p-ADIC REPRESENTATIONS, MODULARITY AND BEYOND

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Abstract. These are my notes from the AIM workshop: p-adic Representations, Modularity, and Beyond, held at AIM in Palo Alto, Feb. 20-24, 2006. Everything here was LaTeXed 'on the fly' during the workshop, so please read with caution, as there may well have been transcription errors. Several talks are missing from this write-up, most notably the talks by: Breuil and Berger on the p-adic local Langlands correspondence; and Wintenberg's talk on Serre's conjecture. However, there are preprints in circulation reporting on all three speakers. To ameliorate this deficiency one can find the necessary preprints on the relevant homepages:

- `http://www.ihes.fr/~lberger/`
- `http://www.ihes.fr/~breuil/`

Notes for the final two talks were provided by Craig Citro.

1. Kisin: Modularity, the Breuil-Mézard conjecture and beyond

Note to the reader: — This first section was added in as a postscript, and thus it is not necessarily what was said at the workshop, but merely my own summary of the paper/talk of Kisin.

1.1. The main results.

Theorem 1. *Let $\mathcal{O}$ be the ring of integers in a finite extension of $\mathbb{Q}_p$, having residue field $\mathbb{F}$, and*

$$\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathcal{O})$$

a continuous representation. Suppose that

1. *$\rho|_{G_{\mathbb{Q}_p}}$ is potentially semi-stable with distinct Hodge-Tate weights.*
2. *ρ becomes semi-stable over an abelian extension of $\mathbb{Q}_p$.*
3. *$\overline{\rho} : G_{\mathbb{Q},S} \xrightarrow{\rho} \mathrm{GL}_2(\mathcal{O}) \to \mathrm{GL}_2(\mathbb{F})$ is modular, and $\overline{\rho}|_{\mathbb{Q}(\zeta_p)}$ is absolutely irreducible.*
4. *$\overline{\rho}|_{G_{\mathbb{Q}_p}} \not\sim \begin{pmatrix} \chi & * \\ 0 & \chi \end{pmatrix}, \begin{pmatrix} \omega\chi & * \\ 0 & \chi \end{pmatrix}$ for any character $\chi : G_{F_v} \to \mathbb{F}^\times$, where ω denotes the mod p cyclotomic character.*

Then (up to twist) ρ is modular.

A consequence of this theorem (using only the case when the representation ρ becomes semi-stable over an abelian extension) is a Galois-theoretic description of the eigencurve, which answers a question of Coleman and Mazur. The last condition on this theorem should soon be abolished; the first half of the third hypothesis should soon be redundant (Khare-Wintenberger); and the second condition should be removable by results of Colmez.

We let μ_{Gal} denote the Hilbert-Samuel multiplicity of the mod p reduction of the local deformation ring. Breuil and Mézard conjectured that this quantity should be given by a certain invariant μ_{Aut} (described - at least conjecturally - later), indeed they conjecture:

For any irreducible, finite dimensional representation of $\mathrm{GL}_2(\mathbb{Z}_p)$ on a vector space over $\mathbb{F}$ is isomorphic to $\mathrm{Sym}^n\overline{\mathbb{F}} \otimes \det^m$ where $n \in \{0, 1, \ldots, p-1\}$ and $m \in \{0, 1, \ldots, p-2\}$ — we write $\sigma_{n,m}$ for this representation. Such a representation much factor through $\mathrm{GL}_2(\mathbb{F}_p)$, thus we have

$$(L_{k,\tau})^{\mathrm{ss}} \otimes_{\mathcal{O}} \mathbb{F} \xrightarrow{\sim} \bigoplus_{n,m} \sigma_{n,m}^{a(n,m)},$$

where $L_{k,\tau}$ is a $\mathrm{GL}_2(\mathbb{Z}_p)$-stable $\mathcal{O}$-lattice.

Then we write

$$\mu_{\mathrm{Aut}} = \mu_{\mathrm{Aut}}(k, \tau, \overline{\rho}) = \sum_{n,m} a(n,m)\mu_{n,m}(\overline{\rho})$$

Then Breuil-Mézard conjectured the following:

Conjecture 1 (Breuil-Mézard)**.**

$$\mu_{\mathrm{Gal}} = \mu_{\mathrm{Aut}}.$$

Generally, one can apply global arguments to prove that $\mu_{\mathrm{Gal}} \geq \mu_{\mathrm{Aut}}$, the reverse inequality is considerably more difficult, and is in essence equivalent to proving a modularity lifting theorem.

Suppose that $\tau : I_{\mathbb{Q}_p} \to \mathrm{GL}_2(E)$ is of Galois type. W let $R^{\square,\psi}(k, \tau, \overline{\rho})$ be a certain (uniquely defined) quotient of $R^{\square}(\overline{\rho}) \otimes_{W(\mathbb{F})} \mathcal{O}$ — where $R^{\square}(\overline{\rho})$ is the universal framed deformation ring, i.e. the ring representing the functor which associates to a local Artin ring A with residue field $\mathbb{F}$ the set of isomorphism classes of deformations V_A of $\overline{\rho}$ to A, together with a lifting to V_A of a some fixed choice of basis for $V_{\mathbb{F}}$.

The following conjecture generalizes the Breuil-Mézard conjecture to the situation where $\overline{\rho}$ has nontrivial endomorphisms and is central in this approach to the Fontaine-Mazur conjecture:

Conjecture 2 (Kisin)**.** *The Hilbert-Samuel multiplicity of $R^{\square,\psi}(k, \tau, \overline{\rho})/(\pi)$ is equal to μ_{Aut}.*

Most cases of this conjecture are proved in Kisin's preprint. Indeed, by the same reasoning as above, the difficulty lies in proving the single *inequality*: $e(R^{\square,\psi}(k, \tau, \overline{\rho})/(\pi)) \leq \mu_{\mathrm{Aut}}$ - where e denotes 'Hilbert-Samuel multiplicity'.

1.2. **Colmez's functor and an expectation.** One of the main inputs into Kisin's proof (without the assumption that the representation becomes semi-stable over an *abelian* extension) of the above inequality is the following construction of Colmez.

Let $G = \mathrm{GL}_2(\mathbb{Q}_p)$, $K = \mathrm{GL}_2(\mathbb{Z}_p)$ and let Z be the center of G. If σ is a representation of KZ on a finite dimensional vector space V_σ over $\mathbb{F}$, then write $I(\sigma) = \mathrm{Ind}_{KZ}^{G}\sigma$ for the compact induction of σ.

Put $\sigma = \mathrm{Sym}^r\mathbb{F}$, and let $\chi : \mathbb{Q}_p^\times \to \mathbb{F}^\times$ be a character, let $\lambda \in \mathbb{F}$. For $x \in \mathbb{F}$ we put $\mu_x : \mathbb{Q}_p^\times \to \mathbb{F}^\times$ — the unramified character sending $p \in \mathbb{Q}_p^\times$ to x. Now set $\pi(r, \lambda, \chi) = I(\sigma)/(T - \lambda)I(\sigma) \otimes \chi \circ \det$.

Let Π be a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ on a $W(\mathbb{F})$-module. The representation Π is *admissible* if Π has finite length and each of its Jordan-Hölder factors has a central character. Equivalently, Π is admissible when it is of finite length and the Jordan-Hölder factors of Π are either one-dimensional or an infinite dimensional subquotient of some $\pi(r, \lambda, \chi)$.

Theorem 2 (Colmez)**.** *There exists an exact contravariant functor V^* from the category of finite length, admissible $\mathrm{GL}_2(\mathbb{Q}_p)$-representations to the category of finite length representations of $W(\mathbb{F})[G_{\mathbb{Q}_p}]$. Moreover, we have*

(1) $V^*(\Pi) = 0$ *if* Π *is one-dimensional,*
(2) $V^*(\pi(r, \lambda, \chi)) = \chi\mu_{\lambda^{-1}}$ *if* $\lambda \neq 0$,
(3) $V^*(\pi(r, 0, \chi)) = \mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2^{r+1} \otimes \chi.$

One can reinterpret Colmez's functor as a covariant functor as follows: Fix a character $\psi : G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ (regarded as a character of $\mathbb{Q}_p^\times$ via local class field theory), suppose that Π is a finite length $\mathcal{O}[\mathrm{GL}_2(\mathbb{Q}_p)]$-module, which is admissible as a $W(\mathbb{F})[1/p]$-module. Define $V_\psi(\Pi) = (V^*(\Pi))^*(\chi_{\mathrm{cyc}}\psi)$ where $V^*(\Pi)^*$ is the Pontryagin dual of the finite length $\mathcal{O}$-module $V^*(\Pi)$.

Suppose that Π is now a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ on a $W(\mathbb{F})$-module, put $\Pi_n = \Pi \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$. Assume that Π is p-adically complete and separated, in particular $\Pi = \mathrm{proj\,lim}_n \Pi_n$, and Π_n is admissible (and of finite length) for each n. We write $V_\psi(\Pi) = \mathrm{proj\,lim}\, V_\psi(\Pi_n)$. Since admissible representations have finite length, projective limits are exact, thus $V_\psi(\Pi)/pV_\psi(\Pi) = V_\psi(\Pi_1)$, in particular $V_\psi(\Pi)$ is a finite generated $W(\mathbb{F})$-module, as it is p-adically separated. Such a representation Π will be called an *admissible lattice*. If in addition Π is an $\mathcal{O}$-module, we call it an *admissible $\mathcal{O}$-lattice*.

The following result (in its full generality) is still pending:

Theorem 3 (Colmez(?)). *Let E'/E be a finite extension and let V be a two-dimensional E'-vector space with a continuous $G_{\mathbb{Q}_p}$-action. Suppose that V is potentially semi-stable of type τ with Hodge-Tate weights 0 and $k-1$ ($k \geq 2$) and that $\det V = \psi\chi$.*

Then there exists an admissible $\mathcal{O}_{E'}$-lattice Π with central character ψ such that $V_\psi(\Pi) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} V$. If Π' is another such lattice, then there exists a continuous isomorphism of $E'[\mathrm{GL}_2(\mathbb{Q}_p)]$-modules $\Pi' \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} \Pi \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Moreover, there exists a $\mathrm{GL}_2(\mathbb{Z}_p)$-equivariant inclusion $\sigma(k,\tau) \hookrightarrow \Pi \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

This result is known for triganuline representations.

2. Emerton: Part one.

Caveat: — This session started with Matthew Emerton fielding questions from the audience, thus this section consisted largely of open discussion, and consequently the narrative suffered.

RIBET: Where does $\mathfrak{m}$ live?

Let N be an integer. Define $\mathbb{T}(N)$ to be the Hecke algebra of level N. We have the following diagram of maps:

$$\begin{tikzcd} \mathcal{H}(N) = \otimes_{\ell \nmid N} \mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_\ell)//\mathrm{GL}_2(\mathbb{Z}_\ell)) \arrow[d] & \\ \mathbb{T} \arrow[r] & \mathbb{T}(N)_{\mathfrak{m}_N} \\ & R_{\Sigma_N} \arrow[u, two heads] \end{tikzcd}$$

This is compact with N enlarging:

$$\begin{tikzcd} \mathcal{H}(N') \arrow[r, two heads] \arrow[d] & \mathbb{T}(N')_{\mathfrak{m}} \arrow[d] & R_{\Sigma_{N'}} \arrow[l, two heads] \arrow[d] \\ \mathcal{H}(N) \arrow[r, two heads] & \mathbb{T}(N)_{\mathfrak{m}} & R_{\Sigma_N} \arrow[l, two heads] \end{tikzcd}$$

BUZZARD: Why let all primes ramify?

This is the whole picture, but in practice, only finitely many primes are used.

TAYLOR: Explain Colmez.

2.1. Definition of Colmez' functor.

$\mathrm{GL}_2(\mathbb{Q}_p)$-representations over A (where A is some artinian ring lifting $\mathbb{F}$)

$$\mathrm{Def}(\overline{\pi}) \tilde{\to} \mathrm{Def}(\overline{\rho})$$

FALSE START

Let $\mathrm{MF} = V^*$ be the functor from Kisin's talk, consider an admissible finite length representation $\pi(r, \lambda, \chi)$. There is a diagram of functors:

$$\begin{tikzcd}[column sep=small]
 & & \{\text{fin. lgth, smth, cntrlly cofin. /w J-H factors in list}\} \\
\{\text{fin. lgth, adms. } W(\mathbb{F})[\mathrm{GL}_2(\mathbb{Q}_p)]\text{-reps}\} \arrow[urr] \arrow[drr, "V^*"] & & \\
 & & \{\text{finite length } W(\mathbb{F})[G_{\mathbb{Q}_p}]\text{-mods}\} \arrow[uu]
\end{tikzcd}$$

Then

$$\{\text{admissible J-H factors }\} \subseteq \{\text{J-H factors of } \pi(r,\lambda,\chi)\}\,. \tag{1}$$

Recall that irreducible admissible is the same as irreducible, smooth with central character.

Smooth: every vector fixed by an open sub-group.

Admissible: above with finite length.

Finite length admissible is equivalent to finite length, smooth and centrally cofinite. These are, in turn, the same as finite length with Jordan-Holder factors in (1).

The finite dimensional clause doesn't matter on the Galois side — this is (maybe?) a local analogue of Ihara's lemma.

2.1.1. *Deformation theory.* Let A be an artin ring. Let π/A be finite free over A. Apply MF, gives ρ/A, deforming:

$$\overline{\pi} \leftrightarrow \overline{\rho}$$

Consider $\mathrm{Hom}(\pi, A)$, if A was killed by A/p^n, then consider $\mathrm{Hom}(\pi, \mathbb{Z}/p^n\mathbb{Z})$. We have

$$\mathrm{Hom}(\varinjlim A^n, A) = \varprojlim \mathrm{Hom}(A^n, A) \cong A^n$$

Definition of MF: Define

$$\mathrm{MF}(\pi) = V^*(\pi \otimes_A \mathrm{Hom}(A, \mathbb{Q}_p/\mathbb{Z}_p)).$$

Take

$$\begin{tikzcd}
\pi = \mathrm{Hom}(\pi^*, A) \arrow[r, leftrightarrow] \arrow[d] & P \arrow[d] & \\
\overline{\pi}_F \arrow[r] & \overline{\pi}^* \arrow[r, "\mathrm{MF}'"] \arrow[d, equal] & \overline{\rho} \\
 & \mathrm{Hom}(\overline{\pi}, \mathbb{F}) &
\end{tikzcd}$$

we have $\mathrm{GL}_2(\mathbb{Q}_p)$ on both sides, where P is category of pro-free A-modules. In fact, action is integral, thus we actually have an action of $\mathbb{F}[[\mathrm{GL}_2(\mathbb{Z}_p)]]$, the latter functor being covariant. Need to be careful about changing scalars - analogous to defining Hom's of sheaves.

2.2. **The** $\pmod p$ **correspondence.** Let $G = \mathrm{GL}_2(\mathbb{Q}_p)$, $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ and $\overline{B} = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$. We want

$$\overline{\rho} = \begin{pmatrix} \chi & * \\ 0 & \psi \end{pmatrix} \xrightarrow{??} \overline{\pi}$$

If $\chi\psi^{-1} \neq \omega \neq 1$ then

$$0 \longrightarrow \mathrm{Ind}_B^G \chi \otimes \psi\omega \longrightarrow \pi \longrightarrow \mathrm{Ind}_B^G \psi \otimes \chi\omega \longrightarrow 0$$

$$0 \subseteq \underbrace{\mathrm{St} \subseteq \cdot}_{1} \quad \underbrace{\subseteq \overline{\pi}}_{\mathrm{Ind}_{\overline{B}}^{G}\omega^{-1}\otimes\omega} \quad .$$

2.3. **Jacquet Modules.** Let $T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$. Consider

$$\mathrm{Ind}_{\overline{B}^G} \chi \otimes \psi\omega$$

Then

$$\mathrm{Hom}_G(V, \mathrm{Ind}_{\overline{B}}^{G}(\chi \otimes \psi\omega)) \cong \mathrm{Hom}_B(V, \psi\omega \otimes \chi)$$

and

$$(\mathrm{Ind}_{\overline{B}}^{G} \chi \otimes \psi\omega)_N = \psi\omega \otimes \chi \quad \text{where } N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$$

There is a action of T on the left-hand side.

We define the *ordinary Jacquet functor* as

$$J^{\mathrm{ord}}(V) = \left(V^{\begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}} \right)^{\mathrm{ord}}$$

with an action of U_p.

$$J^{\mathrm{ord}}(\mathrm{Ind}_{\overline{B}}^{G}(\chi \otimes \chi\omega)) = \chi \otimes \psi\omega$$

$$\mathrm{Hom}(\mathrm{Ind}_{\overline{B}}^{G} U, V) = \mathrm{Hom}_T(U, J^{\mathrm{ord}}(V))$$

One can compute:

$$R^1 J^{\mathrm{ord}}(V) = (V_N)(\omega^{-1} \otimes \omega).$$

Where R^1 is the first derived functor of the ordinary Jacquet functor. N.B. the ordinary Jacquet functor has cohomological dimension 2.

Does

$$H^2(\mathrm{GL}_2(\mathbb{Q}_p), \overline{F}_p) = 0?$$

Is the 'bar' irrelevant?

Computing cohomology difficult because complicated interactions with the topology and the representation theory.

3. Some open problems

3.1. Conjecture: Emerton.

Conjecture 3 (Emerton)**.** *Let $\overline{\rho}$ be absolutely irreducible odd residual two-dimensional representation of $G_{\mathbb{Q}}$. Let $\mathfrak{m}_N \subseteq \mathbb{T}_N \otimes \overline{\mathbb{F}}_p$ correspond to $\overline{\rho}$ ($\mathfrak{m}_N$ generated by $(T_\ell - \mathrm{Tr}_{\overline{\rho}}(\mathrm{Frob}_\ell))_{\ell \nmid N}$)*

$$\operatorname*{inj\,lim}_N \left[H^1(X(N)_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)[\mathfrak{m}] \right] \cong \left({\bigotimes}' \overline{\pi}_\ell^{\mathrm{modified}} \left(\overline{\rho}|_{G_{\mathbb{Q}_\ell}} \right) \right) \otimes \overline{\rho}$$

where $\overline{\pi}_\ell^{\mathrm{modified}} \left(\overline{\rho}|_{G_{\mathbb{Q}_\ell}} \right)$ is a finite length admissible smooth representation of $\mathrm{GL}_2(\mathbb{Q}_\ell)$. (Emerton gives a specific $\overline{\pi}_\ell^{\mathrm{modified}}$)

3.1.1. *Calegari.* — From the explicit list of these $\overline{\pi}_\ell^{\mathrm{mod}}$ can one calculate explicitly the multiplicities of the $\mathfrak{m}$-eigenspaces in $H^1(X(N)_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)$ at finite level? (Emerton: away from p "easy", however at p is a *thesis question.*)

3.1.2. *Diamond.* — What about reducible representations?

Further: what about a version for Shimura curves? (For quaternion algebras unramified over p) Would this help understand what happens at p?

3.2. Berger.

In the notation of Breuil-Berger:

Question 1 (Berger)**.** Let $V_{/\overline{\mathbb{F}}_p}$ be an irreducible Galois representation of $G_{\mathbb{Q}_p}$ of arbitrary dimension. Consider

$$\Omega(V) = \left[\left(\operatorname{proj}\lim_{\psi} D(V)\right)^b\right]^* = \mathrm{Hom}_{\mathrm{Cont}}\left(\left(\operatorname{proj}\lim_{\psi} D(V)\right)^b, \overline{\mathbb{F}}_p\right)$$

Let $P = \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$, and $\Omega(V)$ is a smooth irreducible admissible representation of P. Which such P-representations occur as $\Omega(V)$ for some V?

3.2.1. *Ribet.* — Does the P-action extend to a larger group in any natural way? For a one- or two-dimensional V, the answer is YES.

3.3. Buzzard.

Why (φ,Γ)-modules?! Is there a useful generalization? In particular, can one replace Γ by higher dimensional p-adic Lie groups, and get a "(φ,Γ)-module". This has to classify p-adic Glaois representations, not just modulo p.

Kisin recalls for us that Fontaine gives a procedure using a norm field that produces a theory that does this for mod p representations for any p-adic Lie group Γ.

3.4. Breuil.

Question 2. Let V be a two-dimensional irreducible potentially crystalline representation of $G_{\mathbb{Q}_p}$, assume that it is of supercuspidal type (i.e. the associated WD-group representation is irreducible) Let $B(V)$ be the associated (conjecturally) irreducible admissible Banach space representation. Can one prove that the locally analytic vectors in $B(V)$ determine the Hodge filtration on $D_{\mathrm{pcris}}(V)$?

3.4.1. *Emerton.* More generally: relate $B(V)^{\mathrm{an}}$ to $D_{\mathrm{rig}}(V)$. Can one relate $(B(V)^{\mathrm{an}})'$ to the de Rham cohomology of the coverings of the p-adic upper half-plane? (Where '$'$' is the dual.)

4. Serre's conjecture: Ribet

Let p be a prime, for instance, let $p = 5$. Then suppose we have a Galois representation

$$\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

which is irreducible, odd, the question is, is it modular?

Khare induction on the prime. Tate proved for $p = 2, 3$ Tate and Serre proved that this is vacuously the case.

Start at ρ lift to $\tilde{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(E_p)$ — want $\tilde{\rho}$ to be minimal, i.e. with prescribed Serre level and weight k for $2 \leq k \leq p+1$ — it should be E-rational, compatible. This representation should lift to a Galois representation which is geometric etc... $\tilde{\rho} = \tilde{\rho}_p$ we have a family $(\tilde{\rho}_p)$. Should look as if it comes from a modular form.

Need to interweave Taylor's potential modularity theorem with deformation theory. Then use an analogue of Wiles' 3-5 trick.

One technical obstacle, is you may get a reducible representation, then one has to apply Skinner-Wiles - which means you must check the hypothesis!

Khare inducts simultaneously on the weight and the prime characteristic. Ideally, one wants to move to a lower prime, and simultaneously control the weight, in particular, reduce it. Then induct.

4.1. **Interplay between Taylor's theorem and deformation theory.** Consider level $N = 1$ and residue characteristic $p = 3$ - then it is a theorem of Serre that Serre's conjecture is true in both the strong and weak formulations. In particular, every residual Galois representation is modular (because there are none!). Consider $\overline{\rho}_p$, take a minimal lift ρ_p with level one and weight 2. Then include ρ_p in a strictly compatible family $\{\rho_p\}$. In general one must keep track of ramification. In this case, however $S = \emptyset$.

Taylor's potential modularity states that given a Barsotti-Tate representation ρ_p there exists a totally real field F such that $\rho_p|_{G_F}$ is modular. In particular, there is a Hilbert modular form over F of parallel weight two such that $\rho_p|_{G_F} \cong \rho_{f,\mathfrak{p}}$ for some $\mathfrak{p} \mid p$. One can do this process to insure that the images of ρ_p and $\rho_p|_{G_F}$ have the same image.

5. Buzzard: Serre's conjecture over $\mathbb{Q}$

Fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and let F denote an unramified extension of $\mathbb{Q}_p$. Let $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \supseteq I \subseteq \mathrm{Gal}(\overline{F}/F) = G_F$. Local class field theory gives us a canonical isomorphism $G_F^{\mathrm{ab}} \cong F^\times$, the image of I in G_F^{ab} is identified with $\mathcal{O}_F^\times$. Therefore, there exists a canonical quotient I_n of I identified with $k^\times$ where k denotes the residue field of F (where $\#k = p^n$). We say that a character $\chi : I \to \overline{\mathbb{F}}_p^\times$ has *level* n if it factors as

$$I \to I_n \to \overline{\mathbb{F}}_p^\times.$$

There are $p^n - 1$ characters of level n. We have

$$I \twoheadrightarrow I_n = k^\times.$$

A character of level n is *fundamental* if the induced group homomorphism $k^\times \to \overline{\mathbb{F}}_p^\times$ extends to an injection of fields $k \hookrightarrow \overline{F}_p$.

Let $F/\mathbb{Q}_p$ be an unramified extension.

Lemma 4. *If $\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is continuous, then either*

$$\overline{\rho} \cong \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$$

where $\chi_1|_I$ and $\chi|_I$ have level n_1; or $\overline{\rho}$ is irreducible and

$$\overline{\rho}|_I \cong \begin{pmatrix} \chi & 0 \\ 0 & \chi^{p^n} \end{pmatrix}$$

where χ of level $2n$.

If $f = \sum_{n\geq 1} a_n q^n \in \overline{\mathbb{F}}_p[[q]]$ is a mod p modular cusp form of level N, $p \nmid N$ and $a_1 = 1$ and f is an eigenform. Then there exists a Galois representation $\rho_F = \rho$ associated to F

$$\rho_F : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

continuous, odd, semisimple. If ℓ is a prime and $\ell \nmid Np$, then $\mathrm{Tr}(\rho_F(\mathrm{Frob}_\ell^{\mathrm{arith}}))$ is a_ℓ and $\det(\rho_F) = \chi_{\mathrm{cyc}}^{k-1}$ a dirichlet character of level n.

What can we say about $\rho|_{D_p}$? Good results $2 \leq k \leq p+1$. *Answer:* in this case if a_p is nonzero, then $\rho|_{D_p}$ is reducible

$$\begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$$

and $\chi|_I = \omega^{k-1}$ and $\chi_2|_I =$ trivial, where ω is the mod p cyclotomic character. If $a_p = 0$, then $\rho|_{D_p}$ is irreducible and

$$\rho|_{I_p} \cong \begin{pmatrix} \psi^{k-1} & 0 \\ 0 & \psi^{p(k-1)} \end{pmatrix}$$

where ψ is fundamental of level 2.

Some facts: — If $f = \sum a_n q^n$ is a mod p weight k cusp form, then $Af = \sum a_n q^n$ is a mod p weight $k + (p-1)$ cusp form. And $\Theta f = \sum n a_n q^n$ is a mod p weight $k + (p+1)$ cusp form.

$$\rho_{Af} \cong \rho_f \quad \text{and} \quad \rho_{\Theta f} \cong \rho_f \otimes \omega$$

So if $\rho \cong \rho_f$ for some f of weight k, then $\rho \otimes \omega$ modular weight $k + (p+1)$ and ρ is modular of weight $k + (p-1)$.

These are the ingredients of Serre's precise conjecture. If $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is continuous, odd and irreducible, then Serre predicts ρ is modular and furthermore predicts the precise weight $k(\rho)$ for which there exists an f of weight k such that $\rho \cong \rho_F$.

Idea for $k(\rho)$: — say

$$\rho|_{I_p} \cong \begin{pmatrix} \omega^a & * \\ 0 & \psi^{p(k-1)} \end{pmatrix}$$

and

$$(\omega^{-b} \otimes \rho)|_{I_p} \sim \begin{pmatrix} \omega^{(a-b)} & * \\ 0 & 1 \end{pmatrix}$$

looks modular of weight $a - b + 1$, therefore ρ looks modular of weight $(a - b + 1) + b(p+1)$, if furthermore $* = 0$ then

$$\rho|_{I_p} \sim \begin{pmatrix} \omega^b & * \\ 0 & \omega^a \end{pmatrix}$$

and same trick gives another k.

How do you generalize to totally real fields?

Annoying fact: — if f is a characteristic zero Hilbert modular form of weight $(k_1, \ldots, k_\alpha)$ and all k_i congruent modulo 2, of level prime to p. Then for $w \in \mathbb{Z}$ (which is congruent to k mod 2) there exists an automorphic form $\pi_{f,\alpha}$ associated to f and $\rho_{\pi_{f,\alpha}}$ is crystalline at all places of F above p thne $\det \rho_{\pi_{f,w}} = \omega^{\text{integer}} \times \text{char conductor prime to } p$.

Problem: — typically there are mod p totally odd representations of $\mathrm{Gal}(\overline{F}/F)$ whose determinant is not the reduction of $\omega^{\text{int}} \times$ (prime to p). Therefore, the naive generalization of Serre's conjecture should NOT say that for all $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ continuous totally odd irreducible are modular coming from a Hilbert modular form of level prime to p.

Fred's fix: — totally rethink the notion of weight.

Say f is a weight k classical mod p modular form, where $2 < k < p+1$, of level N prime to p. One can lift f to some characteristic zero form F of weight 2 and level $\Gamma_1(Np)$, of character ω at p.. Can find ρ_f in $\mathrm{Jac}(X_n(Np))[p]$. Assume from now on that everything has an implicit level N. One can find f is a certain subspace of $\mathrm{Pic}^\circ(X(p))[p]$ where $X(p)_{/\mathbb{Q}}$ is the non-geometrically connected modular curve of level $\Gamma_1(N) \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (mod p) over $\mathbb{Q}$.

The group $\mathrm{Pic}^\circ(X(p))[p](\overline{\mathbb{Q}})$ has an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and a commuting action of $\mathrm{GL}_2(\mathbb{F}_p)$, and our modular ρ_F lives in the subspace of this Pic° where $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subseteq \mathrm{GL}_2(\mathbb{F}_p)$ is acting in a certain explicit way.

One can now write down an explicit irreducible mod p representation of $\mathrm{GL}_2(\mathbb{F}_p)$, say V_k, such that $\overline{\rho} \subseteq \mathrm{Hom}_G(V, \mathrm{Pic}^\circ(X(p))[p](\overline{\mathbb{Q}}))$. This latter space is the one which generalizes to the totally real setting.

Emerton: — $\overline{\rho}$ modular at weight V - where V is *any* irreducible mod p representation of $\mathrm{GL}_2(\mathbb{F}_p)$, if $\overline{\rho} \subseteq \mathrm{Hom}_G(V, H^1_{\text{et}}(X(p), \mathbb{F}_p))$.

Diamond: — $\overline{\rho}$ modular of weight V if

$$\overline{\rho} \subseteq (V_{\mathbb{F}_p} \otimes \mathrm{Pic}^\circ(X(p))[p](\overline{\mathbb{Q}}))^G = \mathrm{Hom}_G(V^*, \mathrm{Pic}^\circ(X(p))[p](\overline{\mathbb{Q}}))$$

One gets a reformulation of Serre's conjecture: If ρ is continuous, odd, irreducible $G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$, then ρ is modular, and furthermore it is modular for weight V - for which we have a recipe. The recipe now looks "nicer", because the irreducible mod p representations of $\mathrm{GL}_2(\mathbb{F}_p)$ are all of the form $\det^a \otimes \mathrm{Sym}^{b-1}(\mathbb{F}_p^2)$ where $0 \le a \le p-2$ and $1 \le b \le p$.

Get a simpler picture, e.g. in the irreducible case:

$$\rho|_I = \omega^a \begin{pmatrix} \psi^b & 0 \\ 0 & \psi^{pb} \end{pmatrix}$$

where ω is fundamental of level 1 and ψ fundamental level 2. Fred predicts weight $V = \det^a \otimes \mathrm{Sym}^{b-1}$.

6. Gee: Proof of Buzzard-Diamond-Jarvis

Let F be totally real, and let $p > 2$ be unramified in F. Let

$$\overline{\rho} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p),$$

be modular of weights $\{V\}$ where V are irreducible characteristic p representations of $\mathrm{GL}_2(\mathcal{O}_F/p)$ and $V = \bigotimes_{V|p} V_{\mathbf{a}_v,\mathbf{b}_v}$ where $\mathbf{a}_v, \mathbf{b}_v$ are $[k_v : \mathbb{F}_p]$-tuples indexed by $\sigma : k_v \hookrightarrow \overline{\mathbb{F}}_p$, where $0 \le \mathbf{a}_v \le p-1$, not all $\mathbf{a}_v = p-1$ and $1 \le \mathbf{b}_v \le p$.

$$V_{\mathbf{a}_v,\mathbf{b}_v} = \bigotimes_{\sigma: k_v \hookrightarrow \overline{\mathbb{F}}_p} \left(\det^{\mathbf{a}_v} \mathrm{Sym}^{\mathbf{b}_v - 1} k_v^2\right) \otimes_\sigma \overline{\mathbb{F}}_p.$$

There exists explicit recipe for $\overline{\rho}|_{G_v}$ to $\{V_{v_p}$ respresentations of $\mathrm{GL}_2(k_v)\}$ and $\overline{\rho}$ to $\{V = \otimes V_v\}$.

Assume p inert: If $\overline{\rho}|_{G_p}$ is irreducible then there are 2^f weights, $f = [F : \mathbb{Q}]$. If $\overline{\rho}|_{G_p}$ is reducible, there are $\le 2^f$ weights (generically).

$$\begin{pmatrix} \psi_1 & * \\ 0 & \psi_2 \end{pmatrix}$$

if $* = 0$ you get all the weights, if $*$ is generic, get 1 weights.

Say a weight $\otimes_{v|p} V_{\mathbf{a}_v,\mathbf{b}_v}$ is *regular* if $2 \le \mathbf{b}_v \le p-2$ for all v.

Theorem 5 (Gee)**.** *Assume further that $\overline{\rho}(G_F)$ is non-solvable. (Can be removed). Assume also that for each v: $\overline{\rho}|_{G_v}$ is not scalar. If V is a regular weight, $\overline{\rho}$ is irreducible, $\overline{\rho}$ is modular of weight V if and only if B-D-J predicted that it is.*

Theorem 6 (Gee)**.** *For $p > 2$ and F a totally real field, with p unramified in F. Let $E/\mathbb{Q}_p$ be a finite extension, and let $\mathcal{O}$ denote the ring of integers of E. Let*

$$\rho : G_F \to \mathrm{GL}_2(\mathcal{O}),$$

be continuous, unramified outside of a finite set of primes, and $\det \rho = (\mathrm{cyc})($*finite order*$)$. *Suppose that*

(1) *$\rho|_{G_v}$ is potentially Barsotti-Tate for all $v \mid p$.*

(2) *$\overline{\rho}$ is modular*

(3) *$\overline{\rho}|_{G_{F(\zeta_p)}}$ is absolutely irreducible.*

then ρ is modular.

Assume $2 \le k \le p$, $p > 2$ if a modular newform of level $\Gamma_1(N)$, $p \nmid N$ and weight k.

$$\overline{\rho}_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p),$$

assume $\overline{\rho}_f$ also irreducible. Assume

$$\overline{\rho}_f|_{G_p} \cong \begin{pmatrix} \psi_1\omega^{k-1} & 0 \\ 0 & \psi_2 \end{pmatrix}$$

where ψ_1 and ψ_2 are unramified an $\omega^{k-1}\psi_1 \neq \psi_2$.

then $(\overline{\rho}_f \otimes \omega^{k'-1})|_{G_p} \cong \begin{pmatrix} \psi_2\omega^{k'-1} & 0 \\ 0 & \psi_1 \end{pmatrix}$ where

$$k' = \begin{cases} p+1-k & \text{if } k \neq p \\ p & \text{if } k = p \end{cases}$$

Serre predicts that there exists an eigenform of weight k', of level $\Gamma_1(N)$, such that $\overline{\rho}_g \cong \overline{\rho}_f \otimes \omega^{k'-1}$. If $k = p$, the U_p-eigenvalue of g is congruent to $\psi_2(\mathrm{Frob}_p)$ modulo p.

By using Hida theory it suffices to find g' of level $\Gamma_1(Np)$, and weight 2 with

$$\overline{\rho}_{g'} \cong \overline{\rho}_f \otimes \omega^{k'-1}.$$

then

$$\rho_{g'} \cong \begin{pmatrix} \tilde{\psi}_2\tilde{\omega}^{k'-2}\chi_{\mathrm{cyc}} & * \\ 0 & \tilde{\psi}_1 \end{pmatrix}$$

where ˜ stands for Teichmüller lifts.

Assume that $\overline{\rho}(G_{\mathbb{Q}})$ is non-solvable. Now: (1) find $\rho_{g'}$, then (2) prove $\rho_{g'}$ is modular. For (2) we simply check the hypothesis of the earlier theorem. (1) follows from a theorem of Ramakrishna (and Taylor). In essence on has to check that the local deformation ring at p is large enough - a dimension calculation.

For B-D-J one has to consider many lifts. In fact, the lifts we want to consider are potentially Barsotti-Tate of a specified type. (These types are always tame). Starting the a residual representation considers all lifts of this type. Then using combinatorial arguments you control the weights.

7. Buzzard: p-adic Local Langlands

For $\mathrm{GL}_2(K)$, where $K/\mathbb{Q}_p$ finite: it bijects supercuspidal (infinite dimensional) representations of $\mathrm{GL}_2(K)$ with irreducible 2-dimensional $\mathbb{C}$-representations of the Weil group W_K. This first set is contained in the set of smooth irreducible admissible representations of $\mathrm{GL}_2(K)$. The latter set is contained in the set of F-semisimple 2-dimensional Weil-Delgine representations.

Vigneras: situation is also good when we replace $\mathbb{C}$ by $\overline{\mathbb{F}}_\ell$ for $\ell \neq p$.

What about $\overline{\mathbb{F}}_p$? What about a "mod p local Langlands?"

Objects on the right-hand side: $\{$continuous $\rho : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)\}$, this set contains the irreducible representations.

At least for $K/\mathbb{Q}_p$ unramified, then ρ gives rise to a finite set of irreducible representations of $\mathrm{GL}_2(k)$ where k denotes the residue field of K.

e.g. When $K = \mathbb{Q}_p$ and ρ irreducible

$$\rho|_I = \begin{pmatrix} \psi^b & 0 \\ 0 & \psi^{pb} \end{pmatrix}$$

you get Sym^{b-1} and also twist of Sym^{p+b}.

Take $F/\mathbb{Q}_p$ unramified, with ring of integers $\mathcal{O}$. Let $K = \mathrm{GL}_2(\mathcal{O})$, and $Z = F^\times \hookrightarrow G = \mathrm{GL}_2(F)$ and k be the residue field of F. If V is a finite dimensional representation of $\mathrm{GL}_2(k)$ over $\overline{\mathbb{F}}_p$. Then make V a representation of K by letting K act via $\mathrm{GL}_2(k)$ and then a representation of KZ by letting $\mathcal{O}^\times$ act via K and letting $\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$ act trivially. Define $c - \mathrm{Ind}_{KZ}^G V$ to be the set of functions $f : G \to V$ such that $f(kg) = k * f(g)$ for all $k \in KZ$, where $*$ is the action of KZ on V; and such that the support of f is a finite union of costs of KZ. We define a G-action on $c - \mathrm{Ind}$ by putting $(gf)(g') = f(g'g)$. Note that $c - \mathrm{Ind}$ is an infinite-dimensional representation of G.

Consider all the embeddings $k \overset{\sigma}{\hookrightarrow} \overline{\mathbb{F}}_p$. Now assume that $V = \bigotimes_\sigma \sigma \circ \mathrm{Sym}^{r_\sigma} k^2$ where $0 \leq r_\sigma \leq p-1$. [Up to twists this is all the irreducible representations of $\mathrm{GL}_2(k)$. Explicitly: homogeneous polynomials of degree r_σ in two-variables x and y such that

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} f\right)(x,y) = f(ax+cy, bx+dy)$$

Define an $\overline{\mathbb{F}}_p$-linear map $U = \otimes U_\sigma : V \to V$, by

$$U_\sigma(x^i y^{r_\sigma - i}) = 0$$

if $i > 0$ and

$$U_\sigma(y^{r_\sigma}) = y^{r_\sigma}.$$

Now define a map

$$\varphi : G \to \mathrm{End}_{\overline{\mathbb{F}}_p}(V)$$

by

$$\varphi\left(\begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix}\right) = U$$

Extend to $KZ \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix} KZ$ by $\varphi(ka^{-1}k') = k * \varphi(a^{-1} * k'$. Then extend to G by 0. So φ gives rise to a G-endomorphism of $c - \mathrm{Ind}_{KZ}^G V$ in a natural way — call this T.

Wonderful observations of Barthel-Livné: Let $W_{\mathbf{r}} := c - \mathrm{Ind}_{KZ}^G \otimes \mathrm{Sym}^{r_\sigma}$.

If $\lambda \in \overline{\mathbb{F}}_p$, $\lambda \neq 0$, then $W_{\mathbf{r}}/(T-\lambda)$ is almost always an irreducible smooth admissible representation of G (call these *principal series*), except occasionally it has length 2, $1-d$ subquotient and Steinberg subquotient.

An irreducible representation of G is *supersingular* if its a quotient of $W_{\mathbf{r}}/(T)$. The principal series are never isomorphic to $1-d$ which are never isomorphic to the Steinberg which is never isomorphic to principal series representations and none are ever isomorphic to supersingular. Within the principal series, $1-d$, Steinberg understood. Supersingular case: mysterious $W_{\mathbf{r}}/(T)$ does have infinite length if $K \neq \mathbb{Q}_p$.

Breuil: restricts to $K = \mathbb{Q}_p$ and $\mathbf{r} = r \in \{0, \ldots, p-1\}$, Breuil finishes the story: he shows $W_{\mathbf{r}}/(T)$ is irreducible and $W_{\mathbf{r}}/(T)$ is isomorphic to twist of $W_s/(T)$ if and only if $r = s$ or $r + s = p-1$ and know exactly the twist.

B-L observe that: any smooth irreducible admissible $\overline{\mathbb{F}}_p$-representation of $G = \mathrm{GL}_2(F)$, with a central character is $1-d$, principal series, Steinberg, or supersingular — up to twist.

For $F = \mathbb{Q}_p$ we can write down all smooth admissible irreducible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ and all two-dimensional representations of $G_{\mathbb{Q}_p}$.

Restrict to the semi-simple case: Choose a lift F of Frobenius in $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and restrict to representations ρ of $G_{\mathbb{Q}_p}$ such that $\det \rho(F) = 1$.

If $\rho|_I = \begin{pmatrix} \psi^{r+1} & 0 \\ 0 & \psi^{p(r+1)} \end{pmatrix}$, where ψ is fundamental of level 2, then match with $W_r/(T)$.

If $\rho = \begin{pmatrix} \omega^{r+1} \times \mathrm{unr}(\lambda) & 0 \\ 0 & \mathrm{unr}(\lambda^{-1}) \end{pmatrix}$ then match with

$$(W_r/(T-\lambda))^{\mathrm{ss}} \oplus \left(W_{p-3-r}/(T-\lambda^{-1}) \otimes \omega^{r+1}\right)^{\mathrm{ss}}.$$

This all gives us a semi-simple local Langlands conjecture (theorem for $\mathrm{GL}_2(\mathbb{Q}_p)$).

Matthew Emerton's picture:

$$\begin{tikzcd}
 & H^{\Gamma(p)} \arrow[d, equal] & H \arrow[d, equal] \\
V \arrow[r, hook] & H^1(X(p), \overline{\mathbb{F}}_p)_{\overline{\rho}} \arrow[r, hook] & \operatorname{inj\,lim} H^1(X(p^r), \mathbb{F}_p)_{\overline{\rho}}
\end{tikzcd}$$

These groups all have a $\mathrm{GL}_2(\mathbb{F})$ action on them.

Finally, consider $\mathrm{GL}_2(F)$ where $F = \mathbb{Q}_{p^2}$ and $\mathbf{r} = (r_0, r_1)$, and let $V_{\mathbf{r}}$ be a representation of $\mathrm{GL}_2(\mathbb{F}_{p^2})$. It is known that $W_{\mathbf{r}}/(T)$ has infinite length. This is too big. Barthel-Livné suggest that we should consider quotients of this object.

Paskunas: writes downa particular irreducible quotient of $W_{\mathbf{r}}/(T)$ call it $P_{\mathbf{r}}$. Paskunas showed that $P_{\mathbf{r}}$ is isomorphic to a twist of $P_{\mathbf{s}}$ if and only if $\mathbf{r} = \mathbf{s}$ or $\mathbf{r} = p - 1 - \mathbf{s}$ where $\mathbf{r} \in \{0, \ldots, p-1\}^2$.

Up to unramified twist we get exactly $q(q-1)/2$ irreducible non-isomorphic supersingular representations of $\mathrm{GL}_2(F)$. Now we guess: irreducible representations of $G_F \longleftrightarrow P_{\mathbf{r}}$ —- WRONG.

$V_{\mathbf{a},\mathbf{b}} = \det^{a_0} \times \sigma \circ \det^{a_1} \times \mathrm{Sym}^{b_0 - 1} \otimes \sigma \circ \mathrm{Sym}^{b_1 - 1}$. $\mathbb{F}_{p^2} \hookrightarrow \overline{\mathbb{F}}_p$

ρ irreducible and

$$\rho|_I = \begin{pmatrix} \psi_4^{r_0+1+p(r_1+1)} & 0 \\ 0 & \psi_4 p^2(this) \end{pmatrix} \tag{2}$$

If $\rho \to P_{\mathbf{r}}$ for some $\mathbf{r}$, then because $P_{\mathbf{r}}$ is isomorphic to a twist of $P_{p-1-\mathbf{r}}$ we must see 2 lines such that sum of b's is $(p-1, p-1)$

Fred predicts $V_{a,b}$:

a_0	a_1	$b_0 - 1$	$b_1 - 1$
0	0	r_0	r_1
$r_0 + 1$	-1	$p - 2 - r_0$	$r_1 + 1$
0	$r_1 + 1$	$r_0 - 1$	$p - 2 - r_1$
r_0	$r_1 + 1$	$p - 1 - r_0$	$p - 3 - r_1$

If ρ is reducible, then

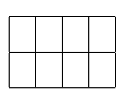
$$\rho|_I \cong \begin{pmatrix} \psi_2^{r_0+1+p(r_1+1)} & 0 \\ 0 & 1 \end{pmatrix} \longleftrightarrow PS \oplus PS \oplus P_{\mathrm{r}}$$

Fred predicts:

a_0	a_1	$b_0 - 1$	$b_1 - 1$
0	0	r_0	r_1
$r_0 + 1$	$r_1 + 1$	$p - 3 - r_0$	$p - 3 - r_1$
$p - 1$	r_1	$r_0 + 1$	$p - 2 - r$
r_0	$p - 1$	$p - 2 - r_0$	$r_1 + 1$

Thus (2) corresponds to a new quotient of $W_{\mathbf{r}}/(T)$.

Matt had an insighful diagram with (too!) much commentary:



8. Kisin: Pseudo-representations

Pseudo-representations: — G a group and A a ring, or more generally an A-algebra R. Consider functions $T : R \to A$, satisfying two conditions: (1) $T(xy) = T(yx)$, (2) depends on d, (3) $T(1) = d \in \mathbb{N}$. Furthermore we require $d!$ is invertible in A. For $d = 2$, thus, we suppose that 2 is invertible in A. If you have an element $\sigma \in R$ then we can define $T(\sigma)$.

Define $S(\sigma) := \frac{1}{2}(T(\sigma)^2 - T(\sigma^2))$, then S is a character, i.e. $S(\sigma_1)S(\sigma_2) = S(\sigma_1\sigma_2)$. We have

$$X^2 - T(\sigma)X + S(\sigma).$$

Define $\mathrm{Ker}(T) = \{x \in R : T(xy) = 0 \forall y \in R\}$. Then we have a map

$$\overline{R} = R/\mathrm{Ker}(T) \to A,$$

for $\sigma \in \overline{R} : P_\sigma(x)$ — the characteristic polynomial of σ. Ask the following question: is $P_\sigma(\sigma) = 0$.

Theorem 7 (Taylor)**.** *If A is an algebraically closed field, then there is a correspondence between: the set of pseudo-representations and the set of semi-simple representations.*

Always genuine representations give pseudo-representations.

8.1. **Problem 1:** Recast the theory so that $d!^{-1}$ is not necessarily in A. [[This is not simply a problem about divided powers]].

8.2. **Problem 2:** The relationship between moduli of pseudo-representations and representations. We have the following theorem:

Theorem 8 (Nyssen, Rouquire)**.** *If the representation is absolutely irreducible, then these moduli are equivalent.*

Let $\mathbb{F}$ be a (finite) field, for example

$$V_{\mathbb{F}} = \omega_1 \oplus \omega_2 \mapsto T_{\mathbb{F}}$$

such that ω_1 and ω_2 are distinct characters. Look at representations whose reduction is a nontrivial extension of ω_1 by ω_2. Let A be an $W(\mathbb{F})$-algebra.

Consider the following diagram:

$$\begin{tikzcd} X_{\omega_2} \arrow[d] \\ \mathrm{Spec}(R(T_{\mathbb{F}})) \end{tikzcd}$$

Note that $\mathrm{Ext}^1(\omega_1, \omega_2) \setminus \{0\}/\mathbb{F}^\times$. We have the following: [[Mark drew a picture of a cone (the special fiber) projecting on to a disc.]]

Taking a point x on the special fiber. Then x gives a representation V_x (= extensions of ω_1 by ω_2). Completing we have

$$\widehat{R}_x = \text{ universal deformation ring of } V_x.$$

Now X_{ω_2} carries a universal rank 2 vector bundle V_{ω_2}. Taking the direct image: $\pi_* V_{\omega_2}$, we get a bundle on $\mathrm{Spec}(R(T_{\mathbb{F}}))$.

Mark suggested the following diagram:

$$\begin{tikzcd} X_{\omega_2} \arrow[dr] & & X_{\omega_1} \arrow[dl] \\ & \mathrm{Spec}(R(T_{\mathbb{F}})) & \end{tikzcd}$$

and that maybe one could glue along the trivial extension and get some geometric object - perhaps an algebraic stack. Richard ask if X_{ω_2} was even proper, Mark insisted it should be. In fact, by the valuative criterion of properness, this is indeed the case.

8.3. **Problem 3:** Classically, if one is looking at deformation rings of Galois representations one has various cohomological tools. One would like analogous of these in the situation of pseudo-representations.

9. Kedlaya: (φ, Γ)-modules

9.1. Motivation.

9.1.1. *Dieudonné-Manin classification.* — Let k be an algebraically closed field of characteristic $p > 0$, and let K be a finite extension of the field of fractions of the Witt vectors $W(k)$, fix a uniformizer π of $W(k)$. Let φ denote a lifting of the Frobenius endomorphism.

Definition 1. A φ-module over K is a finite free K-module M equipped with a semilinear φ-action, i.e. $M \to M$ such that $\varphi^*(M) = M \otimes_\varphi K \tilde{\to} M$.

Theorem 9 (Dieudonné-Manin classification)**.** *For $r = \frac{a}{b} \in \mathbb{Q}$ ($b > 0$ and $(a,b) = 0$), let M_r be the φ-module defined by:*

$$\begin{pmatrix} 0 & & & \pi^a \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}$$

Then every φ-module over K is isomorphic to a direct sum of M_r's. In particular $\mathrm{Ext}^1(M_r, M_s) = 0$*. Moreover $M_r \cong M_s$ if and only if $r = s$.*

Define the *degree* of a φ-module as follows: if M has rank one, say $M = (x)$, i.e. pick $v \in M$, then $\varphi(v) = xv$. Define $\deg(M) = v_p(x)$. Where v_p is the p-adic valuation. Define $\deg(M) = \deg(\bigwedge^{\mathrm{rank}(M)} M)$. If $M_r \cong M_s$ implies that their ranks are equal, in particular their degrees are equal. We define the *slope* of a φ-module M as the quotient $\deg(M)/\mathrm{rank}(M)$.

Note that the decomposition of a φ-module is not unique. For example:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

has fixed vectors

$$K^\varphi e_1 + K^\varphi e_2.$$

Now assume that k is only assumed to be perfect, and not algebraically closed. Apply the D-M classification over $\widehat{K^{\mathrm{unr}}}$ get isotypical decomposition of $M \otimes_K \widehat{K^{\mathrm{unr}}}$, which descends to a decomposition "pure slope decomposition."

Alternative characterization of "pure." Say M has rank r and degree d, then M is *pure* of slope r/d if there exists an $\mathcal{O}_K$-lattice L of M such that $\pi^{-r}\varphi^d$ acts on L and

$$\left(\pi^{-r}\varphi^d\right)^* L \to L$$

i.e. in some basis $\pi^{-r}\varphi^d$ acts via an invertible matrix over $\mathcal{O}_K$.

Exercise: M is pure of slope r/d if and only if $M \otimes_K \widehat{K^{\mathrm{unr}}} \cong (M_{r/d})^{\oplus i}$ for some i. "Pure of slope zero" = "étale" = "unit-root." Also (pure)$\otimes$(pure)=(pure).

Now let k be an *arbitrary* field of characteristic $p > 0$, and let K be a finite extension of the field of fractions of a Cohen ring. For instance, for F a finite extension of $\mathbb{Q}_p$ we can define

$$\mathcal{E} = \mathcal{O}_F\widehat{[[t]][t^{-1}]}[\frac{1}{p}].$$

Apply D-M classification over $\widehat{L^{\mathrm{unr}}}$ where $L = \widehat{\mathrm{inj}\lim_\varphi} K$. We get an isotypical decomposition of $M \otimes L$ but not of M itself.

$$\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & x \\ 0 & p \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}^{\varphi} = \begin{pmatrix} 1 & x + \varphi(y) - py \\ 0 & p \end{pmatrix}$$

which does not split, but $\begin{pmatrix} p & x \\ 0 & 1 \end{pmatrix}$ splits. Get on M a slope filtration:

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_\ell = M$$

where each M_i/M_{i-1} is pure of slope s_i, with $s_1 < s_2 < \cdots < s_\ell$.

Recall that to a p-adic Galois representation $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathbb{Q}_p)$ one can associate an étale (φ, Γ)-module over the ring $\mathcal{E}$, where $\Gamma = \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$. Define $\mathcal{E}^\dagger$ to be those power series in t which converge and are bounded in some annulus $* \leq |t| < 1$. This ring is *not* complete for the p-adic topology, but it is henselian. If you complete it for the p-adic topology, then you get $\mathcal{E}$. Also define the *Robba ring* $\mathcal{R}$ to be those power series in t which converge in some annulus $* \leq |t| < 1$.

Note that $\mathcal{R}$ is not a field, however its units are bounded, i.e. belong to $\mathcal{E}^\dagger$. In particular the concept of degree still makes sense over the Robba ring $\mathcal{R}$ as then does the concept of slope. When you define, however, "pure of some slope" over $\mathcal{R}$, the lattice should be over $\mathcal{O}_{E^\dagger}$.

Theorem 10. *There is a functor:*

$$\{\text{\'etale } \varphi\text{-modules}\}_{/\mathcal{E}^\dagger} \to \{\text{\'etale } \varphi\text{-modules}\}_{/\mathcal{R}}$$

given by "tensor with $\mathcal{R}$", which is an equivalence of categories. The same is then true for (φ, Γ)-modules.

Note that there is still a functor from the category of étale φ-modules over $\mathcal{E}^\dagger$ to the category of étale φ-modules over $\mathcal{E}$, however, this is only fully faithful, but *not* essentially surjective. Restricting this latter functor to (φ, Γ)-modules gives the functor of Colmez. In fact, this restricted functor *is* an equivalence of categories (Theorem of Chernonnier-Colmez).

$$\begin{tikzcd}
\mathcal{R} \arrow[dr, dash] & & (\mathcal{E}^\dagger)^{\mathrm{unr}} \arrow[dl, dash] \\
& \tilde{\mathcal{R}} &
\end{tikzcd}$$

We can get a DM-classification over $\tilde{\mathcal{R}}$, then we descend

Theorem 11. *Let M be a φ-module over $\mathcal{R}$, then there exists a unique filtration*

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_\ell = M$$

where each M_i/M_{i-1} is pure of slope s_i and $s_1 < s_2 < \cdots < s_\ell$.

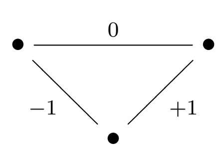
We can also have filtrations going the "wrong way."

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$$

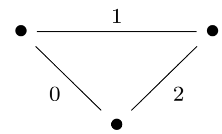
where M_1 is pure of rank 1 slope 1, M is pure of rank 2 and slope 0, and M_2 pure of rank 1 and slope 2. For example, (φ, Γ)-modules, modular form of weight 3 and $a_p \equiv 0 \pmod p$. If, on the other hand, $a_p \neq 0$ modulo p then the representation well be *ordinary*, and you get an exact sequence:

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$$

each φ-module pure of slope zero. The Newton polygon looks like:



where as the Hodge polygon looks like:



In principle: The Newton polygon is not equal to a Hodge polygon.

9.2. (φ,Γ)**-module from Galois representation.** Fix embeddings: $\mathbb{Q}_p \subset \mathbb{Q}_p(\mu_{p^\infty}) \subset \overline{\mathbb{Q}}_p \subset \mathbb{C}_p$. Let $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}(V)$ be a finite dimensional p-adic Galois representation. Consider $\mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}$ — this has a φ-p-power Frobenius action on it. Fix a compatible system of units: $\varepsilon = (\ldots, \varepsilon_1, \varepsilon_0)$, and put $t = [\varepsilon] - 1$, then construct:

$$W\left(\mathrm{Frac}\left(\operatorname*{proj\,lim}_{\varphi} \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}\right)\right)\left[\frac{1}{p}\right] \supset \mathcal{E} := \mathbb{Z}_p[[t]]\left[t^{-1}\right]\left[\frac{1}{p}\right] \supset \widehat{\mathcal{E}^{\mathrm{unr}}}$$

this has a $G_{\mathbb{Q}_p}$ action on it.

Theorem 12 (Fontaine)**.** *Then we define the following étale* (φ,Γ)*-module:*

$$D(V) := \left(V \otimes_{\mathbb{Q}_p} \widehat{\mathcal{E}^{\mathrm{unr}}}\right)^H = \textit{finite free } \mathcal{E}\textit{-module of rank } \dim_{\mathbb{Q}_p}(V)$$

and this is equivalent to

$$\left(D(V) \otimes_{\mathcal{E}} \widehat{\mathcal{E}^{\mathrm{unr}}}\right)^{\varphi=1} \cong V,$$

where Γ *acts on the first factor and* $G_{\mathbb{Q}_p}$ *acts on the second.*

10. Emerton part II

10.1. **Set up.** Let $\mathbb{F}$ be a finite field extension of $\mathbb{F}_p$, fix $\mathcal{O} = \mathcal{O}_K \subset K$, where $\mathbb{F}$ is the residue field of K and K is a finite extension of $\mathbb{Q}_p$. Let $\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$ be an absolutely irreducible, modular Galois representation. Denote by $\mathcal{R}$ the universal deformation ring of $\overline{\rho}$ unramified outside some set Σ. Define $\hat{H}^1 = \hat{H}^1_{\Sigma,\overline{\rho}} = \operatorname{inj\,lim}_{n_1,\ldots n_s} H^1(X(q_1^{n_1}\cdots q_s^{n_s}), \mathcal{O})_{\mathfrak{m}_{\overline{\rho}}}$ where $\mathfrak{m}_{\overline{\rho}}$ is the maximal ideal in $\mathbb{T}(q_1^{n_1}\cdots q_s^{n_s})$ corresponding to $\overline{\rho}$ — where $\mathbb{T}(q_1^{n_1}\cdots q_s^{n_s})$ the algebra generated by T_ℓ for all $\ell \neq \Sigma$.

$$\begin{tikzcd}
\mathcal{R} \arrow[r, two heads] \arrow[d] & \mathbb{T}_{\Sigma,\overline{\rho}} = \operatorname{proj\,lim} \mathbb{T}(q_1^{n_1}\cdots q_s^{n_s})_{\mathfrak{m}_{\overline{\rho}}} \\
\mathcal{R}^{\mathrm{mod}} \arrow[ur] &
\end{tikzcd}$$

Geometrically, we have

$$\mathrm{Spec}(\mathcal{R}^{\mathrm{mod}}) \hookrightarrow \mathrm{Spec}\mathcal{R}.$$

This is the Zariski closure of all $x \in \mathrm{Spec}\mathcal{R}$ corresponding to classical modular forms lifting $\overline{\rho}$ unramified outside Σ.

Theorem 13 (Boeckle)**.** *If* $p > 2$ *and* $\overline{\rho}|_{G_{\mathbb{Q}_p}}$ *is flat or ordinary not* $\begin{pmatrix} \omega^{-1} & * \\ 0 & 1 \end{pmatrix}$, *and if* $\overline{\rho}|_{G_{\mathbb{Q}_p(\sqrt{p^*})}}$ *is irreducible, then we have an isomorphism* $\mathcal{R} \tilde{\to} \mathcal{R}^{\mathrm{mod}}$.

Argument: Taylor-Wiles gets lots of points in $\mathcal{R}^{\mathrm{mod}}$, infinite fern lets you fill this out in families. Let $\mathcal{R}^{\mathrm{mod}}/I$ be the ring of a component of $\mathrm{Spec}\mathcal{R}^{\mathrm{mod}}$, then we have the following conjecture:

Conjecture 4. *There is an equivariant isomorphism:*

$$\hat{H}^1[I] \overset{??}{\cong} \rho_{\mathcal{R}^{\mathrm{mod}}/I} \hat{\otimes}_{\mathcal{R}^{\mathrm{mod}}/I} \hat{\pi}_p \left(\rho_{\mathcal{R}^{\mathrm{mod}}/I}|_{G_{\mathbb{Q}_p}}\right) \hat{\otimes} \left(\widehat{\bigotimes}_{\ell \neq p, \ell \in \Sigma} \hat{\pi}_\ell^{\mathrm{mod}} \left(\rho_{\mathcal{R}^{\mathrm{mod}}/I}|_{G_{\mathbb{Q}_\ell}}\right)\right) \tag{3}$$

where $\rho_{\mathcal{R}^{\mathrm{mod}}/I}$ is the universal deformation over $\mathrm{Spec}(\mathcal{R}^{\mathrm{mod}}/I)$*, and $\hat{\pi}_p$ is an orthonomalizable $\mathcal{R}^{\mathrm{mod}}/I$-Banach module, attached to $\rho_{\mathcal{R}^{\mathrm{mod}}/I}|_{G_{\mathbb{Q}_p}}$ via p-adic local Langlands. Also, π_ℓ^{mod} is the $\mathfrak{m}$-adic completion of modified local Langlands at ℓ applied to "generic fiber of $\rho_{\mathcal{R}^{\mathrm{mod}}/I}$, descended to $\mathcal{R}^{\mathrm{mod}}/I$." Note that the term on the left hand-side has an action of $G_{\mathbb{Q}} \times \prod_{q\in\Sigma} \mathrm{GL}_2(\mathbb{Q}_q)$, and the terms on the right hand-side have an action of $G_{\mathbb{Q}}$, $\mathrm{GL}_2(\mathbb{Q}_p)$, $\mathrm{GL}_2(\mathbb{Q}_q)$ respectively.*

Hypotheses:

(1) Assume that $\overline{\rho}|_{G_{\mathbb{Q}_p}}$ has scalar endomorphisms — irreducible $\begin{pmatrix} \chi & * \\ 0 & \psi \end{pmatrix}$ where $\chi \neq \psi$ and $* \neq 0$ and it is *not* a twist of $\begin{pmatrix} \omega^{-1} & * \\ 0 & 1 \end{pmatrix}$.

(2) Suppose there exists $\hat{\pi}_p(\mathcal{R}^{\mathrm{mod}}/I)$-orthonormalizable $\mathfrak{m}$-adically complete $\mathcal{R}^{\mathrm{mod}}/I$-module. Such that for all classical modular forms f unramified outside $\Sigma - \{p\}$ of weight $k \geq 2$, deforming $\overline{\rho}$ giving rise to
$$\varphi_f : \mathcal{R}^{\mathrm{mod}} \to K_f$$
where ϕ_f factors through $\mathcal{R}^{\mathrm{mod}}/I$, we have $\hat{\pi}_p(\mathcal{R}^{\mathrm{mod}}/I)\hat{\otimes}_{\varphi_f} K_f \cong B(\rho_f|_{G_{\mathbb{Q}_p}})$ — Berger-Breuil-Emerton.

Theorem 14. *Assume the two above hypotheses. Then* (3) *holds.*

Remark 1. Assuming unproved results of Colmez, one could actually remove the second hypothesis.

Corollary 15. *Under the same hypotheses of the previous theorem. If ρ is a deformation of $\overline{\rho}$ to E (some extension of $\mathbb{Q}_p$), such that the map $\mathcal{R} \xrightarrow{\varphi_\rho} E$ factors through $\mathcal{R}^{\mathrm{mod}}/I$, then*

(1) *If $\rho|_{G_{\mathbb{Q}_p}}$ is potentially semi-stable, trianguline with distinct Hodge-Tate weights, then ρ arises from a classical modular form.*

(2) *If $\rho|_{G_{\mathbb{Q}_p}}$ is trianguline, then ρ comes from a twist of an overconvergent finite-slope eigenform (of tame level equal to the tame conductor of ρ).*

Remark 2. The "trianguline" in the first statement should be removable according to correspondence with Colmez. This gives a different approach (see Kisin's talk) to the Fontaine-Mazur conjecture for GL_2.

Furthermore, this actually yields a 2-variable p-adic L-functions over all of the eigencurve. (Actually, the part of the eigencurve mapping to $\mathcal{R}^{\mathrm{mod}}/I$.) It also gives a mod p statement.

11. Schien

We fix the following notations: let F be a finite field extension of $\mathbb{Q}_p$, write $\mathcal{O}_K$ for its ring of integers, π for a fixed uniformizer and $k = \mathcal{O}_F/\pi$ for the residue field. Write $G = \mathrm{GL}_2(F)$, $K = \mathrm{GL}_2(\mathcal{O}_F)$. Define the following congruence subgroups:

$$\begin{aligned} I &= \left\{\gamma \in K : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{\pi}\right\} \\ I_1 &= \left\{\gamma \in K : \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{\pi}\right\} \end{aligned}$$

Then we have:
$$I_1 \subset I \subset K.$$

Let X denote the Bruhat-Tits tree of G. Let V be a two-dimensional vector space over F, a *lattice* in V is an $\mathcal{O}_F$-module L satisfying $L \otimes_{\mathcal{O}_F} F = V$. Note that $G = \mathrm{Aut}_F(V)$ for a choice of basis. Define

$$\begin{aligned} 0-\text{simplices} &\stackrel{\text{def}}{=} \{\mathcal{O}_F\text{-lattices } L \text{ in } V\}_{/\text{similarity}} \\ 1-\text{simplices} &\stackrel{\text{def}}{=} \{(L_0, L_1) : \pi L_0 \subset L_1 \subset L_0\} \end{aligned}$$

Definition 2. For $\sigma \in X$, let $R(\sigma)$ denote the stabilizer of σ in G, i.e.

$$\{g \in G : g\sigma = \sigma\},$$

note that $R(\sigma)$ acts on the vector space V. We define a *G-equivariant coefficient system* $\mathbf{V}$ *on* X to be given by the following data

1. for all simplices $\sigma \in X$ an $\overline{\mathbb{F}}_p$-vector space V_σ;
2. for $\sigma \subset \sigma'$ there is a restriction map $r_\sigma^{\sigma'} : V_{\sigma'} \to V_\sigma$;
3. for all $g \in G$ and $\sigma \in X$, there is a map $g_\sigma : V_\sigma \to V_{g\sigma}$ which is compatible with the restriction maps, such that
$$h_{g_\sigma} \circ g_\sigma = (hg)_\sigma;$$
4. for all $\sigma \in X$ and $g \in R(\sigma)$, V_σ is a smooth $R(\sigma)$-representation.

Let $\mathrm{COEFF_G}$ denote the category of all G-equivariant coefficient systems.

Example 1. For π a smooth representation of G, with underlying space W, let $K(W)$ be the constant coefficient system: $W_\sigma = W$ for all σ, the restriction maps $r_\sigma^{\sigma'} = \mathrm{id}$ for all $\sigma, \sigma' \in X$, and for $w \in W_\sigma$ we define $g_\sigma(w) = gw$.

We have a simplicial complex, so we can take homology: Let $X_0 = \mathrm{Vertices}(X)$.

$$\text{0-chains:} = C_0(X, \mathbf{V}) = \left\{ f : X_0 \to \coprod_{\sigma \in X_0} V_\sigma \mid \forall\, \sigma \in X_0, f(\sigma) = V_{\sigma_0},\ \text{finite support} \right\}.$$

and for the 1-chains, we take $X_{(1)}$ to be the set of oriented edges and X_1 to be the set of unoriented edges, then we define $C_1(X, \mathbf{V})$ to be the following:

$$\left\{ f : X_{(1)} \to \coprod_{\sigma \in X_0} V_\sigma \mid \text{finite support}, \forall (\tau_0, \tau_1) \in X_{(1)} : f(\tau_0, \tau_1) \in V_{\{\tau_0, \tau_1\}},\ f(\tau_0, \tau_1) = f(\tau_1, \tau_0) \right\}$$

One can view the coefficient system as a sheaf, and this is just homology with coefficients in this sheaf.

Boundary map: $\sigma \in X_0$

$$\partial : C_1(X, \mathbf{V}) \to C_0(X, \mathbf{V}) : f \mapsto \left\{ \sigma \mapsto \sum_{\sigma' \in X_0} r_\sigma^{\{\sigma, \sigma'\}} f(\sigma, \sigma') \right\};$$

and set $H_0(X, \mathbf{V}) = \mathrm{coker}\partial$. Everything in sight is compact respecting the G-action, whence $H_0(W, \mathbf{V})$ is a *smooth* G-module.

Recall we chose a basis $\{v_1, v_2\}$ of V. Pick a distinguished vertex and edge:

$$\begin{aligned} \sigma_0 &= \mathcal{O}_F v_1 \oplus \mathcal{O}_F v_2 \\ \sigma_1 &= \{\mathcal{O}_F v_1 \oplus \mathcal{O}_F v_2, \mathcal{O}_F v_1 \oplus \pi_F \mathcal{O}_F v_2\}. \end{aligned}$$

Then

$$\begin{aligned} R(\sigma_0) &= KZ \\ R(\sigma_1) &= \langle IZ, \Pi \rangle, \quad \text{where } \Pi = \begin{pmatrix} 0 & 1 \\ \pi_F & 0 \end{pmatrix}. \end{aligned}$$

Let DIAG be the category of diagrams. The objects look like:

$$D_1 \xrightarrow{y} D_0$$

where $D_0 \in R(\sigma_0) - \mathrm{MOD}$, $D_1 \in R(\sigma_1) - \mathrm{MOD}$ and y is a map of IZ-modules. Here everything is smooth and an $\overline{\mathbb{F}}_p$-vector space.

If $\mathbf{V}$ is a coefficient system, we have

$$r_{\sigma_0}^{\sigma_1} : V_{\sigma_1} \to V_{\sigma_0}$$

thus we have a functor $\mathrm{COEFF}_G \to \mathrm{DIAG}$.

Claim 1. *This is an equivalence of categories.*

Pascunas does this by constructing the reverse functor: it comes down to the fact that G acts transitively on vertices, so that ultimately the diagram is like "$\mathbf{V}/G$".

Say $k = \mathbb{F}_{p^n}$, let $\Gamma = \mathrm{GL}_2(\mathbb{F}_{p^n})$, $H \subseteq \Gamma$ diagonal matrices, let $\mathrm{res} : K \to \Gamma$ be the obvious map: $T = \mathrm{res}^{-1}(H)$, and let

$$\chi : H \to \overline{\mathbb{F}}_p^{\times}$$

be a character. View χ as a representation of I. So any such representation is a representation of $I/I_1 \cong H$. Make it a representation of IZ by decreeing that $\begin{pmatrix} \pi & 0 \\ 0 & \pi \end{pmatrix}$ acts trivially. So to χ we associate an irreducible representation of $\mathrm{GL}_2(k)$: Kevin wrote down all such yesterday: call them $\rho_{\mathbf{a},\mathbf{r}}$, where

$$\begin{aligned} \mathbf{a} &= a_0 + a_1 p + a_2 p^2 + \cdots + a_{n-1} p^{n-1} \\ \mathbf{r} &= r_0 + r_1 p + r_2 p^2 + \cdots + r_{n-1} p^{n-1} \\ & \quad 0 \le a \le q-1 \quad q = p^n \end{aligned}$$

Then we have

$$\begin{aligned} \chi \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} &= \lambda^{\mathbf{a}} \quad , \\ \chi \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} &= \lambda^{\mathrm{r}} \end{aligned}$$

(there is a special case when $\mathrm{r} = 0$; ignore this for now, thought Pascunas deals with this.)

Let $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $\chi^S = S \chi S^{-1}$. Let $\gamma = \{\chi, \chi^S\}$. So if

$$\chi : \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mapsto \prod_{\tau : k \to \overline{\mathbb{F}}_p} \tau(a)^{m_\tau} \tau(d)^{n_\tau},$$

then χ^S switches m_τ, n_τ. So if

$$\chi \longleftrightarrow \rho_{\mathbf{a},\mathbf{r}}$$

then

$$\chi^S \longleftrightarrow \rho_{\mathbf{a}+\mathbf{r}, p-1-\mathbf{r}}$$

call these ρ and ρ' respectively.

Claim 2. *There exists a unique way to extend the IZ-action on $(\rho \oplus \rho')^{I_1}$ to an $R(\sigma_1)$-action and $\Pi^{-1} v = v'$, $\Pi^{-1} v' = v$, $\rho_{\mathbf{a},\mathbf{r}}^{\mathrm{res}(I_1)}$ is 1-dimensional (generated by $X_0^{r_0} X_1^{r_1}$ — as yesterday), where v is a generator of ρ^{I_1}, v' is a generator of $(\rho')^{I_1}$.*

Moreover, $(\rho \oplus \rho')^{I_1} \cong \mathrm{Ind}_{IZ}^{R(\sigma_1)} \chi$.

Again $\gamma = \{\chi, \chi^S\}$. Note that there are $\frac{q(q-1)}{2}$ such representations. Then we get $D_\gamma \in$ DIAG

$$R(\sigma_1) - \text{MOD} \to KZ - \text{MOD} = R(\sigma_0) - \text{MOD} : (\rho \oplus \rho')^{I_1} \to (\rho \oplus \rho')$$

Let $\mathbf{V}_\gamma$ be the corresponding coefficient system.

Exercise: This is all well-defined (i.e. we chose a basis above).

If π is any G-representation, then $\pi^{I_1} \neq 0$. So write

$$\pi^{I_1} = \text{Hom}_{I_1}(\mathbf{1}, \pi|_{I_1}) \overset{\text{Frob reciprocity}}{\cong} \text{Hom}_G(c - \text{Ind}_{I_1}^G \mathbf{1}, \pi).$$

Let $\mathcal{H} = \text{End}_G(c - \text{Ind}_{I_1}^G \mathbf{1}, \pi)$, then π^{I_1} is a ring $\mathcal{H}$-module.

Now VIGNÉRAS has classified these (the irreducible ones): If π is an irreducible G-representations which is not supersingular then π^{I_1} is an irreducible $\mathcal{H}$-module.

For $\gamma = \{\chi, \chi^S\}$, $\lambda \in \overline{\mathbb{F}}_p^\times$, we have a standard $\mathcal{H}$-module M_γ^λ, with

$$M_\gamma^\lambda \cong M_{\gamma'}^{\lambda'} \longleftrightarrow \lambda = \lambda',\ \gamma = \gamma'.$$

If M is an irreducible $\mathcal{H}$-module, $M \not\cong \pi^{I_1}$ for any non-supersingular. G-representation π, then $M \cong M_\gamma^\lambda$. Let $M_\gamma = M_\gamma^1$.

Corollary 16. *If π (a G-representation) is an irreducible nonzero quotient of $H_0(X, \mathbf{V}_\gamma)$, then is is supersingular, and moreover $M_\gamma \subset \pi^{I_1}$.*

So what does he actually do?

For each γ, construct a diagram V_γ and an embedding $D_\gamma \hookrightarrow Y_\gamma$. This corresponds to a map of coefficient systems $\mathbf{V}_\gamma \hookrightarrow I_\gamma$. This induces a map on homology: let

$$\pi_{\mathbf{V}} = \text{im}(H_0(X, \mathbf{V}_\gamma) \to H_0(X, I_\gamma)).$$

Then $\pi_{\mathbf{V}}$ irreducible supersingular G-module. Then

$$(s \circ c(\pi_{\mathbf{V}}|_K))^{I_i} = M_\gamma.$$

Constructing the Y_γ is the hard (technical) work in the paper. He makes a choice in the process:

$$0 \longrightarrow \rho \oplus \rho' \longrightarrow (\text{inj}\rho \oplus \text{inj}\rho')$$

$$\begin{array}{ccccc} 0 \longrightarrow & (\rho \oplus \rho')^{I_1} & \longrightarrow & (\text{inj}\rho \oplus \text{inj}\rho')|_{IZ} \\ & \| & & \| \\ & D_\gamma & & Y_\gamma \end{array}$$

12. Taylor: Florian Herzig's Thesis

"Florian would probably give a better talk." — Richard Taylor.

Notation. — Let $p > 2$ be a prime and we write Frob for the arithmetic Frobenius.

Goal. — To generalize Serre's conjecture to GL_n, specifically the question of the weight.

Recall the $\text{GL}_2(\mathbb{Q})$ case: we have a Galois representation:

$$\rho : G_{\mathbb{Q}} \to \text{GL}_2(\overline{\mathbb{F}}_p),$$

which is odd and irreducible. Then there is an integer

$$N(\overline{\rho}) \in \mathbb{Z}_{>0},$$

determined by $\{\overline{\rho}|_{I_\ell}, \ell \neq p\}$. Then

$$\text{Wt}(\overline{\rho}|_{I_p}) \subset \{\text{irreducible mod } p \text{ representations of } \text{GL}_2(\mathbb{F}_p)\}.$$

Conjecture 5. *Let p be a prime not dividing N. If σ is a modular representation of $\mathrm{GL}_2(\mathbb{F}_p)$ then: $H^1(\Gamma_1(N),\sigma)$ contains a Hecke eigenclass with eigenvalue of T_ℓ equal to $\mathrm{Tr}(\overline{\rho}(\mathrm{Frob}_\ell))$ for all $\ell \nmid Np$ if and only if $N(\overline{\rho}) \mid N$ and $JH(\sigma) \cap \mathrm{Wt}(\overline{\rho}|_{I_p}) \neq \emptyset$.*

Think about $\mathrm{Wt}(\overline{\rho}|_{I_p}) \subset \mathrm{Wt}(\overline{\rho}|_{I_p}^{\mathrm{ss}})$; Matt gave another way to think about this earlier.

How to generalize this? ASH et. al.:

$$\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$$

irreducible. We require that

$$| \dim \overline{\rho}^{c=1} - \dim \overline{\rho}^{c=-1} | \leq 1$$

Then $N(\overline{\rho}) \in \mathbb{Z}_{>0}$ correpsonds with $\{\overline{\rho}|_{I_\ell} : \ell \neq p\}$, and

$$\mathrm{Wt}(\overline{\rho}|_{I_p}) \subset \mathrm{Wt}(\mathrm{GL}_n(\mathbb{F}_p)$$

the latter being an irreducible mod p representation of $\mathrm{GL}_n(\mathbb{F}_p)$.

Set

$$\Gamma_1(N) = \left\{ g \in \mathrm{SL}_n(\mathbb{Z}) \mid g \equiv \left(\begin{array}{c|c} * & * \\ \hline 0 & 1 \end{array}\right) \pmod N \right\}.$$

For $\ell \nmid Np$ we write

$$T_\ell = \left[\Gamma_1(N) \begin{pmatrix} \ell & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \Gamma_1(N) \right].$$

Conjecture 6 (approx.)**.** *If σ is any mod p representation of $\mathrm{GL}_n(\mathbb{F}_p)$, then: there is a Hecke eigenclass x in $H^d(\Gamma_1(N),\sigma)$ for some d with $T_\ell x = \mathrm{Tr}(\overline{\rho}(\mathrm{Frob}_\ell))x$ for all $\ell \nmid Np$, and $p \nmid N$ if and only if $N(\overline{\rho}) \mid N$ and $JH(\sigma) \cap \mathrm{Wt}(\overline{\rho}|_{I_p}) \neq \emptyset$.*

Remark 3. Is there a specific choice of d? Not sure. There is a natural interesting range of choices "in the middle," just it is not clear if one can/should pick a d.

Hope:

$$\mathrm{Wt}(\overline{\rho}|_{I_p}) \subset \mathrm{Wt}(\overline{\rho}|_{I_p}^{\mathrm{ss}}).$$

ASH et. al. tend not to specify $\mathrm{Wt}(\overline{\rho}|_{I_p})$. In the case $n = 3$, there is some complicated recipe.

Let $E/\mathbb{Q}$ be an imaginary quadratic field, let G be a unitary group which becomes an inner form of GL_n over E, with $G(\mathbb{Q}_p) \cong \mathrm{GL}_n(\mathbb{Q}_p)$, one can make the same time of conjecture, namely: on the Galois side, one would have

$$\overline{\rho} : G_E \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$$

and

$$\rho^c \sim \rho^\vee \otimes \mathrm{char}$$

In this case, we can build Galois representations from eigenclasses. In this case, can we prove either direction above? Maybe...

Florian generally looks at semisimple case.

Let's look at irreducible representations of GL_n in characteristic zero. These are parametrized by the highest weight $\mathbf{a} \in \mathbb{Z}^n$, where $\mathbf{a} = (a_1, \ldots, a_n)$ and $a_1 \geq \cdots \geq a_n$. Let $W_{\mathbf{a}}$ be the corresponding module.

Ex.: $n = 2$:

$$W_{a_1,a_2} = \mathrm{Sym}^{a_1-a_2}(\mathrm{Std}) \otimes \det^{a_2}.$$

If $x \in H^d(\Gamma_1(N), W_{\mathbf{a}}(\overline{\mathbb{Q}}_p))$ is an eigenclass, one expects that there exists a continuous representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p) \ni \mathrm{Tr}(\rho(\mathrm{Frob}_\ell)) =$ eigenvalue of T_ℓ for all $\ell \nmid Np$, with ρ de Rham with Hodge-Tate numbers:

$$\underbrace{a_1 + (n-1), a_2 + (n-2), \ldots, a_{n-1} + 1, a_n.}_{\text{note that this means these are distinct.}}$$

Now $W_{\mathbf{a}}/\mathbb{Z}_p$ has a natural representation — Weyl module. Then we can look at $W_{\mathbf{a}} \otimes \overline{\mathbb{F}}_p$ — a representation of GL_n over $\overline{\mathbb{F}}_p$ (or $\mathbb{F}_p$, etc...). These are irreducible in characteristic zero, but not in characteristic p. The representation $W_{\mathbf{a}} \times \overline{\mathbb{F}}_p$ has a unique irreducible submodule $F_{\mathbf{a}}$.

As $\mathbf{a}$ varies, the $F_{\mathbf{a}}$ are distinct, and exhaust all the irreducible representations of GL_n.

Let $q = p^r$, and think about the irreducible representations of $\mathrm{GL}_n(\mathbb{F}_q)$. The irreducible mod p representation of $\mathrm{GL}_n(\mathbb{F}_q)$ are the $F_{\mathbf{a}}(\overline{\mathbb{F}}_p)$ and

$$a_1 - a_2, a_2 - a_3, \ldots, a_{n-1} - a_n < q.$$

The only coincidences are $F_{\mathbf{a}}$ and $F_{\mathbf{a}+m(q-1,\ldots,q-1)}$. Thus

$$\begin{array}{rcl} \mathrm{Wt}(\mathrm{GL}_n(\mathbb{F}_q)) & = & \{\text{isomorphism classes of irreducible representations of } \mathrm{GL}_n(\mathbb{F}_q)\} \\ & \bigcup & \\ \mathrm{Wt}(\mathrm{GL}_n(\mathbb{F}_q))_{\mathrm{reg}} & = & \left\{\mathbf{a} : a_i - a_{i+1} = b_0 + b_1 p + \cdots + b_{r-1}p^{r-1} \text{ with } 0 \le b_j < p - 1 \forall i, j\right\}. \end{array}$$

We have a "map": we sometimes won't associate a regular weight.

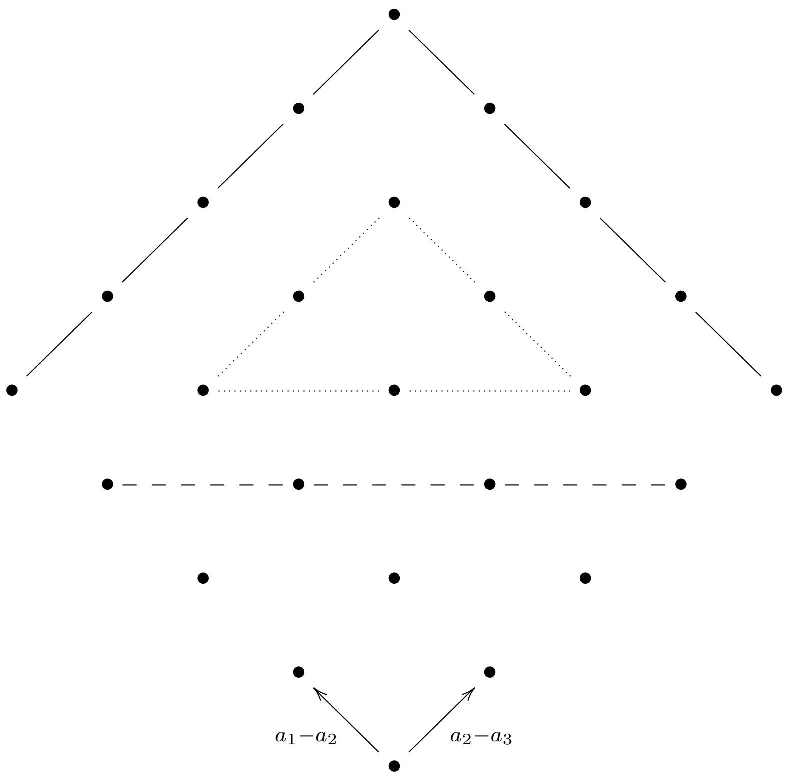
$$R : \mathrm{Wt}(\mathrm{GL}_n(\mathbb{F}_q)) \to \mathrm{Wt}(\mathrm{GL}_n(\mathbb{F}_q))_{\mathrm{reg}} : \mathbf{a} \mapsto \mathbf{b},$$

where

$$\mathbf{b} \equiv \left(a_n - (n-1)\frac{p^r - 1}{p-1}, a_{n-1} - (n-2)\frac{p^r-1}{p-1}, \ldots, a_2 - \frac{p^r-1}{p-1}, a_1\right) \pmod{q-1}.$$

For $q = p$, we are simply saying $(a_n - (n-1), a_{n-1} - (n-2), \ldots, a_1)$ — which is similar to the characteristic zero case.

Ex. $n = 3, q = p$.



Here the "top node" is the point $(p-1, p-1)$ and the "bottom node" is $(0,0)$, and the top 'inverted V' of nodes correspond to the irregular weights and the 'inner triangle' corresponds to the reducible weights. The operator R acts by reflection across the dashed line.

Recipe: $p = q$. We have

$$\overline{\rho}|_{I_p}^{\mathrm{ss}} = \bigoplus_i \left(\chi_i \oplus \chi_i^p \oplus \cdots \oplus \chi_i^{p^{s_i - 1}} \right),$$

where the niveau s_i factors through:

$$I_p \to \mathbb{F}_{p^{s_i}}^{\times},$$

and s_i is the minimal such number, also $n = \dim \overline{\rho} = \sum_i s_i$.

There is a natural:

$$\begin{tikzcd}
 & & w(\overline{\mathbb{F}}_p^{\times}) \arrow[d] \\
\mathrm{GL}_n(\mathbb{F}_p) \supset T = \prod_i \mathbb{F}_{p^{s_i}}^{\times} \arrow[urr, "\tilde{\chi}"] \arrow[rr, "\prod \chi_i"] & & \overline{\mathbb{F}}_p^{\times}
\end{tikzcd}$$

and we have $V(T, \tilde{\chi})$ a representation of $\mathrm{GL}_n(\mathbb{F}_p)$ in characteristic zero, (usually irreducible), so

$$\overline{\rho}|_{I_p}^{\mathrm{ss}} \to V(\overline{\rho}|_{I_p}^{\mathrm{ss}}) \stackrel{\Delta}{=} V(T, \tilde{\chi}),$$

which is full induced from a product of cuspidals.

Ex. $n = 3$:

- $\overline{\rho}|_{I_p}^{\mathrm{ss}} = \chi_1 \oplus \chi_2 \oplus \chi_3$ niveau 1.

$$\mathrm{Ind}_{B_3(\mathbb{F}_p)}^{\mathrm{GL}_3(\mathbb{F}_p)} (\tilde{\chi}_1 \oplus \tilde{\chi}_2 \oplus \tilde{\chi}_3).$$

- $\overline{\rho}|_{I_p}^{\mathrm{ss}} = \chi \oplus \chi^p \oplus \chi^{p^2}$ niveau 3.

$$\alpha \in \mathbb{F}_{p^3}^\times \setminus \mathbb{F}_p^\times : \mathrm{Tr}(\alpha|_{V(T,\tilde{\chi})}) = \left(\chi(\alpha) + \chi(\alpha)^p + \chi(\alpha)^{p^2}\right).$$

Conjecture 7 (Herzig(?))**.**

$$\mathrm{Wt}(\overline{\rho}|_{I_p}^{\mathrm{ss}})_{\mathrm{reg}} = R\left(JH\left(v\left(\overline{\rho}|_{I_p}^{\mathrm{ss}}\right) \pmod p\right)\right).$$

Ex:

- $n = 3, q = p$: Can extend R, it is a one-to-many map, and you get all weights predicted.
- $n = 2, q = p^r$: Consistent with Fred's conjecture, can extend R as above, get all weights, same as Fred's prediction.

(Currently, there are multiple extensions, and it's unclear which is right.)

(Remarks on the data.) His data seems (experimentally) better than what Ash et. al. had found.