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Normal Forms
noise in an Andronov-Hopf bifurcation (Azimuth Project)

with limit cycle

w/o limit cycle
theta/SNIC model

Gutkin & Ermentrout, *Neural Comp*, 1998: add comparable noise to theta model vs. AH model $\Rightarrow$ theta model CV is twice as large

Planar Models
planar model – active phase as single spike

Fig. 2. The singular periodic orbit.
A mathematical framework for critical transitions: Bifurcations, fast–slow systems and stochastic dynamics

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\[
\begin{align*}
\frac{dx_\tau}{\tau} &= \frac{1}{\epsilon} (y - x^2) d\tau + \frac{\sigma}{\sqrt{\epsilon}} dW_\tau, \\
\frac{dy_\tau}{\tau} &= g(x_\tau, y_\tau) d\tau,
\end{align*}
\] (34)

**Theorem 6.1.** Consider the SDE (34) and suppose \( g \equiv 1 \). If \( \sigma \ll \sqrt{\epsilon} \) then critical transitions before the deterministic fold bifurcation point occur with very small probability. For \( \sigma \gg \sqrt{\epsilon} \) critical transitions before the deterministic fold bifurcation occur with very high probability.
coherence resonance vs. self-induced stochastic resonance
DeVille et al., *Phys. Rev. E*, 2005

noise in slow variable  
noise in fast variable
coupling between planar models – escape (Skinner et al., JCNS, 1994)
coupling between planar models – release (Skinner et al., JCNS, 1994)
half-center oscillator (Brown, 1911): components *not intrinsically rhythmic*; generates rhythmic activity, without rhythmic drive

reciprocal inhibition

active phase

silent phase
time courses for half-center oscillations from 3 mechanisms: persistent sodium, post-inhibitory rebound (T-current), adaptation (Ca/K-Ca)
**simulation results:**
*unequal constant drives*

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**A**

Persistent sodium

- relative silent phase duration for cell with varied drive
- relative silent phase duration for cell with fixed drive

**B**

Post-inhibitory rebound

**C**

Intermediate

- Adaptation

**D**

Adaptation - nonmonotonic

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Daun, Rubin, and Rybak, *JCNS*, 2009
Bursting
rigorous framework for bursting – **Terman, 1991-2**
noise in bursting (example)

elliptic (subAH-SNPO) bursting:
Su, R. and Terman, Nonlinearity, 2004

existence of invariant tube
estimate of passage time
metastability of solutions
introduce noise $\sigma$

Su, R. and Terman, Nonlinearity, 2004

$\sigma = O(\varepsilon^n)$

$\sigma = O(e^{-c/\varepsilon})$
A mathematical framework for critical transitions: Bifurcations, fast–slow systems and stochastic dynamics

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\[ dx_{\tau} = \frac{1}{\varepsilon} f(x_{\tau}, y_{\tau}) d\tau + \frac{\sigma_f}{\sqrt{\varepsilon}} dW_{\tau}, \]
\[ dy_{\tau} = g(x_{\tau}, y_{\tau}) d\tau + \sigma_g dW_{\tau}, \]
\[ (\sigma_f^2 + \sigma_g^2)^{1/2} = \sigma = \sigma(\varepsilon) \]

The first goal is an estimate on the concentration of solutions to (16) near the deterministic slow manifold. To identify a neighborhood containing most sample paths we define the process

\[ \xi_{\tau} := x_{\tau} - h_\varepsilon(y_{\tau}). \]

(17)
Then define $X_t := \sigma_f^{-2} \text{Var}(\xi^0_t)$ which satisfies a fast–slow ODE \cite{32} given by
\begin{align*}
    \epsilon \dot{X} &= 2A_\epsilon(y)X + 1, \\
    \dot{y} &= g(h_\epsilon(y), y).
\end{align*}
\tag{20}

The slow manifold of (20) is
\begin{align*}
    C_\epsilon^X = \left\{(X, y) \in \mathbb{R}^2 : x = H_\epsilon(y) = -\frac{1}{2A_\epsilon(y)} + O(\epsilon)\right\}.
\end{align*}

The neighborhood of $C_\epsilon$ is then defined as
\begin{align*}
    N(r; C_\epsilon) := \left\{(x, y) \in \mathbb{R}^2 : \frac{(x - h_\epsilon(y))^2}{H_\epsilon(y)} < r^2 \right\}.
\end{align*}
\tag{21}

\textbf{Theorem 4.1.} Sample paths starting on $C_\epsilon$ stay in $N(r; C_\epsilon)$ with high probability for times approximately given by $O(\epsilon e^{r^2/(2\sigma_f^2)})$. 
Nils Berglund · Barbara Gentz

Pathwise description of dynamic pitchfork bifurcations with additive noise

\[ \frac{dx_t}{\varepsilon} = \frac{1}{\varepsilon} f(x_t, t) \, dt + \frac{\sigma}{\sqrt{\varepsilon}} \, dW_t. \]
Network Architecture + Dynamics
example Hartelt network – how can this synchronize?

**larger network simulation study**


- 100 square-wave bursting neurons, 90 sec/sim
- fixed distribution of neuron intrinsic dynamics
  
  \[ \frac{1}{3} \text{Q}, \frac{1}{3} \text{B}, \frac{1}{3} \text{T} \]  
  \( (E_L \text{ varied}) \)

- variety of connection architectures with fixed total number of links (same total \( g_{syn} \) for each neuron)
  
  -- nearest neighbor (1-d and 2-d)
  -- random
  -- small world (1-d and 2-d)
  -- scale-free
  -- Hartelt

- varied cell-type hierarchies: placement of particular types of intrinsic dynamics within each network, based on *betweenness centrality*

  random, TBQ, TQB, BTQ, BQT, QTB, QBT
results: network burst synchrony vs. cell-type hierarchy

Network burst synchrony is generally less sensitive to which cell type goes where than to network topology, esp. for strong synapses (exception: scale-free networks)