + \beta w \\ w_t =& -\Delta v - f(x, v^2 + w^2)v - \beta v \end{aligned}$$

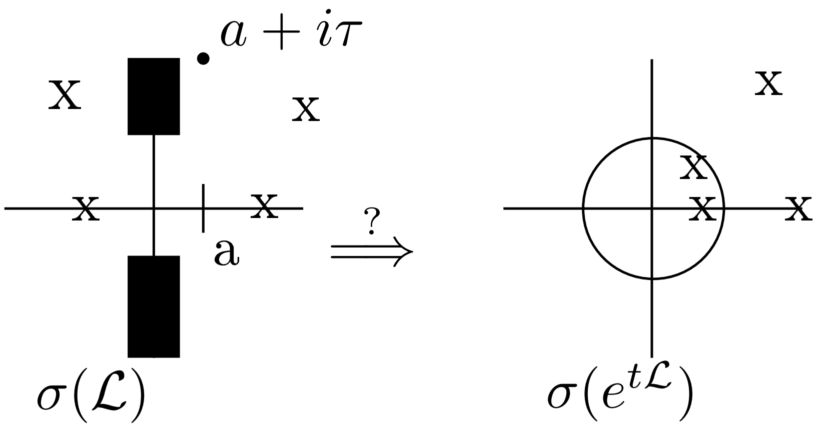
Linearize about a standing wave of frequency β (real-valued steady state) to get $\mathcal{L} = \begin{bmatrix} 0 & -L_R \\ L_I & 0 \end{bmatrix}$ where
$L_R = -\Delta - \beta + Q_{1,}$, $L_I = -\Delta - \beta + Q_2$
and Q_1, Q_2 given potentials

Known: $\sigma_{\text{ess}}(\mathcal{L}) = \{i\xi | \xi \in \mathbb{R}, |\xi| \geq -\beta\}$ PLUS discrete non imaginary eigenvalues.

Invariant manifolds for NLS

Want: stable/center/unstable invariant manifolds for the NLS locally around the standing wave

Invariant manifolds exist provided $\sigma(e^{t\mathcal{L}})$ is split



Gearhart-Prüss gives $\Longrightarrow$

Want to show that $\|(\xi - \mathcal{L})^{-1}\|$ is bounded for $\xi = a + i\tau$, $a \neq 0$ for large enough τ

$$\mathcal{L} = \begin{bmatrix} 0 & D + Q_1 \\ -D + Q_2 & 0 \end{bmatrix}, \; D = -\Delta - \beta$$

$$\xi - \mathcal{L} = \begin{bmatrix} \xi & D \\ -D & \xi \end{bmatrix} \left(I + \begin{bmatrix} \xi & D \\ -D & \xi \end{bmatrix}^{-1} \begin{bmatrix} 0 & Q_1 \\ -Q_2 & 0 \end{bmatrix} \right)$$

Resolvent estimates

$$\xi - \mathcal{L} = \begin{bmatrix} \xi & D \\ -D & \xi \end{bmatrix} (I + A(\xi) \cdot B)$$

where $D = -\Delta - \beta$, $\xi = a + i\tau$, $a \neq 0$, $\tau \to \infty$

$$A(\xi) = \begin{bmatrix} \xi & D \\ -D & \xi \end{bmatrix}^{-1} \begin{bmatrix} 0 & Q_1^{\frac{1}{2}} \\ -Q_2^{\frac{1}{2}} & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} |Q_2|^{\frac{1}{2}} & 0 \\ 0 & |Q_1|^{\frac{1}{2}} \end{bmatrix}$$

$$\mathcal{L} = \begin{bmatrix} 0 & D + Q_1 \\ -D + Q_2 & 0 \end{bmatrix}$$

Lemma 1. *The norm of* $\begin{bmatrix} \xi & D \\ -D & \xi \end{bmatrix}^{-1}$ *is bounded as* $\tau \to \infty$*; here* $\xi = a + i\tau,\ a \neq 0,\ D = -\Delta - \beta$

Lemma 2. $\|B \cdot A(\xi)\| \to 0$ *as* $\tau \to \infty$.

Here:

$$BA(\xi) = \begin{bmatrix} |Q_2|^{\frac{1}{2}}(\xi^2+D^2)^{-1}DQ_2^{\frac{1}{2}} & |Q_2|^{\frac{1}{2}}\xi(\xi^2+D^2)^{-1}Q_1^{\frac{1}{2}} \\ -|Q_1|^{\frac{1}{2}}\xi(\xi^2+D^2)^{-1}Q_2^{\frac{1}{2}} & |Q_1|^{\frac{1}{2}}(\xi^2+D^2)DQ_1^{\frac{1}{2}} \end{bmatrix}$$

The proof of the resolvent estimateis is based on
$(I + A(\xi) \cdot B)^{-1} = I - A(\xi)(I + B \cdot A(\xi))^{-1}B$
Thus $\|(\xi - \mathcal{L})^{-1}\|$ is bounded as $\tau \to \infty$
because $(\xi - \mathcal{L})^{-1} = \Big(I + A(\xi) \cdot B\Big)^{-1} \begin{bmatrix} \xi & D \\ -D & \xi \end{bmatrix}^{-1}$

The Evans function

Evans function $E(z)$ is a Wronskian-type determinant used to detect the point spectrum of differential operators obtained by linearizing PDEs about special solutions such as pulses, fronts, traveling waves, etc.

General rule:

- $E(\cdot)$ is analytic outside of $\sigma_{\text{ess}}(\mathcal{L})$
- $E(z) = 0$ if and only if $z \in \sigma_p(\mathcal{L})$

where $\mathcal{L}$ is the "linearized" operator

Main results

1. Coordinate-free definition of the Evans function in the context of perturbation theory for general first order matrix nonautonomous equations.

2. The Evans function is equal to the modified Fredholm determinant of the “sandwiched resolvent".

3. Schrödinger equations: The Evans function is equal to the classical Jost function.

The Evans function: an example

Reaction-diffusion equation

$$u_t = u_{xx} - u + 2u^3, \quad u = u(x,t), \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+$$

Pulse solution: $u_0(x) = \operatorname{sech} x$ **Linearize!**

Operator $\mathcal{L}$:

$$\mathcal{L}v = v'' - (1 - 6u_0^2(x))v, \quad v = v(x)$$

Rewrite the eigenvalue problem $\mathcal{L}v = zv$ as

$$\begin{aligned} y'(x) &= [A_z + R(x)]y(x), \quad x \in \mathbb{R}, \\ A_z &= \begin{bmatrix} 0 & 1 \\ 1+z & 0 \end{bmatrix}, \quad R(x) = \begin{bmatrix} 0 & 0 \\ -6u_0^2(x) & 0 \end{bmatrix}. \end{aligned}$$

Note $R(x) \to 0$ as $x \to \pm\infty$ exponentially

Want to detect z's that yield bounded on $\mathbb{R}$ solution y then $z \in \sigma_p(\mathcal{L})$

Fix z with $\operatorname{Re} z > -1$

$y'(x) = [A + R(x)]y(x)$, $x \in \mathbb{R}$ perturbed equation

$A = \begin{bmatrix} 0 & 1 \\ 1+z & 0 \end{bmatrix}$, $R(x)$ exp decaying (2×2) perturbation

Eigenvalues of A are $-\sqrt{1+z}$ and $\sqrt{1+z}$

Eigenvectors of A are

$$\eta_+ = \begin{bmatrix} 1 \\ -\sqrt{1+z} \end{bmatrix} \text{ and } \eta_- = \begin{bmatrix} 1 \\ -\sqrt{1+z} \end{bmatrix}$$

Solutions of the unperturbed equation

$y'(x) = Ay(x)$, $x \in \mathbb{R}$, (plane waves) are

$$\begin{aligned} e^{-(\sqrt{1+z})x}\eta_+ &\longrightarrow 0 \text{ as } x \to +\infty \\ e^{(\sqrt{1+z})x}\eta_- &\longrightarrow 0 \text{ as } x \to -\infty \end{aligned}$$

Take solutions $y_\pm(x,z)$ of $y'(x) = [A + R(x)]y(x)$,
$y_\pm(x) \to 0$, $x \to \pm\infty$
asymptotic to solutions
$e^{\mp(\sqrt{1+z})x}\eta_\pm$ of $y'(x) = Ay(x)$

so that

$$\begin{aligned} e^{(\sqrt{1+z})x} y_+(x) &\longrightarrow \eta_+, \quad x \to +\infty \\ e^{-(\sqrt{1+z})x} y_-(x) &\longrightarrow \eta_-, \quad x \to -\infty \end{aligned}$$

Evans function $E(z) = \det[y_+(x,z) \quad y_-(x,z)]$

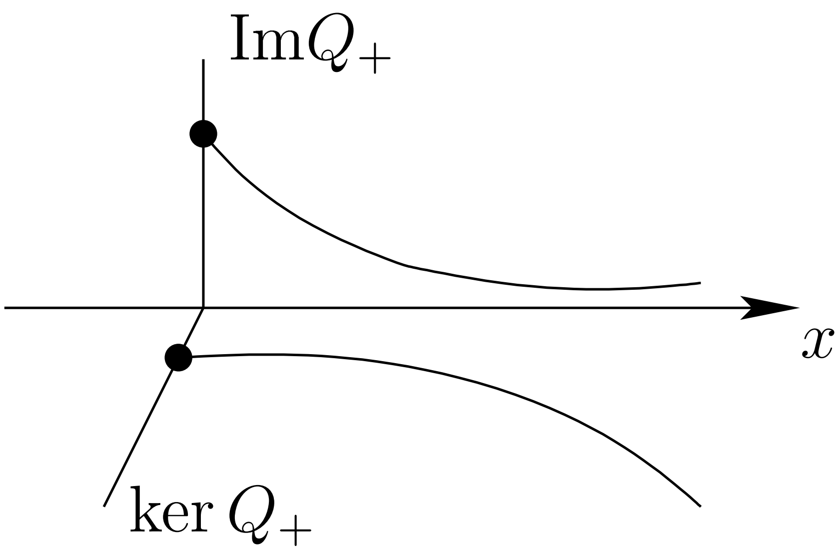
$$\begin{aligned} E(z) = 0 \qquad & \text{if and only if there is a bounded } y \\ & \text{if and only if } z \in \sigma_p(\mathcal{L}) \end{aligned}$$

Not a function!

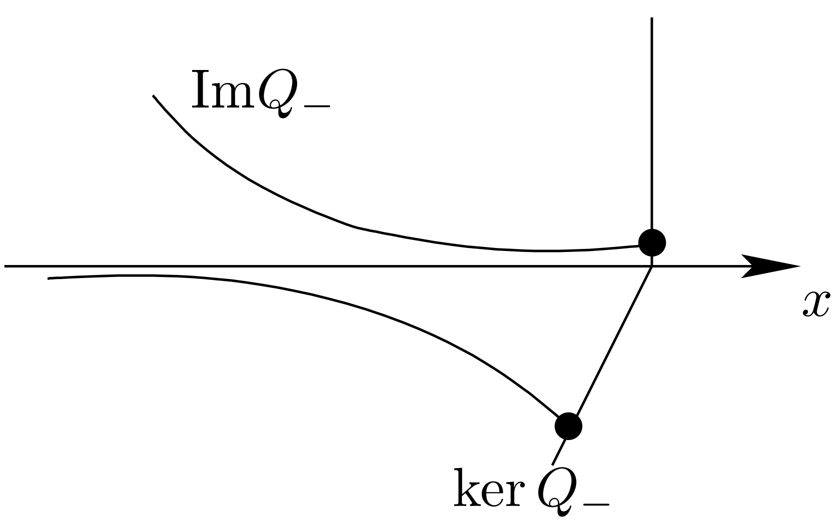
Exponential dichotomies

$y' = A(x)y \quad (d \times d)$ matrix equation, $A \in L^\infty(\mathbb{R};\mathbb{C}^{d\times d})$

Q_+ an exponential dichotomy on $\mathbb{R}_+$

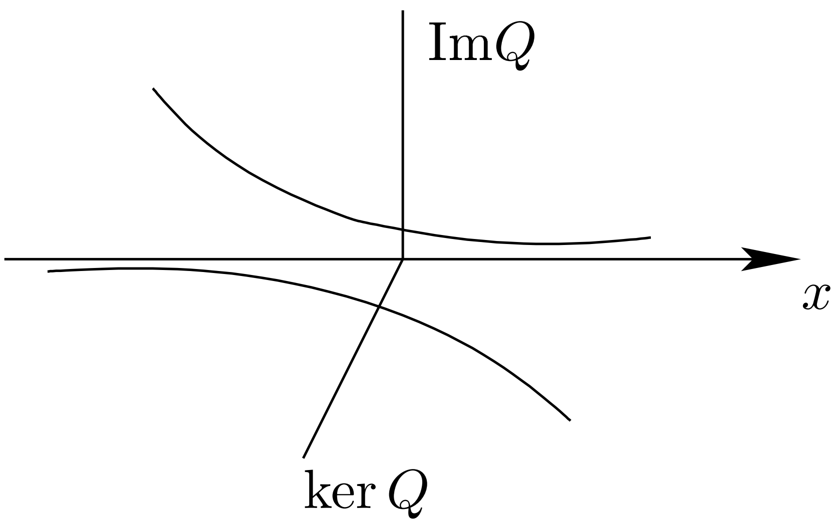


Q_- an exponential dichotomy on $\mathbb{R}_-$



$Q_\pm$ are not unique! only $\operatorname{Im} Q_+$ and $\operatorname{Ker} Q_-$!

Q ED on $\mathbb{R}$ for $y' = A(x)y$



Q is unique!

Dichotomy and operators

Theorem. *My favorite operator*

$$G_A = -\frac{d}{dx} + A(x)$$

is invertible on $L^2(\mathbb{R};\mathbb{C}^d)$ *if and only if* $\frac{dy}{dx} = A(x)y$ *has an exponential dichotomy on* $\mathbb{R}$

(old theorem. Holds for infinite dimensional spaces and unbounded operators $A(x)$. See [Chicone/Latushkin] for many different proofs that go back to Daleckij/Krein, etc.)

Question: When $G_A = -\frac{d}{dx} + A(x)$ is Fredholm?

Dichotomy Theorem: G_A is Fredholm if and only if $\frac{dy}{dx} = A(x)y$ has an exponential dichotomy Q_+ on $\mathbb{R}_+$ AND an exponential dichotomy Q_- on $\mathbb{R}_-$

Also $\operatorname{Ind} G_A = \dim \operatorname{Im} Q_- - \dim \operatorname{Im} Q_+$

Finite dimensional Dichotomy Theorem:

R. Sacker,

Splitting index for linear differential systems, 1979

K. Palmer,

Dichotomies & transversal homoclinic points, 1984

Dichotomies & Fredholm operators, 1988

Palmer's Theorem

A Ben-Artzi & I. Gohberg, Dichotomy of systems, 1992

Important for heteroclinic orbits
Key ingredient in travelling waves

Infinite dimensional Dichotomy Theorem

A. Baskakov (1996-2002)

J. Harterich, B. Sandstede, A. Scheel (2002)

X.-B. Lin (1986)

J. Mallet-Paret (1999)

D. Peterhof, B. Sandstede, A. Scheel (1997)

H. Rodrigues and J. Ruas-Filho (1995)

B. Sandstede, A. Scheel (2001)

W.N. Zhang (1995)

Recent:

Abbondandolo-Majer (2001,2003)

Di Giorgio, A. Lunardi (2003)

Di Giorgio, A. Lunardi, R. Schnaubelt (2004)

(parabolic infinite dimensional)

P. Rabier (2004)

Assume that...

[e.g., unstable subspaces are finite dimensional]

Then

G is Fredholm

$$\iff$$

there exist dichotomies on $\mathbb{R}_+$ and $\mathbb{R}_-$

Infinite dimensional Dichotomy Theorem

What is $G_A = -\frac{d}{dx} + A(x)$ on $L^2(\mathbb{R}; \mathcal{Y})$?
here $A(x)$ are unbounded and $\mathcal{Y}$ is infinite dimensional

Two approaches

1. Assume $\operatorname{Dom} A(x) \equiv W$ compactly embedded in $\mathcal{Y}$
   Treat $G_A : L^2(\mathbb{R}; W) \cap H^2(\mathbb{R}, \mathcal{Y}) \to L^2(\mathbb{R}; \mathcal{Y})$ as bounded operator [Robbin/Salamon, recently Rabier]
   Plus: $\frac{dy}{dx} = A(x)y$ is NOT assumed to be well-posed
   Important: Papers by Sandstede and Scheel on NOT well-posed

2. Assume $\frac{dy}{dx} = A(x)y$ is well-posed.
   Take the propagator $\Phi(x, x')$
   Define the evolution semigroup on $L^2(\mathbb{R}, \mathcal{Y})$
   Treat G_A as its generator [Tomilov/YL]

In other words: $u \in \operatorname{Dom} G_A$, $G_A u = f$ if and only if

$$u(x) = \Phi(x, x')u(x') - \int_{x'}^{x} \Phi(x, s)f(s)ds, \; x \geq x', \; f \in L^2(\mathbb{R}; \mathcal{Y})$$

mild solutions

$G_A = -\frac{d}{dx} + A(x)$ for general parabolic
[Lunardi/Schnaubelt]

General Dichotomy Theorem

Given: $\{\Phi(x,x')\}_{x\geq x'}$, x, $x' \in \mathbb{R}$

strongly continuous exponentially bounded
evolution family on a **reflexive** Banach space $\mathcal{Y}$
generator $\mathbf{G}$ of the evolution semigroup

Dichotomy Theorem

$\mathbf{G}$ is Fredholm on $L_p(\mathbb{R};\mathcal{Y})$, $1 \leq p < \infty$

$$\iff \text{for some } b \geq a \text{ in } \mathbb{R}$$

(i) There exist dichotomies $\{P_x^-\}_{x\leq a}$ on $(-\infty, a]$ and $\{P_x^+\}_{x\geq b}$ on $[b, +\infty)$, and

(ii) The **node operator** $N(b,a)$

$$N(b,a) = (I - P_b^+)\Phi(b,a)\,|_{\ker P_a^-} : \ker P_a^- \to \ker P_b^+$$

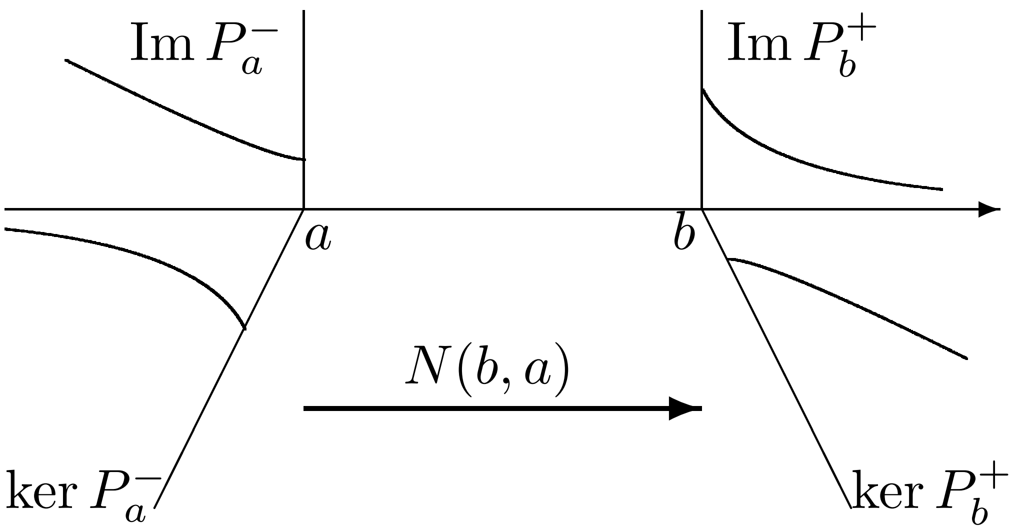
is Fredholm.

Also

$$\begin{aligned} \dim\ker \mathbf{G} &= \dim\ker N(b,a) \\ \operatorname{codim}\ \operatorname{Im}\mathbf{G} &= \operatorname{codim}\ \operatorname{Im} N(b,a) \\ \operatorname{Ind}\mathbf{G} &= \operatorname{Ind} N(b,a) \end{aligned}$$

$\mathbf{G} = \text{closure}\left(-\frac{d}{dx} + A(x)\right)$ is Fredholm

$$\Longleftrightarrow$$



Node Operator

$$N(b,a) = (I - P_b^+)\Phi(b,a)\,|_{\mathrm{Ker}\, P_a^-} : \mathrm{Ker}\, P_a^- \to \mathrm{Ker}\, P_b^+$$

Another form of the Dichotomy Theorem

$\mathbf{G}$ is Fredholm on $L_p(\mathbb{R};\mathcal{Y})$, $1 \le p < \infty$

$$\Longleftrightarrow$$

(i') There exist dichotomies $\{P_x^-\}_{x\le 0}$ on $\mathbb{R}_-$ and $\{P_x^+\}_{x\ge 0}$ on $\mathbb{R}_+$, and

(ii') The pair of subspaces $(\operatorname{Ker} P_0^-, \operatorname{Im} P_0^+)$ is Fredholm.

Also,

$$\begin{aligned} \dim \operatorname{Ker} \mathbf{G} &= \dim(\operatorname{Ker} P_0^-) \cap \operatorname{Im} P_0^+) \\ \operatorname{codim}\ \operatorname{Im} \mathbf{G} &= \operatorname{codim}\ (\ker P_0^- + \operatorname{Im} P_0^+) \\ \operatorname{Ind} \mathbf{G} &= \operatorname{Ind}(\operatorname{Ker} P_0^-, \operatorname{Im} P_0^+) \end{aligned}$$

Fredholm pair (W, V):

$$\dim(W \cap V) < \infty, \quad W + V \text{ is closed,}$$

$$\operatorname{codim}\ (W + V) < \infty,$$

$$\operatorname{Ind}(W, V) = \dim(W \cap V) - \operatorname{codim}\ (W + V)$$

Special Cases

Recall: $\mathbf{G}$ is Fredholm $\Longleftrightarrow$

(i) dichotomies on $\mathbb{R}_\pm$

(ii) node operator is Fredholm

$$N(0,0) = (I - P_0^+)\,|_{\mathrm{Ker}\,P_0^-} : \mathrm{Ker}\,P_0^- \to \mathrm{Ker}\,P_0^+$$

why (ii) is new?

Lemma *If $P_0^+ - P_0^-$ is compact then $N(0,0)$ is Fredholm*

Proof. Consider Fredholm operator

$$L = I - (P_0^+ - P_0^-) : \mathrm{Im}\,P_0^- \oplus \mathrm{Ker}\,P_0^- \to \mathrm{Im}\,P_0^+ \oplus \mathrm{Ker}\,P_0^+$$

$$L = \begin{bmatrix} P_0^+ P_0^- & 0 \\ * & N(0,0) \end{bmatrix} \text{ Fredholm } \Rightarrow N(0,0) \text{ is Fredholm}$$

If $A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$ is Fredholm then

(general fact, Litvinchuk/Spitkovskiy)

$\mathrm{Im}\,A_{22}$ is closed and $\dim \mathrm{Ker}\,A_{22} < \infty$

Connections to spectral flow

Atiyah-Patodi-Singer Theorem

Robbin-Salmon Spectral Flow=Index Theorem (1995)

Abbondandolo-Majer (2001,2003)

DiGiorgio-Lunardi-Schnaubelt (2003,2004)

Rabier (2004)

Consider $\{A(x)\}_{x=-\infty}^{\infty}$ operator path
$A(x)$ selfadjoint, $A_{\pm} = \lim_{x\to\pm\infty} A(x)$,
$B(x) = A(x) - A_0(x)$ is compact

$$A_0(x) = \begin{cases} A_+, & \text{if } x \geq 0; \\ A_-, & \text{if } x < 0; \end{cases}$$

Assume compact resolvents

Assume $0 \notin \sigma(A_{\pm})$.

Then $\mathbf{G}$ is Fredholm. Also

$$\begin{aligned} \text{spectral flow} &= (\#\text{ positive eigenvalues of } A_-) \\ &- (\#\text{ positive eigenvalues of } A_+) \\ &= \dim \operatorname{Ker} P_{A_-} - \dim \operatorname{Ker} P_{A_+} \\ &= \operatorname{Ind} \mathbf{G}_{A_0} = \operatorname{Ind} \mathbf{G}_A \end{aligned}$$

$$A(x) = A_0(x) + (A(x) - A_0(x))$$

General dichotomy theorem for $\frac{dy}{dt} = A(x)y(x)$
on Banach $\mathcal{Y}$

$\mathbf{G}$ is Fredholm $\Longleftrightarrow$

(i) Dichotomies on $\mathbb{R}_{\pm}$

(ii) $(\operatorname{Ker} P_0^-, \operatorname{Im} P_0^+)$ is Fredholm

Also, $\operatorname{Ind} \mathbf{G} = \operatorname{Ind}(\operatorname{Ker} P_0^-, \operatorname{Im} P_0^+)$

Particular case on Hilbert $\mathcal{Y}$

$A(x)$ are selfadjoint with compact resolvent,

$$A_{\pm} = \lim_{x \to \pm\infty} A(x), \quad \operatorname{Dom} A_+ = \operatorname{Dom} A_-$$

$B(x) = A(x) - A_0(x)$ is compact

$\mathbf{G}$ is Fredholm $\Longleftrightarrow$

$$0 \notin \sigma(A_{\pm}) \tag{$*$}$$

$$\operatorname{Ind} \mathbf{G} = \dim \operatorname{Ker} P_{A_-} - \dim \operatorname{Ker} P_{A_+} = \text{ spectral flow } \tag{$**$}$$

Why? $P_{A_{\pm}}$ are compact, so (ii) holds

Note (*) and (**) are “spectral" objects

(i) and (ii) are “geometric" objects

Connections to Morse Theory

$\mathcal{Y} = \mathbb{R}^d$, v heteroclinic solution

$$\frac{dv}{dx} = f(v(x)), \quad \begin{aligned} v_- &= \lim_{x\to-\infty} v(t) \\ v_+ &= \lim_{x\to+\infty} v(t) \end{aligned}$$

$f = -DF$ gradient vector field, F Morse functional

after linearization

$$Gu = -u' + A(x)u, \quad \begin{aligned} A(x) &= Df(v(x)) \\ A_\pm &= Df(v_\pm) \end{aligned}$$

$$\begin{aligned} \dim \operatorname{Ker} P_{A_+} &= \#\text{unstable eigenvalues } Df(v_+) \\ \dim \operatorname{Ker} P_{A_-} &= \#\text{unstable eigenvalues } Df(v_-) \end{aligned}$$

These are Morse indices of v_+ and v_-

Dichotomy Theorem:

$$\operatorname{Ind} G = \text{ Morse index } (v_-) - \text{ Morse index } (v_+)$$

Infinite dimensional setting:

$\mathcal{Y}$ =Hilbert

$F : \mathcal{Y} \to \mathbb{R}$ Morse functional (energy functional)

$v_\pm$ critical points of F

hyperbolic: $D^2F(v_\pm) \cap \{|z| = 1\} = \emptyset$

typical $F(v) = \frac{1}{2}\langle Av, v\rangle + b(v)$

where $A =$ 2nd order self-adjoint

$D^2b(v)$ =compact operator

So, formula

$$\operatorname{Ind}\mathbf{G} = \operatorname{Ind}(\operatorname{Ker} P_{A_-}, \operatorname{Im} P_{A_+})$$

is a generalization of

$$\operatorname{Ind}\mathbf{G} = \text{ difference of Morse indices}$$

Hyperbolicity is necessary

The Evans function: a definition via EDs

$y' = A(x)y$, $A = A(z, x)$ analytic in z, $(d \times d)$

Assume ED on $\mathbb{R}_+$ and $\mathbb{R}_-$ and also assume

$$k := \dim \operatorname{Im} Q_+ = \dim \operatorname{Im} Q_-$$

$Q_\pm = Q_\pm(z)$ are analytic

Select

$$\eta_1^+, \ldots, \eta_k^+ \text{ a basis in } \operatorname{Im} Q_+(z)$$
$$\eta_{k+1}^-, \ldots, \eta_d^- \text{ a basis in } \operatorname{Ker} Q_-(z)$$

Define the Evans function:

$$E(z) = \det[\eta_1^+ \ldots \eta_k^+ \eta_{k+1}^- \ldots \eta_d^-]$$

Then

- $E(z) = 0$ if and only if $\operatorname{Im} Q_+ \cap \operatorname{Ker} Q_- \neq \{0\}$
- $E(z) = 0$ if and only if there is a bounded solution
- $E(z) = 0$ if and only if $z \in \sigma_p(\mathcal{L})$

E depends on the bases. What is a natural choice?

Bohl and Lyapunov exponents

$y'(x) = A(x)y(x)$, $(d \times d)$, $A \in L^1_{\mathrm{loc}}$ on $\mathbb{R}$, on $\mathbb{R}_+$ or on $\mathbb{R}_-$

$\Phi(x)$ fundamental matrix solution, $\Phi(0) = I$

$\Phi(x, x') = \Phi(x)\Phi(x')^{-1}$ propagator
assumed to be exponentially bounded
P projector on $\mathbb{C}^d$

Bohl exponents

$$\varkappa(P) = \limsup_{x\to\infty, x'\to\infty} \frac{1}{x'} \log \|\Phi(x + x')P\Phi(x)^{-1}\| \text{ upper Bohl}$$

$$\varkappa'(P) = -\limsup_{x\to\infty, x'\to\infty} \frac{1}{x'} \log \|\Phi(x)P\Phi(x + x')^{-1}\| \text{ lower Bohl}$$

Lyapunov exponents

$$\lambda(P) = \limsup_{x\to\infty} \frac{1}{x} \log \|\Phi(x)P\| \text{ upper Lyapunov}$$

$$\lambda'(P) = -\limsup_{x\to\infty} \frac{1}{x} \log \|P\Phi(x)^{-1}\| \text{ lower Lyapunov}$$

Bohl and Lyapunov exponents (in other words)

$$\varkappa(P) = \inf\{\alpha \in \mathbb{R} : \sup_{x \geq x'} e^{-\alpha(x-x')} \|\Phi(x) P \Phi(x')^{-1}\| < \infty\}$$

$$\varkappa'(P) = \sup\{\alpha \in \mathbb{R} : \sup_{x \leq x'} e^{-\alpha(x-x')} \|\Phi(x) P \Phi(x')^{-1}\| < \infty\}$$

$$\lambda(P) = \inf\{\alpha \in \mathbb{R} : \sup_{x} e^{-\alpha x} \|\Phi(x) P\| < \infty\}$$

$$\lambda'(P) = \sup\{\alpha \in \mathbb{R} : \sup_{x} e^{\alpha x} \|P \Phi(x)^{-1}\| < \infty\}$$

Upper (lower) Bohl measures the best (worst)

exp growth of the propagator on $\operatorname{Im} P$

Upper (lower) Lyapunov measures the best (worst)

exp growth of the fundamental solution on $\operatorname{Im} P$

$$\varkappa'(P) \leq \lambda'(P) \leq \lambda(P) \leq \varkappa(P)$$

Generally strictly smaller (Perron!)

Similarly on $\mathbb{R}_\pm$: $\varkappa'^{\pm}(P) \leq \lambda'^{\pm}(P) \leq \lambda^{\pm}(P) \leq \varkappa^{\pm}(P)$

Exponential splitting

Disjoint projectors $Q_1, Q_2, \ldots, Q_{d'}$, $d \geq d'$, so that $Q_1 + \cdots + Q_{d'} = I$ form an *exponential splitting* for $y' = A(x)y$ on $\mathbb{R}$ if:
Bohl segments $[\varkappa'(Q_k), \varkappa(Q_k)]$ are disjoint

Bohl spectrum $\mathcal{B} = \cup_{k=1}^{d'}[\varkappa'(Q_k), \varkappa(Q_k)]$

- $\alpha \notin \mathcal{B}$ if and only if $y' = [A(x) - \alpha]y$ has ED
- $\mathcal{B} = \sigma\left(\frac{d}{dx} - A(x)\right) \cap \mathbb{R}$ (a theorem)
- $\mathcal{B} =$ Sacker Sell spectrum (on $\mathbb{R}$)

If $A \equiv A(x)$ autonomous then:
Q_k spectral projections for A
$\sigma(A \mid \operatorname{Im} Q_k) = \{\lambda \in \sigma(A) : \operatorname{Re}\lambda = \varkappa(Q_k)\}$
$\varkappa(Q_k) = \lambda(Q_k) = \lambda'(Q_k) = \varkappa'(Q_k)$

Similarly on $\mathbb{R}_\pm$.
If $\{Q_k\}_{k=1}^{d'}$ is a splitting on $\mathbb{R}$ then it also a splitting on $\mathbb{R}_\pm$

Dichotomy and L^1-perturbations

Assume $y' = A(x)y$ has exponential dichotomy Q_+ on $\mathbb{R}_+$

Consider $y' = [A(x) + R(x)]y$ with $\|R(\cdot)\| \in L^1(\mathbb{R}_+)$

Lemma 3 (Daleckij-Krein,Coppel). *There exists exponential dichotomy P_+ on $\mathbb{R}_+$ for $y' = [A(x) + R(x)]y$*

Even better:

$$\varkappa'^+(P_+) = \varkappa'^+(Q_+) \text{ and } \varkappa^+(I - P_+) = \varkappa'^+(I - Q_+)$$

Also

$$\dim \operatorname{Im} P_+ = \dim \operatorname{Im} Q_+, \quad \dim \operatorname{Ker} P_+ = \dim \operatorname{Ker} Q_+$$

Lemma 4. *Let $Q_1^+, \ldots, Q_{d'}^+$ be an exponential splitting for $y' = A(x)y$ on $\mathbb{R}_+$ and $\|R(\cdot)\| \in L^1(\mathbb{R}_+)$.*
Then there exists an exponential splitting $P_1^+, \ldots, P_{d'}^+$ for $y' = [A(x) + R(x)]y$ on $\mathbb{R}_+$ with the same Bohl spectrum:

$$\varkappa'^+(Q_k^+) = \varkappa'^+(P_k^+) \text{ and } \varkappa^+(Q_k^+) = \varkappa^+(P_k^+), \quad k = 1, \ldots, d'.$$

Also, $\dim \operatorname{Im} Q_k^+ = \dim \operatorname{Im} P_k^+$, $k = 1, \ldots, d'$.

Towards the Evans function

Assumptions: the unperturbed equation $y' = A(x)y$, $A \in L^\infty(\mathbb{R}; \mathbb{C}^{d\times d})$ has:

- exponentially bounded propagator $\Phi(x, x')$
- exponential dichotomy Q on $\mathbb{R}$
- exponential splitting $Q_1, \ldots, Q_{d'}$ so that

$$Q = \sum_{k=1}^{k_0} Q_k, \quad \varkappa(Q_1) < \cdots < \varkappa(Q_{k_0}) < 0$$

$$I - Q = \sum_{k=k_0+1}^{d'} Q_k, \quad \varkappa'(Q_{d'}) > \cdots > \varkappa'(Q_{k_0+1}) > 0$$

Perturbed equation $y' = [A(x) + R(x)]y$ has:
- for some $\delta > 0$

$$\int_0^\infty e^{(\lambda^+(Q_k) - \varkappa'^+(Q_k)+\delta)x} \|R(x)\|\, dx < \infty, \ k = 1, \ldots, k_0$$

$$\int_{-\infty}^0 e^{-(\varkappa^-(Q_k) - \lambda'^-(Q_k)+\delta)x} \|R(x)\|\, dx < \infty, \ k = k_0+1, \ldots, d'$$

Take the exponential splitting $P_1^\pm, \ldots, P_{d'}^\pm$ on $\mathbb{R}_\pm$ so that

$$\varkappa'^\pm(P_k^\pm) = \varkappa'^\pm(Q_k), \ \varkappa^\pm(P_k^\pm) = \varkappa^\pm(Q_k)$$

Jost solutions and the Evans determinant

Definition 1. *Matrix* $(d \times d)$ *solutions of* $Y' = [A(x) + R(x)]Y(x)$

$$Z_1^+, \ldots, Z_{k_0}^+ \quad \text{on } \mathbb{R}_+$$
$$Z_{k_0+1}^-, \ldots, Z_{d'}^- \quad \text{on } \mathbb{R}_-$$

are called the generalized Jost solutions *provided*

$$\limsup_{x \to +\infty} \frac{1}{x} \log \|Z_k^+(x) - \Phi(x)Q_k\| < \varkappa'^+(Q_k), \text{ and}$$
$$Z_k^+(0) = Z_k^+(0)Q_k, \quad k = 1, \ldots, k_0,$$
$$\liminf_{x \to -\infty} \frac{1}{x} \log \|Z_k^-(x) - \Phi(x)Q_k\| > \varkappa^-(Q_k), \text{ and}$$
$$Z_k^-(0) = Z_k^-(0)Q_k, \; k = k_0 + 1, \ldots, d'$$

Here: Φ and $\{Q_k\}_1^{d'}$ is the fundamental matrix solution and the *unique* exponential splitting for $y' = A(x)y$ on $\mathbb{R}$ (and hence on $\mathbb{R}_+$ and $\mathbb{R}_-$)
so that $\varkappa^\pm(Q_{k_0}) < 0 < \varkappa'^\pm(Q_{k_0+1})$.

Definition 2. *The Evans determinant* E *is defined by*

$$E = \det\left[\sum_{k=1}^{k_0} Z_k^+(0) + \sum_{k=k_0+1}^{d'} Z_k^-(0)\right]$$

Jost solutions: comments

- Jost solutions of $Y' = [A(x) + R(x)]Y$ are matrix solutions so that

$$\lambda^+(\|Z_k^+(\cdot) - \Phi(\cdot)Q_k\|) < \varkappa'^+(Q_k), \quad k = 1, \dots, k_0$$

$$\lambda'^-(\|Z_k^-(\cdot) - \Phi(\cdot)Q_k\|) > \varkappa^-(Q_k), \quad k = k_0 + 1, \dots, d'$$

  This means that $Z_k^\pm$ approximate solutions $\Phi(\cdot)Q_k$ of $y' = A(x)y$ up to terms of lower than k-th exponential order

- Condition $Z_k^\pm(0) = Z_k^\pm(0)Q_k$ means that $Z_k^\pm$ have many zero column

- Lyapunov exponents of the Jost solutions belong to k-th Bohl segment

- Jost solutions are unique up to the terms of lower exponential order: Z_k^+ and $\tilde{Z}_k^+$ are Jost solutions implies $Z_k^+(0) - \tilde{Z}_k^+(0)$ belongs to $\mathrm{Im} \sum_{j=1}^{k-1} P_j^+$, *does not* belong to $\sum_{j=1}^{k} P_j^+$

Lemma 5. *The Evans determinant*
$E = \det\left[\sum_{k=1}^{k_0} Z_k^+(0) + \sum_{k=k_0+1}^{d'} Z_k^-(0)\right]$ *does not depend on the choice of the Jost solutions, that is, all lower order terms can be dropped.*

Thus, the Evans determinant is uniquely determined by the equation $y' = [A(x) + R(x)]y(x)$.

Proof

$Z = \sum_{k=1}^{k_0} Z_k^+(0) + \sum_{k=k_0+1}^{d'} Z_k^-(0)$ as a block-matrix in

$$\operatorname{Im} Q \oplus \operatorname{Im}(I - Q) \to \operatorname{Im} \sum_{k=1}^{k_0} P_k^+ \oplus \operatorname{Im} \sum_{k=k_0+1}^{d'} P_k^-$$

is block-diagonal: $Z = \begin{bmatrix} Z_{11} & 0 \\ 0 & Z_{22} \end{bmatrix}$; also

$Z_{11} : \bigoplus_{k=1}^{k_0} \operatorname{Im} Q_k \to \bigoplus_{k=1}^{k_0} \operatorname{Im} P_k^+$ is upper-triangular

$Z_{22} : \bigoplus_{k=k_0+1}^{d'} \operatorname{Im} Q_k \to \bigoplus_{k=k_0+1}^{d'} \operatorname{Im} P_k^-$ is lower-triangular

Lemma 6. *There exists a unique choice of Jost solutions such that* $Z_k^{\pm}(0) \in \operatorname{Im} P_k^{\pm}$ *for the splitting* $\{P_k^{\pm}\}_{k=1}^{d'}$ *of* $y' = [A(x) + R(x)]y$ *with the same Bohl spectrum as* $\{Q_k\}_{k=1}^{d'}$.

Lemma 7. *Columns of* $Z^+(0)$ *form a basis in* $\operatorname{Im} P^+ = \operatorname{Im} \sum_{k=1}^{k_0} P_k^+$. *Columns of* $Z^-(0)$ *form a basis in* $\operatorname{Ker} P^- = \operatorname{Im} \sum_{k=k_0+1}^{d'} P_k^-$

Thus, the "new Evans" = "old Evans".

Jost solutions: existence

Recall assumption

$$\int_0^\infty e^{(\lambda^+(Q_k)-\varkappa'^+(Q_k)+\delta)x}\|R(x)\|dx < \infty, \quad k=1,\ldots,k_0.$$

WLOG $\delta < \frac{1}{2}\min_k\{\varkappa'(Q_k)-\varkappa(Q_{k-1})\}$

Consider equation

$$Z(x)-(T_\tau Z)(x)=\Phi(x)Q_k \text{ on } [\tau,\infty), \quad \tau>0,$$

where

$$(T_\tau Z)(x) = -\int_x^\infty \Phi(x)\left[\sum_{j=k}^{d'} Q_j\right]\Phi(x')^{-1}R(x')Z(x')dx'$$

$$+\int_\tau^x \Phi(x)\left[I-\sum_{j=k}^{d'} Q_j\right]\Phi(x')^{-1}R(x')Z(x')dx', \quad x\ge\tau$$

Lemma 8. *For τ large enough T_τ is a contraction on $C_b([\tau,\infty);e^{-(\lambda^+(Q_k)+\delta/2)x})$. Also,*

$$\sup_{x\ge\tau} e^{-(\varkappa'^+(Q_k)-\delta/2)x}\|Z(x)-\Phi(x)Q_k\|<\infty$$

for the unique solution of $Z-T_\tau Z=\Phi(\cdot)Q_k$.

Full system for Jost solutions

$$Z_1^+(x) - \Phi(x)Q_1 = -\int_x^\infty \Phi(x)\Phi(x')^{-1}R(x')Z_1^+(x')dx',$$

$$Z_k^+(x) - \Phi(x)Q_k = -\int_x^\infty \Phi(x)\left[\sum_{j=k}^{d'} Q_j\right]\Phi(x')^{-1}R(x')Z_k^+(x')dx$$

$$+\int_\tau^\infty \Phi(x)\left[I - \sum_{j=k}^{d'} Q_j\right]\Phi(x')^{-1}R(x')Z_k^+(x')dx',$$

$$x \geq \tau,\ k = 2, \ldots, k_0, \varkappa_{k_0} < 0,$$

$$Z_k^-(x) - \Phi(x)Q_k = \int_{-\infty}^x \Phi(x)\left[\sum_{j=1}^{k} Q_j\right]\Phi(x')^{-1}R(x')Z_k^-(x')dx'$$

$$-\int_x^{-\tau} \Phi(x)\left[I - \sum_{j=1}^{k} Q_j\right]\Phi(x')^{-1}R(x')Z_k^-(x')dx',$$

$$x \leq -\tau,\ k = k_0+1, \ldots, d'-1,\ \varkappa'_{k_0+1} > 0,$$

$$Z_{d'}^-(x) - \Phi(x)Q_{d'} = \int_{-\infty}^x \Phi(x)\Phi(x')^{-1}R(x')Z_{d'}^-(x')dx'.$$

Volterra-Fredholm mixture!

Dichotomies and operators

Recall assumptions

- $y' = A(x)y$ has ED Q on $\mathbb{R}$
- $\|R(\cdot)\| \in L^1(\mathbb{R})$

This implies

- $y' = [A(x) + R(x)]y$ has EDs $P_\pm$ on $\mathbb{R}_\pm$
- $\dim \operatorname{Im} P_+ = \dim \operatorname{Im} Q$, $\dim \operatorname{Ker} P_- = \dim \operatorname{Ker} Q$

Consider operator
$G_{A+R}u(\cdot) = u'(\cdot) - [A(\cdot) + R(\cdot)]u(\cdot)$ on $L^2(\mathbb{R}; \mathbb{C}^d)$.
Dichotomy Theorem. (Palmer, Ben-Artzi/Gohberg, Sandstede/Scheel, ... YL/Tomilov)

G_{A+R} is Fredholm if and only if $y' = [A(x) + R(x)]y$ has exponential dichotomies $P_\pm$ on $\mathbb{R}_\pm$. Also,
$\operatorname{Ind} G_{A+R} = \dim \operatorname{Im} P_+ - \dim \operatorname{Im} P_-$

Thus we know G_{A+R} is Fredholm and $\operatorname{Ind} G_{A+R} = 0$.

Either G_{A+R} is invertible or

$$\operatorname{Ker} G_{A+R} \neq \{0\}$$

that is, there exists a bounded solution of
$y' = [A(x) + R(x)]y(x)$.
Back to the "linearized" ODE operator $\mathcal{L}$:

$z \in \sigma_p(\mathcal{L})$ means $\mathcal{L}v = zv$ for a bounded v

means $y' = [A_z(\cdot) + R_z(\cdot)]y$ has a bounded solution

means $G_{A_z+R_z}$ is not invertible

Sandwiched resolvent

Recall that $y' = A(x)y,\ A \in L^\infty(\mathbb{R}; \mathbb{C}^{d\times d}\}$ has ED Q on $\mathbb{R}$.

Equivalently (a theorem!):

$G_A u(\cdot) = u'(\cdot) - A(\cdot)u(\cdot)$, $\operatorname{Dom} G_A = H^2(\mathbb{R}; \mathbb{C}^d)$ is invertible and $G_A^{-1}u(x) = \int_{\mathbb{R}} K(x,x')u(x')dx'$ where

$$K(x,x') = \begin{cases} \Phi(x)Q\Phi(x')^{-1}, & x > x' \\ -\Phi(x)(I-Q)\Phi(x')^{-1}, & x < x' \end{cases}$$

Use polar decomposition $R(x) = V_{R(x)}|R(x)|$ to write

$$R(x) = R_\ell(x)R_r(x)$$

Lemma 9. *If $\|R(\cdot)\| \in L^1(\mathbb{R})$ then the "sandwiched resolvent"*

$$H = R_r G_A^{-1} R_\ell$$

is a Hilbert-Schmidt integral operator with the kernel $R_r(x)K(x,x')R_\ell(x')$.

Modified Fredholm determinant

$$\det{}_2[I-H] = \det[(I-H)e^H] = \prod_{\lambda\in\sigma(H)} (1-\lambda)e^\lambda$$

Dichotomies and sandwiched resolvents

Since $y' = A(x)y$ has ED on $\mathbb{R}$ if and only if $G_A u = u' - A(\cdot)u$ is invertible,

Then $G_{A+R}u = u' - [A(\cdot) + R(\cdot)]u$ satisfies

$$G_{A+R} = G_A - R(\cdot) = G_A(I - G_A^{-1}R)$$
$$= G_A(I - G_A^{-1}R_\ell \cdot R_r)$$

because $R = R_\ell \cdot R_r$.

Then G_{A+R} is not invertible (has nontrivial kernel) if and only if

$$1 \in \sigma_p(R_r G_A^{-1} R_\ell) = \sigma \text{ (sandwiched resolvent)}$$

Conclusion: $y' = [A(x) + R(x)]y$ has a bounded on $\mathbb{R}$ solution if and only if

$$\det{}_2(I - H) = 0, \quad H = R_r G_A^{-1} R_\ell$$

Evans determinant $E = 0$ if and only $y' = [A(x) + R(x)]y$ has a bounded on $\mathbb{R}$ solutions.

So E and $\det_2(I - H)$ must be related.

The main result

Assume:

- $y' = A(x)y$ has ED Q on $\mathbb{R}$, $\{Q_k\}_{k=1}^{d'}$ exp splitting $\Phi(x)$ fundamental matrix solution
- $\|R(\cdot)\|$ is in L^1 with the exponential weight

$$e^{(\lambda^+(Q_k)-\varkappa'^+(Q_k)+\delta)x}, x \geq 0, k = 1, \dots, k_0$$

$$e^{-(\varkappa^-(Q_k)-\lambda'^-(Q_k)+\delta)x}, x \leq 0, k = k_0 + 1, \dots, d'$$

Define the Evans determinant

$$E = \det \left[\sum_{k=1}^{k_0} Z_k^+(0) + \sum_{k=k_0+1}^{d'} Z_k^-(0) \right]$$

Define the sandwiched resolvent $H = R_r G_A^{-1} R_\ell$

$$G_A u = u' - A(\cdot)u, \quad R(x) = R_\ell(x) R_r(x)$$

Theorem. Evans=Fredholm

$$\det{}_2(I - H) = \theta E,$$

where

$$\theta = \exp\left(\int_0^\infty \mathrm{tr}[\Phi(x) Q \Phi(x)^{-1} R(x)] dx - \int_{-\infty}^0 \mathrm{tr}[\Phi(x)(I - Q)\Phi(x)^{-1} R(x)] \right) dx$$

Plan of the proof

Want $\det_2(I-H) = \theta E$

Truncate: $R^{(n)}(x) = \begin{cases} R(x), & |x| \le n \\ 0, & |x| > n \end{cases}$ and let $n \to \infty$.

Then $H^{(n)} = R_r^{(n)} G_A^{-1} R_\ell^{(n)} \to H$ in Hilbert-Schmidt norm

Thus $\det_2(I-H^{(n)}) \to \det_2(I-H)$ and also

$$\theta^{(n)} = \exp\Big(\int_0^\infty \mathrm{tr}[\Phi Q \Phi^{-1} R^{(n)}]dx \\ - \int_{-\infty}^0 \mathrm{tr}[\Phi(I-Q)\Phi^{-1}]R^{(n)}\Big)dx \to \theta$$

Need to see $E^{(n)} \to E$ or $Z_k^{(n)} \to Z_k$ for Jost solutions $Z_k^{(n)}$ of $y' = [A(x) + R^{(n)}(x)]y$

But Jost solutions satisfy integral equations

$$Z_k - T_\tau Z_k = \Phi(\cdot) Q_k$$

For contractions $T_\tau, T_\tau^{(n)}$ check $(I - T^{(n)})^{-1} \to (I - T_\tau)^{-1}$

Compactly supported perturbation

If $\operatorname{supp} R$ is compact, then the integral equations

$$Z_k - T_\tau Z_k = T(\cdot)Q_k, \quad k = 1, \ldots, d'$$

for the Jost solutions become Volterra equations (after exp lower order terms are dropped)

$$Z^+(x) = \Phi(x)Q - \int_x^\infty \Phi(x)\Phi(x')^{-1}R(x')Z^+(x')dx',$$

$$Z^-(x) = \Phi(x)(I-Q) + \int_{-\infty}^x \Phi(x)\Phi(x')^{-1}R(x')Z^-(x)dx',$$

where

$$Z^+(x) = \sum_{k=1}^{k_0} Z_k^+(x) \text{ and } Z^-(x) = \sum_{k=k_0+1}^{d'} Z_k^-(x)$$

Recall that $H = R_r G_A^{-1} R_\ell$ is an integral operator with a semi-separable kernel.

Recent papers:

[Gesztesy/Makarov], [Gohberg/Kaashoek/Krupnik]

relate $\det_2(I-H)$ and solutions of the Volterra equations.

Gesztesy/Makarov and Gohberg/Kaashoek/Krupnik

$$\mathcal{H}(x,x') = \begin{cases} f_1(x)g_1(x'), & x' < x \\ f_2(x)g_2(x'), & x \le x' \end{cases}$$

where $f_j \in L^2(\mathbb{R}, \mathbb{C}^{d\times d_j})$, $g_j \in L^2(\mathbb{R}, \mathbb{C}^{d_j\times d})$

Theorem.

$$\begin{aligned} \det_2(I-H) &= \det U(a) \exp \int_{-\infty}^{\infty} \operatorname{tr}(f_1(x)g_1(x))dx \\ &= \det U(b) \exp \int_{-\infty}^{\infty} \operatorname{tr}(f_2(x)g_2(x))dx \end{aligned}$$

where

$$U(x) = \begin{bmatrix} I - \int_x^\infty g_1(x)\hat{f}_1(x)dx & \int_{-\infty}^x g_1(x)\hat{f}_2(x)dx \\ \int_x^\infty g_2(x)\hat{f}_1(x)dx & I - \int_{-\infty}^x g_2(x)\hat{f}_2(x)dx \end{bmatrix}$$

and $\hat{f}_i$ *solve the Volterra equations*

$$\hat{f}_1(x) = f_1(x) - \int_x^\infty \mathcal{S}(x,x')\hat{f}_1(x')dx'$$

$$\hat{f}_2(x) = f_2(x) + \int_{-\infty}^x \mathcal{S}(x,x')\hat{f}_2(x')dx'$$

with $\mathcal{S}(x,x') = f_1(x)g_1(x') - f_2(x)g_2(x')$

Semi-separable kernels: why does it work?

$(Hu)(x) = \int_{-\infty}^{\infty} \mathcal{H}(x,x')u(x')dx'$ with semi-separable

$$\mathcal{H}(x,x') = \begin{cases} f_1(x)g_1(x'), & x' < x \\ f_2(x)g_2(x'), & x < x' \end{cases}$$

Then

$$(Hu)(x) = \int_{-\infty}^{x} f_1(x)g_1(x')u(x')dx' + \int_{x}^{\infty} f_2(x)g_2(x')u(x')dx'$$

$$= \int_{-\infty}^{x} \mathcal{S}(x,x')u(x')dx' + \int_{-\infty}^{\infty} f_2(x)g_2(x')u(x')dx'$$

where $\mathcal{S}(x,x') = f_1(x)g_1(x') - f_2(x)g_2(x')$

In other words $H = S + F_2G_2$ where
$Su(x) = \int_{-\infty}^{x} \mathcal{S}(x,x')u(x')dx'$ Volterra

$$(F_2\mathbf{u})(x) = f_2(x)\mathbf{u},\ F_2 : \mathbb{C}^{d_2} \to L^2(\mathbb{R};\mathbb{C}^d)$$

$$G_2u = \int_{-\infty}^{\infty} g_2(x')u(x')du',\ G_2 : L^2(\mathbb{R};\mathbb{C}^d) \to \mathbb{C}^{d_2}$$

Semi-separable: why it works

$H = S + F_2 G_2$ where:

S Volterra, G_2 finite rank

$$\begin{aligned}\det(I - H) &= \det(I - S + F_2 + G_2)\\ &= \det(I - S)\det(I + (I - S)^{-1} F_2 G_2)\\ &= \det(I + (I - S)^{-1} F_2 G_2)\\ &= \det(I + G_2 (I - S)^{-1} F_2)\end{aligned}$$

finite dimensional determinant!

Similarly for $\det_2(I - H)$

Similarly

$$(Hu)(x) = \int_{-\infty}^{x} f_1(x)g_1(x')u(x')dx' + \int_{x}^{\infty} f_2(x)g_2(x')u(x')dx'$$
$$+ \int_{x}^{\infty} f_1(x)g_1(x')u(x')dx' - \int_{x}^{\infty} f_1(x)g_1(x')u(x')dx'$$

implies $H = S' + F_1G_1$ and

$$\det(I - H) = \det(I + G_1(I - S')^{-1}F_1)$$

$$(S'u)(x) = -\int_{x}^{\infty} \mathcal{S}(x, x')u(x')dx'$$
$$(F_1\mathbf{u})(x) = f_1(x)\mathbf{u}, \ F_1 : \mathbb{C}^{d_1} \to L^2(\mathbb{R}; \mathbb{C}^d)$$
$$G_1u = \int_{-\infty}^{\infty} g_2(x')u(x')dx', \ G_2 : L^2(\mathbb{R}; \mathbb{C}^d) \to \mathbb{C}^{d_1}$$

Determinants

$$\begin{aligned}\det(I-H) =& \det(I+G_2(I-S)^{-1}F_2)\\ =& \det(I+G_1(I-S')^{-1}F_1)\end{aligned}$$

can be expressed via solutions $\hat{f}_1$ and $\hat{f}_2$ of the Volterra equations

$$(I-S')\hat{f}_1 = f_1 \text{ and } (I-S)\hat{f}_2 = f_2$$

or

$$\begin{aligned}\hat{f}_1(x) =& f_1(x) - \int_x^\infty \mathcal{S}(x,x')\hat{f}_1(x')dx'\\ \hat{f}_2(x) =& f_2(x) + \int_{-\infty}^x \mathcal{S}(x,x')\hat{f}_2(x')dx'\end{aligned}$$

Gesztesy-Makarov works because

$$G_1(I - S')^{-1}F_1 = \int_{-\infty}^{\infty} g_1(x')\hat{f}_1(x')dx'$$

$$G_2(I - S)^{-1}F_2 = \int_{-\infty}^{\infty} g_2(x')\hat{f}_2(x')dx'$$

$$U(-\infty) = \begin{bmatrix} I - G_1(I - S')^{-1}F_1 & 0 \\ \int_{-\infty}^{\infty} g_2(x')\hat{f}_1(x')dx' & I \end{bmatrix}$$

$$U(+\infty) = \begin{bmatrix} I & \int_{-\infty}^{\infty} g_1(x')\hat{f}_2(x')dx' \\ 0 & I - G_2(I - S)^{-1}F_2 \end{bmatrix}$$

Semi-separable vs Evans

Jost solutions $Z^{\pm}(x)$ satisfy Volterra equations

$$
\begin{aligned}
Z^{+}(x) =& \Phi(x)Q - \int_x^{\infty} \Phi(x)\Phi(x')^{-1}R(x')Z^{+}(x')dx' \\
Z^{-}(x) =& \Phi(x)(I-Q) + \int_{-\infty}^{x} \Phi(x)\Phi(x')^{-1}R(x')Z^{-}(x')dx'
\end{aligned}
$$

Recall that $Z^{+}(x) = Z^{+}(x)Q$ and $Z^{-}(x) = Z^{-}(x)(I-Q)$
The Evans determinant

$$
\begin{aligned}
E =& \det[Z^{+}(0) + Z^{-}(0)] \\
=& \det[U^{+}(0) + U^{-}(0)]
\end{aligned}
$$

where $U^{\pm}(x) = \Phi(x)^{-1}Z^{\pm}(x)$ satisfy Volterra equations

$$
\begin{aligned}
U^{+}(x) =& Q - \int_x^{\infty} \Phi(x')^{-1}R_{\ell}(x') \cdot R_r(x')\Phi(x')U^{+}(x')dx' \\
U^{-}(x) =& (I-Q) + \int_{-\infty}^{x} \Phi(x')^{-1}R_{\ell}(x') \cdot R_r(x')\Phi(x')U^{-}(x')dx'
\end{aligned}
$$

Evans vs Gesztesy/Makarov:

Define $U(x) = U^+(x) + U^-(x)$ so that $E = \det U(0)$
Use $U^+(x) = U^+(x)Q$ and $U^-(x) = U^-(x)(I - Q)$ to write $U(x)$ in the decomposition $\mathbb{C}^d = \operatorname{Im} Q \oplus \operatorname{Im}(I - Q)$ as

$$U(x) = \begin{bmatrix} Q - \int_x^\infty g_1(x')\hat{f}_1(x')dx' & \int_{-\infty}^x g_1(x')\hat{f}_2(x')dx' \\ \int_x^\infty g_2(x')\hat{f}_1(x')dx' & (I-Q) - \int_{-\infty}^x g_2(x')\hat{f}_2(x')dx' \end{bmatrix}$$

where we denote

$$\begin{aligned} g_1(x) =& Q\Phi(x)^{-1}R_\ell(x) : \mathbb{C}^d \to \operatorname{Im} Q \\ g_2(x) =& -(I - Q)\Phi(x)^{-1}R_\ell(x) : \mathbb{C}^d \to \operatorname{Im}(I - Q) \\ \hat{f}_1(x) =& R_r(x)\Phi(x)U^+(x)Q : \operatorname{Im} Q \to \mathbb{C}^d \\ \hat{f}_2(x) =& R_r(x)\Phi(x)U^-(x)(I - Q) : \operatorname{Im}(I - Q) \to \mathbb{C}^d \end{aligned}$$

This is the Gesztesy/Makarov $U(x)$!

Remark that $U(x)$ is defined using
$\hat{f}_1(x) = R_r(x)Z^+(x)Q$ and $\hat{f}_2(x) = R_r(x)Z^-(x)(I-Q)$
They solve

$$\begin{aligned}\hat{f}_1(x) =& f_1(x) - \int_x^\infty \mathcal{S}(x,x')\hat{f}_1(x')dx' \\ \hat{f}_2(x) =& f_2(x) + \int_{-\infty}^x \mathcal{S}(x,x')\hat{f}_2(x')dx'\end{aligned}$$

where $\mathcal{S}(x,x') = R_r(x)\Phi(x)\Phi(x')^{-1}R_\ell(x')$

$$\begin{aligned}f_1(x) =& R_r(x)\Phi(x)Q : \operatorname{Im} Q \to \mathbb{C}^d \\ f_2(x) =& R_r(x)\Phi(x)(I-Q) : \operatorname{Im}(I-Q) \to \mathbb{C}^d\end{aligned}$$

Semi-separable:

Sandwiched resolvent $H = R_r G_A^{-1} R_\ell$ has the kernel

$$\mathcal{H}(x,x') = \begin{cases} R_r(x)\Phi(x)Q\cdot Q\Phi(x')^{-1}R_\ell(x'), & x>x' \\ -R_r(x)\Phi(x)(I-Q)\cdot(I-Q)\Phi(x')^{-1}R_\ell(x'), & x<x' \end{cases}$$

$$= \begin{cases} f_1(x)\cdot g_1(x'), & x>x' \\ f_2(x)\cdot g_2(x'), & x<x' \end{cases}$$

By Gesztesy-Makarov, $\det_2(I-H)$ and $\det U(x)$ are related. Also, $U(x)$ satisfies

$$\frac{dU}{dx} = \begin{bmatrix} g_1(x)f_1(x) & g_1(x)f_2(x) \\ -g_2(x)f_1(x) & -g_2(x)f_2(x) \end{bmatrix} U(x)$$

Using that $\det U(0) = E$ (the Evans determinant), and Liouville's-Abel formula we have the required assertion

$$\det_2(I-H) = \theta E,$$

where θ contains the traces

Constant A: Evans vs Fredholm

$y' = Ay$ on $\mathbb{R}$

$A =$constant $(d \times d)$ =sum of Jordan blocks

$$\begin{bmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{bmatrix} \text{ or } \begin{bmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & \ddots & \lambda \end{bmatrix}$$

$m =$ max size of nondiagonal Jordan block

split eigenvalues $\sigma(A) = \cup_{k=1}^{d'} \sigma_k$

eigenvalues λ's in σ_k have the same $\operatorname{Re} \lambda =: \varkappa_k$

$\varkappa_k$ =Bohl exponents=Lyapunov exponents

Bohl projectors Q_k =spectral projections of A with $\sigma(A\,|_{\operatorname{Im} Q_k}) = \sigma_k$

Assume $\sigma(A) \cap i\mathbb{R} = \emptyset$
that is dichotomy $y' = Ay$ on $\mathbb{R}$:

$$\varkappa_1 < \cdots < \varkappa_{k_0} < 0 < \varkappa_{k_0+1} < \cdots < \varkappa_{d'}$$

Constant A: assume less

Matrix Jost solutions of $Z' = [A + R(x)]Z$ satisfy Volterra-Fredholm

$$Z_k^+(x) - e^{xA}Q_k = -\int_x^\infty e^{(x-x')A} \left[\sum_{j=k}^{d'} Q_j\right] R(x')Z_k^+(x')dx'$$

$$+ \int_\tau^\infty e^{(x-x')A} \left[I - \sum_{j=k}^{d'} Q_j\right] R(x')Z_k^+(x')dx',$$

$$k = 1, \ldots, k_0, \varkappa_{k_0} < 0$$

$$Z_k^-(x) - e^{xA}Q_k = \int_{-\infty}^x e^{(x-x')A} \left[\sum_{j=1}^{k} Q_j\right] R(x')Z_k^-(x')dx'$$

$$- \int_x^{-\tau} e^{(x-x')A} \left[I - \sum_{j=1}^{k} Q_j\right] R(x')Z_k^-(x')dx',$$

$$k = k_0 + 1, \ldots, d', \varkappa_{k_0+1} > 0$$

Constant A: the result

Jost solutions exist provided

$$\int_{-\infty}^{\infty} |x|^m \|R(x)\| dx < \infty \qquad (*)$$

where m is the maximal size of nondiagonal Jordan blocks
No need in exponential decay of $\|R(x)\|$ as in

$$\int_0^{\infty} e^{(\lambda^+(Q_k) - \varkappa'^+(Q_k) + \delta)x} \|R(x)\| dx < \infty$$

Need only $\|R(\cdot)\| \in L^1(\mathbb{R})$ if all Jordan blocks are diagonal

Theorem. *Evans=Fredholm provided:*

$$A = \text{constant and } (*) \text{ holds}$$

Difference equations

Unperturbed equation

$y_{j+1} = A_j y_j$, $j \in \mathbb{Z}$, $y_j \in \mathbb{C}^d$, A_j are $(d \times d)$ matrices

Perturbed equation

$y_{j+1} = A_j^{\times} y_j$, $j \in \mathbb{Z}$,
$A_j^{\times} = A_j + B_j C_j$, B_j and C_j are matrices

Assumptions:

A_j and $A_j^{\times}$ are invertible

Fundamental matrix solutions

$$\Phi_j = \text{product of } A_j \text{ and } A_j^{-1},\ \Phi_0 = I$$

$$\Psi_j = \text{product of } A_j^{\times} \text{ and } (A_j^{\times})^{-1},\ \Psi_0 = I$$

Note: propagator $\Phi(x, x') = \Phi(x)\Phi(x')^{-1}$ for $y' = A(x)y$ with bounded $A(x)$

Also assume:

Exponential dichotomy for $y_{j+1} = A_j y_j$ on $\mathbb{Z}$

Bohl exponents: similar to differential equations

Gesztesy/Makarov vs difference equations

Assume: (A_j), $(A_j^{-1}) \in \ell^\infty(\mathbb{Z};\mathbb{C}^{d\times d})$

$y_{j+1} = A_j y_j$ has exponential dichotomy Q on $\mathbb{Z}$

$(C_j\Phi_j),\ (\Phi_{j+1}^{-1}B_j) \in \ell^2(\mathbb{Z},\mathbb{C}^{d\times d})$

$A_j^\times = A_j + B_jC_j$ is invertible for each $j \in \mathbb{Z}$

On $\ell^2(\mathbb{Z};\mathbb{C}^d)$ consider difference operator

$$H : (y_j)_{j\in\mathbb{Z}} \mapsto \left(\sum_{k=-\infty}^{\infty} \mathcal{H}_{jk} y_k\right)_{j\in\mathbb{Z}}$$

with a semi-separable kernel

$$\mathcal{H}_{jk} = \begin{cases} C_j\Phi_jQ\Phi_{k+1}^{-1}B_k, & j > k \\ -C_j\Phi_j(I-Q)\Phi_{k+1}^{-1}B_k, & j \le k \end{cases}$$

Main identity

$$\begin{aligned}(H\mathbf{y})_j &= \sum_{k=-\infty}^{j-1} C_j\Phi_jQ\Phi_{k+1}^{-1}B_ky_k - \sum_{k=j}^{\infty} C_j\Phi_j(I-Q)\Phi_{k+1}^{-1}B_ky_k \\ &= \sum_{k=-\infty}^{j-1} C_j\Phi_j\Phi_{k+1}^{-1}B_ky_k - \sum_{k=-\infty}^{\infty} C_j\Phi_j(I-Q)\Phi_{k+1}^{-1}B_ky_k \\ &= -\,C_j\Phi_j\Phi_{j+1}^{-1}B_jy_j \\ &\quad - \sum_{k=j+1}^{\infty} C_j\Phi_j\Phi_{k+1}^{-1}B_ky_k + \sum_{k=-\infty}^{\infty} C_j\Phi_jQ\Phi_{k+1}^{-1}B_ky_k\end{aligned}$$

Theorem.

$$\begin{aligned}\det\nolimits_2(I-H) &= \textit{finite dimensional determinant}\\ &\qquad \textit{involving solutions of Volterra}\\ &\quad \times \exp \sum_{j\in\mathbb{Z}} -\operatorname{tr}(I-Q)\Phi_{j+1}^{-1}B_jC_j\Phi_j(I-Q)\\ &= \prod_{j\in\mathbb{Z}} \det A_j^{-1} \det A_j^{\times}\\ &\quad \times \textit{ finite dimensional determinant}\\ &\qquad \textit{involving solutions of Volterra}\\ &\quad \times \exp \sum_{j\in\mathbb{Z}} -\operatorname{tr}(I-Q)\Phi_{j+1}^{-1}B_jC_j\Phi_j(I-Q)\end{aligned}$$

Evans determinant E : similarly via Jost solutions

Theorem (Evans=Fredholm). *Assume*

$$\sum_{j=0}^{\infty} e^{(\varkappa_k^+ - {\varkappa'}_k^- + \delta)j} \|B_j C_j\| < \infty,\ k = 1, \dots, k_0$$

$$\sum_{j=-\infty}^{\infty} e^{-(\varkappa_k^- - {\varkappa'}_k^- + \delta)j} \|B_j C_j\| < \infty,\ k = k_0 + 1, \dots, d'$$

Then $E = \theta \mathrm{det}_2(I - H)$, *where* $\theta = \exp(\dots)$ *contains traces.*

Similarly for $A \equiv A_j$ assume less

Schrödinger equations

Eigenvalue problem on $L^2(\mathbb{R})$

$$-v'' + V(x)v = zv,\ z \in \mathbb{C} \setminus \mathbb{R}_+,\ x \in \mathbb{R},\ V \in L^1(\mathbb{R};\mathbb{R})$$

Re-write for $y(x) = [v(x) \quad v'(x)]^T$ as

$$y' = [A + R(x)]y$$

$$A = \begin{bmatrix} 0 & 1 \\ -z & 0 \end{bmatrix} \text{ (depends on } z\text{) and } R(x) = \begin{bmatrix} 0 & 0 \\ V(x) & 0 \end{bmatrix}$$

Eigenvalues of $A = A_z$

$$\sigma(A) = \{iz^{\frac{1}{2}}, -iz^{\frac{1}{2}}\},\ \operatorname{Im} z^{\frac{1}{2}} \geq 0$$

assuming $z \notin [0, \infty)$ we have $\operatorname{Re}(iz^{\frac{1}{2}}) < 0$
So $y' = Ay$ has exponential dichotomy Q on $\mathbb{R}$
$Q = Q_z$ spectral projection for A with $\sigma(A\,|_{\operatorname{Im} Q}) = \{iz^{\frac{1}{2}}\}$
Bohl-Lyapunov exponents

$$\varkappa_1 = \operatorname{Re}(iz^{\frac{1}{2}}) < 0 < \varkappa_2 = \operatorname{Re}(-iz^{\frac{1}{2}})$$

Evans vs Jost

Evans $E = \det[Z^+(0) + Z^-(0)]$ where $Z^\pm$ satisfy

$$Z^+(x) - e^{xA}Q = -\int_x^\infty e^{(x-x')A}R(x')Z^+(x')dx',$$

$$Z^-(x) - e^{xA}(I-Q) = \int_{-\infty}^x e^{(x-x')A}R(x')Z^-(x')dx;$$

where $A = \begin{bmatrix} 0 & 1 \\ -z & 0 \end{bmatrix}$, $R(x) = \begin{bmatrix} 0 & 0 \\ V(x) & 0 \end{bmatrix}$

use $W = \begin{bmatrix} 1 & 1 \\ iz^{\frac{1}{2}} & -iz^{\frac{1}{2}} \end{bmatrix}$, $W^{-1} = \frac{1}{2iz^{\frac{1}{2}}} \begin{bmatrix} iz^{\frac{1}{2}} & 1 \\ iz^{\frac{1}{2}} & -1 \end{bmatrix}$

to diagonalize $A = A_z$:

$$\tilde{A} = W^{-1}AW = \begin{bmatrix} iz^{\frac{1}{2}} & 0 \\ 0 & -iz^{\frac{1}{2}} \end{bmatrix}, \quad e^{x\tilde{A}} = \begin{bmatrix} e^{iz^{\frac{1}{2}}x} & 0 \\ 0 & e^{-iz^{\frac{1}{2}}x} \end{bmatrix}$$

$$\tilde{Q} = W^{-1}QW = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\tilde{R} = W^{-1}RW = \frac{1}{2iz^{\frac{1}{2}}}V(x)S, \quad S = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

Change of variables $\tilde{Z}^{\pm}(x) = W^{-1}Z^{\pm}(x)W$

Conditions $\tilde{Z}^{+}\tilde{Q} = \tilde{Z}^{+}$ and $\tilde{Z}^{-}(I - \tilde{Q}) = \tilde{Z}^{-}$ imply

$$\tilde{Z}^{+}(x) = \begin{bmatrix} u_1^{+}(x) & 0 \\ u_2^{+}(x) & 0 \end{bmatrix}, \ \tilde{Z}^{-}(x) = \begin{bmatrix} 0 & u_1^{-}(x) \\ 0 & u_2^{-}(x) \end{bmatrix}$$

Going back to

$$y^{\pm}(x) = \begin{bmatrix} f^{\pm}(x) \\ f^{\pm\prime}(x) \end{bmatrix} = W \begin{bmatrix} u_1^{\pm}(x) \\ u_2^{\pm}(x) \end{bmatrix}$$

The system for the (generalized) Jost solutions $Z^{\pm}$ is equivalent to

$$f^{+}(x) = e^{iz^{\frac{1}{2}}x} - \int_x^{\infty} \left[z^{-\frac{1}{2}} \sin(z^{\frac{1}{2}}(x - x'))\right] V(x')f^{+}(x')dx'$$

$$f^{-}(x) = e^{-iz^{\frac{1}{2}}x} + \int_{-\infty}^{x} \left[z^{-\frac{1}{2}} \sin(z^{\frac{1}{2}}(x - x')\right] V(x')f^{-}(x')dx'$$

Also, $\det W = -2iz^{\frac{1}{2}}$ implies

Wronskian $(f^{+}, f^{-}) = -2iz^{\frac{1}{2}} \det[Z^{+}(0) + Z^{-}(0)]$

Jost function $J = -\dfrac{\text{Wronskian } (f^{+}, f^{-})}{2iz^{\frac{1}{2}}}$

Evans function $E = \det[Z^{+}(0) + Z^{-}(0)]$

We proved Jost = Evans!

Scattering theory

Consider Schrödinger $v'' + k^2 v = V(x)v$,
V real potential, $\int_{-\infty}^{\infty}(1+|x|)|V(x)|dx < \infty$

For brevity $k = z^{\frac{1}{2}}$, $\operatorname{Im} k \geq 0$.

Plane waves $e^{\pm ikx}$ satisfy $v'' + k^2 v = 0$

Seek scattering solutions $\psi_1(k,x)$ and $\psi_2(k,x)$ of $v'' + k^2 v = V(x)v$ such that

$$\psi_1(k,x) \sim \begin{cases} e^{ikx} + s_{12}(k)e^{-ikx}, & x \to -\infty \\ s_{11}(k)e^{ikx}, & x \to +\infty \end{cases}$$

$$\psi_2(k,x) \sim \begin{cases} s_{22}(k)e^{-ikx}, & x \to -\infty \\ e^{-ikx} + s_{21}(k)e^{ikx}, & x \to +\infty \end{cases}$$

$s_{12}(k)e^{-ikx}$ part of e^{ikx} reflected to the left

$s_{12}(k) = R(k)$ reflection coefficient

$s_{11}(k)e^{ikx}$ part of e^{ikx} transmitted to the right

$s_{11}(k) = T(k)$ transmission coefficient

$S = [s_{ij}(k)]$ is called the S-matrix

Jost solutions

Jost solutions $f_\pm(k,x)$ solve Volterra equations and

$$|f_+(k,x) - e^{ikx}| = O\Big(|k|^{-1}\Big),\ |k| \to \infty,\ \mathrm{Im}\, k > 0$$
$$|f_-(k,x) - e^{-ikx}| = O\Big(|k|^{-1}\Big),\ |k| \to \infty,\ \mathrm{Im}\, k < 0$$

Denote $f_+(-k,x) = \overline{f_+(k,x)}$ and $f_-(-k,x) = \overline{f_-(k,x)}$
For $k \neq 0$, $k \in \mathbb{R}$, solutions

$$f_+(k,x),\ f_+(-k,x) \text{ are fundamental}$$
$$f_-(k,x),\ f_-(-k,x)\text{are fundamental}$$
$$\text{Wronskian } (f_\pm(k,x), f_\pm(-k,x)) = \mp 2ik \neq 0$$

Then

$$f_-(k,x) = c_{11}(k) f_+(k,x) + c_{12}(k) f_+(-k,x)$$
$$f_+(k,x) = c_{22}(k) f_-(k,x) + c_{21}(k) f_-(-k,x)$$

Here
$c_{12}(k) =$Wronskian $(f_-(k,x), f_+(k,x))/2ik = c_{21}(k)$
is called the Jost function (and as we know, it is equal to the Evans function)

Transmission coefficient

chose s_{11} and s_{12} so that

$$\begin{aligned}\psi_1(k,x) :=& f_-(-k,x) + s_{12}(k) f_-(k,x)\\ =& s_{11}(k) f_+(k,x) \end{aligned} \qquad (*)$$

Then

$$\begin{aligned} f_-(-k,x) =& \overline{f_-(k,x)} \sim \overline{e^{-ikx}} = e^{ikx},\ x \to -\infty \\ f_+(k,x) \sim& e^{ikx},\ x \to +\infty,\ f_-(k,x) \sim e^{-ikx},\ x \to -\infty \end{aligned}$$

gives

$$\begin{aligned}\psi_1(k,x) \sim& e^{ikx} + s_{12}(k) e^{-ikx},\ x \to -\infty \\ \sim& s_{11}(k) e^{ikx},\ x \to +\infty \end{aligned}$$

Thus $\psi_1(k,x)$ is a scattering solution

express $f_+(k,x) = c_{22}(k) f_-(k,x) + c_{21}(k) f_-(-k,x)$

Then $(*)$ is possible only if

$$s_{11}(k) = \frac{1}{c_{21}(k)},\ s_{12}(k) = \frac{c_{22}(k)}{c_{21}(k)}$$

Conclusion:

Jost= 1/transmission coefficient

1/transmission coefficient=Evans function

[Kapitula/Sandstede]

Plan

$d/dx + A(x) + R(x)$	and "sandwiched resolvent"
main results:	Evans=Fredholm, Evans=Jost
Evans function:	old defn and example
coordinate free Evans:	Bohl/Lyapunov, splittings
Jost solutions	and Evans determinants
Evans=Fredholm:	dichotomy vs sandwiched resolvent
Evans=Fredholm:	constant coefficients
Evans=Jost:	Schrödinger eqns

Plan

$d/dx + A(x) + R(x)$	and "sandwiched resolvent"
main results:	Evans=Fredholm, Evans=Jost
"bended" d/dx :	linearized 2D Euler: σ_{ess}
spectral mapping thms:	Gearhart-Prüss, stability/instability
NLS invar mnflds:	resolvent estimates
Evans function:	old defn and example
exp dichotomies:	Dichotomy Theorem
connections:	Morse, spectral flow/index
coordinate free Evans:	Bohl/Lyapunov, splittings
Jost solutions	and Evans determinants
Evans=Fredholm:	dichotomy vs sandwiched resolvent
Gesztesy/Makarov:	semi-separable kernels
Evans=Fredholm:	constant coefficients
difference equations:	Gesztesy/Makarov,Evans=Fredholm
Evans=Jost:	Schrödinger eqns
scattering theory:	transmission coeff