OPEN PROBLEMS IN NON-NEGATIVE SECTIONAL CURVATURE

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Abstract. We compile a list of the open problems and questions which arose during the Workshop on Manifolds with Non-negative Sectional Curvature held at the American Institute of Mathematics, Palo Alto, CA, in September '07.

1. Diameter Pinching

Problem 1. Does the Grove-Petersen-Wu Finiteness Theorem hold in dimension 4, i.e. are there only finitely many diffeomorphism types in the class of 4-dimensional Riemannian manifolds satisfying

$$\sec_M \geq -\Lambda^2, \quad \operatorname{diam}(M) \leq D, \quad \operatorname{Vol}(M) \geq V?$$

Problem 2. If $\sec_M \geq 1$ and $\operatorname{diam}(M) > \frac{\pi}{2}$, must M be diffeomorphic to S^n?

Problem 3. Let M^n be a manifold for which $\sec_M \geq 1$ and $\operatorname{diam}(M) \geq \frac{\pi}{2} - \varepsilon(n)$. Is M homeomorphic to a manifold M', where $\sec_{M'} \geq 1$, $\operatorname{diam}(M') \geq \frac{\pi}{2}$? This problem may be easier to address if we also assume that $\operatorname{Vol}(M) \geq V$ and $\varepsilon = \varepsilon(n, V)$.

2. Collapse and Alexandrov Geometry

Problem 4. Perelman's Stability Theorem yields that manifolds in a given sequence of non-collapsing manifolds are eventually pairwise homeomorphic. Are they also PL-homeomorphic or diffeomorphic?

Problem 5. Understand DC-structures on manifolds. In particular, does Perelman's Stability Theorem hold in the DC-category. Is PL = DC always?

Problem 6. Extend the Wilking Connectivity Theorem to Alexandrov spaces, i.e. if X is a positively curved Alexandrov space and $Y \subset X$ a totally geodesic subspace of codimension k, is it true that $X - Y$ has homology only up to dimension $2k - 2$?

Problem 7. Suppose X is the non-collapsed Gromov-Hausdorff limit of (M_i^n, g_i), where $|\sec_{M_i}| \leq 1$, and that along every geodesic on M_i one hits a conjugate point before $t = \pi + \frac{1}{i}$. Is X rigid in any sense?

Problem 8. Is there a sequence of simply-connected, pointwise strictly $\frac{1}{4}$-pinched manifolds M_i^n, $n > 2$, that collapse?

Problem 9. Find an appropriate definition of Morse functions on Alexandrov spaces and construct examples.

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Problem 10. Study the collapse of Alexandrov spaces.

Problem 11. Consider finite towers

$$M_0 \xrightarrow{F_1} M_1 \xrightarrow{F_2} \cdots \xrightarrow{F_k} M_k$$

of fiber bundles, where the fibers $\{F_1, \ldots, F_k\}$ and M_k are fixed topological manifolds. Loosen this notion to get a "brotherhood" on the manifolds M_0 and a property of such M_0 *not* known to be possessed by all manifolds of $\sec \geq K$, $\operatorname{diam} \leq 1$.

Problem 12. Give an Alexandrov analogue of rational ellipticity. In particular, are manifolds with almost non-negative sectional curvature rationally elliptic?

Problem 13. Given a non-collapsing Gromov-Hausdorff convergence $M_i \longrightarrow X$, can one find a "tangent bundle" structure on X that is sensitive to the diffeomorphism class of the M_i?

Problem 14. Is there a Gauss formula for Alexandrov spaces, i.e. must a convex hypersurface Y of an Alexandrov space X have $\sec_Y \geq \sec_X$?

Problem 15. Is every finite dimensional Alexandrov space a limit of Riemannian manifolds with $\sec \geq K$?

Problem 16. Study Alexandrov (almost) submetries.

Problem 17. Is there an alternate approach to homotopy groups that is adapted to Alexandrov spaces?

Problem 18. Study collapse to a ray.

Problem 19. Can an n-dimensional torus collapse to an interval? (The answer to this question is "No", essentially settled at the workshop.)

Problem 20. Study the collapse of Riemannian manifolds with boundary which have $\sec \geq K$ on the interior and controlled boundary concavity.

Problem 21. Find an application where infinite-dimensional Alexandrov spaces appear as limits of manifolds of increasing dimension.

3. Group Actions and Submersions

Problem 22. Let M^n be a manifold with $\sec > 0$ or $\sec \geq 0$ or almost non-negative curvature. Does M^n have a positive symmetry degree, i.e. is there an $S^1 \subset \mathrm{Diff}(M^n)$?

Problem 23. Is there a principal T^2-bundle whose total space admits $\sec > 0$?

Problem 24. Given a fat G-principal bundle, must G be S^1, S^3 or $SO(3)$?

Problem 25. Can one reduce the structure group of a fat principal G-bundle?

Problem 26. Given a homogeneous space G/H, with G compact, classify all homogeneous metrics with $\sec \geq 0$.

Problem 27. Is there a positively curved 5-manifold with a free isometric S^3 or $SO(3)$ action?

Problem 28. Given a Riemannian submersion with positively curved total space, is the dimension of the fiber less than the dimension of the base? It is perhaps simpler to decide if there is a bound on the dimension of the fiber in terms of the dimension of the base. Is the image of the A-tensor large at some point?

Problem 29. Classify Riemannian submersions from a Lie group with a bi-invariant metric. (As was observed at the workshop, they are not necessarily biquotient submersions.)

Problem 30. Suppose (M^n, g) is simply-connected and $\sec_M \geq 0$. Is $\mathrm{rank}(\mathrm{Iso}(M^n, g)) \leq \frac{2}{3}n$?

Problem 31. Suppose we have an isometric group action on M. If one changes the metric on the orbit space, does it lift to an invariant metric on M?

Problem 32. If a group acts isometrically on a manifold of $\sec \geq 1$ and the fixed point set is a circle, is it's length $\leq 2\pi$?

4. Manifolds of Cohomogeneity-one and Polar Actions

Problem 33. Study the existence and non-existence of metrics with non-negative curvature on manifolds of cohomogeneity-one.

Problem 34. Find cohomogeneity-one manifolds with "interesting" topology, in particular not homeomorphic to a symmetric space. Study curvature properties of these manifolds.

Problem 35. Classify cohomogeneity-one manifolds with $\sec \geq 0$ and at least one totally geodesic principal orbit.

Problem 36. Compute topological invariants of cohomogeneity-one manifolds. Classify topologically the new candidates for positive curvature.

Problem 37. Study the existence of Einstein metrics on manifolds of cohomogeneity-one.

Problem 38. Suppose M is a polar manifold with $\sec > 0$. Must M be diffeomorphic to a compact rank-one symmetric space?

Problem 39. Suppose $\Sigma \subset M^n$ is the section of a polar action. If Σ is rationally elliptic, must M^n be rationally elliptic?

5. Vector Bundles

Problem 40. Is there a metric with $\sec \geq 0$ on $\mathbb{R}^6$-bundles over $S^3 \times S^3$ with non-trivial Euler class? Wilking has shown that the answer is "No" if the soul is $S^3 \times S^3$ with the product metric.

Problem 41. Which vector bundles over $S^2 \times S^2$ or $\mathbb{C}P^2 \# \pm \mathbb{C}P^2$ where the structure group does not reduce to a torus admit $\sec \geq 0$?

Problem 42. Classify metrics with $\sec \geq 0$ on $S^2 \times \mathbb{R}^4$ (or, more generally, on $S^n \times \mathbb{R}^k$).

Problem 43. Suppose $E \longrightarrow M$ is a vector bundle over a compact, simply-connected manifold M for which $\sec_M \geq 0$. Does $E \oplus \mathbb{R}^k \longrightarrow M$ have $\sec \geq 0$ for k large?

6. Quasi-positive Curvature and Positive Curvature on an Open Dense Set

Problem 44. Which theorems from $\sec > 0$ carry over to positive curvature on an open dense set?

Problem 45. Suppose G is a compact Lie group with a left-invariant metric. Are there any new examples $H\backslash G$ with quasi-positive curvature?

Problem 46. Find new examples of fundamental groups in quasi-positive curvature or positive curvature on an open dense set.

Problem 47. Fix $k \in \mathbb{N}$. Is there a $n_0 = n_0(k)$ such that for any quasi-positively curved manifold (M^n, g) with $n \geq n_0$ and $\mathrm{cohom}(M^n, g) \leq k$, there exists a chain

$$M_0 = M^n \subset M_1^{n+k} \subset M_2^{n+2k} \subset \cdots$$

such that all inclusions are totally geodesic, the manifolds M_i are quasi-positively curved, $\cup M_i$ is the classifying space of a Lie group, and $M_i/\operatorname{Iso}(M_i, g)$ is isometric to $M/\operatorname{Iso}(M, g)$?

7. Ricci Flow

Problem 48. Let M^n be a compact manifold with positive isotropic curvature. Does any blow-up limit of the Ricci flow have non-negative curvature operator?

Problem 49. Suppose G acts freely and isometrically on M. What kind of flows on M/G are induced by the Ricci flow on M?

Problem 50. Can one improve the Hsiang-Kleiner theorem on positively curved 4-manifolds with symmetry from homeomorphism to diffeomorphism by using the Ricci flow?

8. Miscellaneous Problems

Problem 51. Is a "generic" manifold a $K(\pi, 1)$-space (where "generic" is to be determined)?

Problem 52. Does positive sectional curvature imply that the manifold is formal (in the sense of Sullivan's minimal model)?

Problem 53. For compact, odd dimensional, positively curved manifolds is there a cyclic subgroup of the fundamental group whose index is bounded only in terms of the dimension?

Problem 54. Is there a finiteness result for n-dimensional, positively curved manifolds with $\pi_1 = \pi_2 = 0$?

Problem 55. Is there a $\delta(n) > 0$ such that any n-dimensional, positively curved manifold carries a $\delta(n)$-pinched metric?

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