THE GEOMETRY OF THE OUTER AUTOMORPHISM GROUP OF A FREE GROUP

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Summary statements of some problems raised during the AIM Workshop *The geometry of the outer automorphism group of a free group*, October 25-29, 2010.

1. Curve Complex Analogues

Because the curve complex has proved so fruitful for studying the mapping class group, much interest and effort has centered on finding an analogous complex for $Out(F_n)$. Throughout the group discussion, desired properties for such a complex were collected in the statement of the problem given below. Because the choice of these properties depends on what we intend to do with this complex, the group's development of Problem 1.1 intermingled with discussion on its potential applications. We try to capture some of this conversation in the remark. Finally, we listed several current candidate complexes, to ask which ones fulfill which of the sought properties.

Problem 1.1. *Construct a δ-hyperbolic graph on which $Out(F_n)$ acts so that:*
(1) Every iwip γ has positive translation distance ($|\gamma| > 0$)
(2) Polynomially growing γ are elliptic ($|\gamma| = 0$)
(3) The construction is elegant/beautiful/useful/truthful
(4) Local structure allows for inductive arguments
(5) The graph is highly connected (with highly connected links)

Remark. Lucas Sabalka began the discussion by stating that infinite-order elements should act on the complex in a reasonable way. This was distilled into properties (1) and (2), although there may be some question about the elements corresponding to block-upper-triangular matrices, or "non-hyperbolic elements with hyperbolic mapping torus."

Regarding the question of motivation, one could ask for specific properties of candidate complexes, e.g., does this one admit an acylindrical action of $Out(F_n)$? Or one could seek specific applications, e.g., suggested Martin Bridson, proving that the second cohomology of every infinite subgroup is infinite-dimensional. Thierry Coulbois wanted the complex to be a natural object, as Teichmüller space is. Said Martin Lustig: truth in beauty, is that enough? Juan Souto reminded us that there have been so many unforeseen consequences of curve complex hyperbolicity. On the other hand, Yair Minsky said that he and Masur had certain applications in mind when they proved that theorem.

The group came up with several putative applications of a complex for $Out(F_n)$:

(i) Bounded asymptotic dimension
(ii) Uniformity in the theorem of Dahmani-Guirardel-Osin
(iii) Rigidity of maps from lattices (bounded cohomology)
(iv) Quasi-isometric rigidity

The DGO theorem mentioned in (ii) states that, for all iwips $g \in Out(Fn)$, there exists n such that for all $k \geq 1$, the normal closure of g^{nk} is free and purely iwip. The proof uses the hyperbolic graphs of Bestvina and Feighn. Vincent Guirardel commented that "there are already some hyperbolic graphs that do the job"—does this include uniformity? Bridson, I think, said that the desired complex would yield "lovely" proof of (iii).

Yair, who first brought up the issue of rigidity properties of $Out(F_n)$, suggested that the local structure of the desired complex should allow inductive arguments, leading to (4) in the problem above. Specifically, he mentioned a δ-hyperbolic or CAT(0) structure on links, recalling that the map from the curve complex to its links has nice properties more or less like the visual map in CAT(0) space. For (4), he suggested "CAT(0)-like global-to-local quasi-projections."

Karen Vogtmann suggested the property of highly connectedness in (5), considering possible cohomological groups, though she commented that we may have enough highly connected complexes.

The group drew up a list of candidate complexes:

(1) The free-factor complex, for which vertices are conjugacy classes of free factors, and edges are inclusion up to conjugacy.
(2) The splitting complex with trivial edge groups (which is quasi-isometric to the sphere complex).
(2') Variants allowing more possibilities for splittings, e.g. splittings over $\mathbb{Z}^n$, or combinations, e.g. splittings of rank less than k. (Matt Clay suggested these variants, and Kasra Rafi appreciated that this idea relates to length.)
(3) The poset of commensurability classes of abelian subgroups consisting of linearly growing automorphisms. (Juan and Bridson expressed interest in analogizing, from the mapping class group, correspondence between Dehn multi-twists and cut systems which relate to curve complexes)
(4) The Kapovich-Lustig graph.

Problem 1.2. *Do various candidate complexes satisfy the conditions of Problem 1.1?*

2. Geodesics in Outer Space

Juan Souto prompted another major discussion by asking to what extent Teichmüller and WP geodesic analogies are comparable to geodesics in Outer Space with the Lipschitz metric. On the board, we compiled the multi-part question give below; the text that follows is meant to give a sense of the surrounding conversation.

Problem 2.1. *Understand analogies of Teichmüller and WP geodesic behavior with behavior of geodesics of $Out(F_n)$ in the Lipschitz metric. Specifically,*

(1) *What are the "geodesics"/"lines" having strongly contracting projection functions?*
(2) *Is there a "good" thick part?*
(3) *Describe the relationship between behavior of geodesics and boundary theory.*
(4) *Given an iwip ϕ, what is $Min(\phi) := \{x | d(x, \phi(x)) = \inf_y d(y, \phi(y))\}$? Is it, or is it in, a CAT(0) space, or an axis bundle? Is it quasi-isometric to a line?*
(5) *How close are $Min(\phi)$ and $Min(\phi^{-1})$?*

Parts (1) and (2) were borne out of Juan's inquiry: is there a "thick part" which is hyperbolic? It is not enough for the definition to be a lower bound for injectivity radius. What about a thick geodesic? Every iwip axis should be thick with quantifiers. Can one find geodesics that go from thick to thin part? In general, to what extent are analogies truthful or not?

Yael Algom-Kfir, to clarify Juan's question, said an axis is hyperbolic with some quantifiers (projection is exponentially contractible). Yael: So Juan wants, if you fix the quantifiers, find the geodesics for which the projection map is strongly contractible. Juan again: notion of "thick geodesic" is better than "thick part." Thick part is too easy an analogy: you can run along strata, two rays parallel in thick part; this should not happen. As for motivation for the problem? Happiness, says Juan. Yael: he wants to understand the phenomena, for two iwips, when is one more contracting then the other?

Meanwhile, Yair Minsky suggested (3).

Kasra Rafi: if subsurface projections (meaning local structure) are bounded, heuristically, if you have an iwip spending not too much time in any one subsurface, then you are in the thick part. Juan disagreed, saying one should find which geodesics are contracting to figure out what subsurface projections should be.

Regarding (4): this was originally stated as "what is the set of axes of an iwip, what is its topology and how thick is it." The set of axes were clarified to mean the min set named in the problem. Handel's result gives that there are axes in outer space, but he was not sure if it is in Lipschitz metric homotopic to a line, or continuous vs. discrete. Yair asked, if you take powers, do the min sets change? Handel: there are graphs here that don't stretch by λ? Then there was some talk of train tracks for powers; axis bundles are associated to endpoints; having to take powers to make them train tracks. Handel: elements of axis bundle are train tracks with powers included. For irreducibles, are there graphs that are stretched by λ? Mladen Bestvina: no actual examples, but you're not done because of illegal turns. Handel: axis bundle diameter varies. Karen Vogtmann: understand geometry of the set. Handel: it is known what makes diameter bigger, quasi-isometric to a line. He thinks there's not a uniform bound to the diameter.

Thierry Coulbois asked if (5) was too specific. Martin Lustig, perhaps in reference to (5)?, brought up the possibility of an analogue to the Masur criterion for non-uniquely ergodic points.

3. Group Properties

Problem 3.1. *Does $Out(F_n)$ have Property (T) or (τ) if $n \geq 4$?*

$Out(F_3)$ does not have Property (T) because it has a finite index subgroup Γ with $|H_1(\Gamma, \mathbb{Z})| = \infty$. This motivates the question,

Problem 3.2. *Does there exist a finite index subgroup $\Gamma < Out(F_n)$ with infinite abelianization?*

More generally,

Problem 3.3. *What are and what is known about the finite index subgroups of $Out(F_n)$?*

One fact about finite-index subgroups concerns principal congruence subgroups. Given H a characteristic finite-index subgroup of F_n, one has $\Gamma = \ker(Aut(F_n) \to Aut(F_n/H))$.

Problem 3.4. *Does every finite index subgroup contain one of these Γ?*

Mladen Bestvina asked if there is an effective way to construct these subgroups. Verbal subgroups, or pass to Burnside group, suggested Martin Bridson. Karen Vogtmann commented that you would classify characteristic subgroups of $Out(F_n)$.

Bridson offered a baby question in that direction: if $n \geq 4$, is $Out(F_n)$ large? Meaning, does there exist a finite-index subgroup which acts on a tree, thus maps onto a free group? On the board, he writes:

Problem 3.5. *Does there exist a finite-index subgroup $\Gamma < Out(F_n)$ such that $\Gamma \to F_2$?*

Thierry Coulbois asked, what techniques would apply to these questions, and how does this relate to our machinery? Bridson offered a general point: the questions regard subgroups, whereas train tracks allow normal forms for understanding group elements.

Juan Souto asked if one can find an infinite torsion quotient. This would imply non-bounded generation, he said, and don't we know this? Because we know bounded cohomology. The question is one way to ask how hyperbolic is $Out(F_n)$:

Problem 3.6. *Does there exist $p \to 0$ such that $Out(F_n)/\langle\langle \gamma^p \rangle\rangle$, all $\gamma \in Out(F_n)$, is infinite?*

Remark. DGO implies $Out(F_n)$ is SQ-universal.

Another group property that arose is uniform exponential growth of the exponentially growing subgroups of $Out(F_n)$, with possibly a second level of uniformity if the growth rate can be shown to be independent of the particular subgroup.

Problem 3.7. *Prove (uniform) uniform exponential growth for two-generator subgroups of $Out(F_n)$, or in particular two Dehn twists.*

Matt Clay in his talk had proposed a "baby question" towards proving the Dehn twist case: bounding a certain ratio **(which ought to be recalled here)** appearing in his joint work with Alexandra Pettet.

Martin Bridson suggested using linear quotients, as it is potentially easier to prove uniform exponential growth by mapping to well-understood subgroups.

4. Homomorphisms with Mapping Class Groups

Juan Souto asked if there are any non-obvious ways for the mapping class group to sit in $Out(F_n)$.

Problem 4.1. *If S is closed, what is $Hom(MCG(S), Out(F_n))$? What about injective homomorphisms?*

In the other direction, there are no injective homomorphisms from $Out(F_n)$ to $MCG(S)$, because $Out(F_n)$ has the Poison subgroups that preclude linearity, and these never appear in $MCG(S)$ by the result of Brendle and Hamidi-Tehrani. Nonetheless one can ask about non-injective homomorphisms. An induction argument produces many homomorphisms with infinite image from $Out(F_3)$ to $MCG(S)$, but what about for $n \geq 4$?

Problem 4.2. *What is $Hom(Out(F_n), MCG(S))$? Can homomorphisms have infinite image if $n \geq 4$?*

Remark. Juan warned that you need bounds on S and n because otherwise the answer is the infinite abelianization problem.

If there exists a finite index subgroup $\Gamma < Out(F_n)$ which surjects onto $\mathbb{Z}$, then $Out(F_n)$ surjects onto A a virtually abelian subgroup of $MCG(S)$. So a warm-up question would be

Problem 4.3. *Does every homomorphism $\phi : Out(F_n) \to MCG(S)$ have virtually abelian image?*

Martin Bridson, I think, explained that if you fix n, the genus of S must be greater than or equal to $n!$. This is because $Out(F_n)$ has the Weyl group $\mathbb{Z}_2 \rtimes S_n$ (where S_n is the symmetries of the rose). $Z^n \cdot n! = \mathbb{Z}_2 \rtimes S_n$.

It is known, by Martin and Juan's results, that all the Poison groups in $Out(F_n)$ would give virtually abelian image under a homomorphism to $MCG(S)$. However, subgroups of finite index change the story.

Problem 4.4. *Same questions above for subgroups of finite index.*

5. Currents, Laminations, and Horofunction Boundary

Arnaud Hilion was seeking a certain inequality. Take T a tree in outer space, and associate to it a Patterson-Sullivan current μ_T, with normalization choice $i(T, \mu_T) = 1$.

Problem 5.1. *For all $T, T' \in X_n$ as above, is it true $i(T, \mu_{T'}) \cdot i(T', \mu_T) \geq 1$? At least, where is the inequality true?*

The hope is to use this to define a WP-metric on X_n.

Kasra Rafi: Teichmüller space embeds in currents, and Bonahon defines a metric which turns out to be the WP metric. If you try this with quadratic differentials and the Lipschitz metric, then inequality does not hold. The metric you get is not positive—Bonahon's construction doesn't work! That's a downer. But hope is that there is perhaps a subspace of outer space, perhaps a manifold, in which the inequality does hold. The subspace should be $Out(F_n)$-invariant.

The next question comes from Martin Lustig, with the second part added by Arnaud. Lustig recalled that Huber[?] and Besson decompose an R-tree into components, and also we know an R-tree has a dual lamination characterizing its topological side in some sense.

Problem 5.2. *Can one understand the GL-theory of* FINE DECOMPOSITION *in terms of dual laminations and vice versa? And in terms of systems of partial isometries?*

Interested in the horofunction boundary of outer space X_n in the Lipschitz metric, Cormac Walsh explained that it is coarser than the usual boundary. Mapping $x \mapsto d(-, x)$, outer space embeds as a subset X of $C_0(X_n)/\sim$, where $\sim$ is homotopy to a constant, and the topology is uniform convergence on compact subsets.

Problem 5.3. *What is the closure $\overline{X}$? What is $\overline{X} \backslash X$, the horofunction boundary of outer space in the Lipschitz metric?*

Thierry Coulbois asked if we know the horofunction boundary in the simplicial metric, and also if it is reasonable Busemann points could be the whole thing. For the latter, Cormac said yes, as it is true for Teichmüller space. Specifically, a Busemann point B_σ is determined by a 'good' ray $\sigma : [0, \infty) \to X_n$ by $B_\sigma(x) = \lim_{n\to\infty}(d(x, \sigma(n)) - n)$. If X were CAT(0) or Teichmüller space with the Lipschitz metric, then $\overline{X} \backslash X$ would consist entirely of Busemann points. However, warned Cormac, even for normed spaces you can get points that are not Busemann points.