## PROBLEMS FROM THE WORKSHOP ON PHASE TRANSITIONS

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- (1) Find the densest packings of 37-gons. (37-gons are almost round circles, so the point is the relation between the known densest packing of round circles and the unknown densest packing of 37-gons.)
- (2) Let  $\mathcal{F}_k(n, m = rn)$  denote a random k-SAT formula, with n variables and m = rn clauses, chosen with replacement (say) from the  $2^k \binom{n}{k}$  possible clauses. Suppose, as we believe, that for each  $k \geq 3$  there is a critical density  $r_c$  at which  $\Pr[\mathcal{F}_k(n, rn) \text{ is satisfiable}]$  jumps from 1 to 0. How does the scaling window near this transition behave? In particular, is it the case that, for each k, there is a critical exponent  $\psi$  and a continuous universal function g, such that for all constant real x we have

 $\lim_{n \to \infty} \Pr[\mathcal{F}_k(n, r_c + n^{-\psi}x) \text{ is satisfiable}] = g(x) ?$ 

If so, what is g(0), i.e., the probability of satisfiability when we are exactly at the critical density? (The probability of percolation at the critical density in the plane is known, but of course this is for very special reasons involving conformal fields.)

- (3) There are examples known of nearest neighbor interactions between variables  $s_j \in \{1, 2, \dots, K\}$ , for j on the lattice  $\mathbb{Z}^d$  (d > 1), such that the limiting Gibbs state  $m_0$  (=  $\lim_{T\to 0} m_T$ ), as temperature T goes to 0, is concentrated on configurations without any translational symmetry. (These models are based on "aperiodic tilings".) The question is whether this ground state survives to positive temperature, that is, whether such a model can have a low-temperature phase (i.e.  $m_T$  for T > 0) which has no translational symmetry in an appropriate sense. (There is some simulation evidence of this by Leuzzi and Parisi: J. Phys. A: Math. Gen. 33 (2000) L203-L206.)
- (4) Let  $\Lambda$  be a finite "box" in  $\mathbb{Z}^d$ . Fix a finite collection of sites  $M = \{x_1, x_2, \cdots, x_k\}$  with each  $x_j \in \Lambda$ . Consider the usual ferromagnetic Ising model (with periodic boundary conditions) on  $\pm 1$  spin configurations indexed by  $\Lambda$ . Start heat bath (= Glauber) dynamics with all sites = +1. Update sites in M sequentially: first  $x_1$ , then  $x_2, \cdots$ . If there are two values of  $\beta$ , say  $\beta_1 > \beta_2$ , is it true that for each stage,

$$E_{\beta_1}(\sigma_{x_i}) \ge E_{\beta_2}(\sigma_{x_i}), \ 1 \le i \le k.$$

(5) Let G be a finite graph with maximum degree  $\Delta$ . This problem is about choosing a random coloring from the uniform distribution. Work with Glauber dynamics: pick a site at random, choose a color at random, and try to change the color at the chosen site. If this results in a proper coloring, make the change. If not, keep the current coloring. For  $q \geq \Delta + 2$  it is known that this Markov chain is ergodic, i.e., that it connects the set of proper colorings. Let t = t(q) be the mixing time: the smallest time such that the total variation distance from the uniform distribution is at most 1/e from any starting configuration. The question is: does t(q) decrease as q increases?

Even on the square lattice with n sites there are curious gaps in our knowledge: we know that t(3) is polynomial in n (Luby, Randall and Sinclair: Proc. FOCS 1995) and we know that  $t(q) = O(n \log n)$  for  $q \ge 6$ . But, for q = 4 and q = 5 we do not even have a polynomial upper bound, even though there is overwhelming numerical evidence that there is a finite correlation length, and therefore that the mixing time is  $O(n \log n)$ .

- (6) Is there a variant of k-SAT that describes the scaling properties of jamming? In other words, is k-SAT a good "Boolean idealization" of the jamming process? Should this involve k-SAT on a lattice? (See Schwartz and Middleton, Phys. Rev. E 70 (2004), 035103(R).)
- (7) Find a model with constraints and a phase transition, which we can perturb by "softening" the constraints. Study the sensitivity of this model's behavior with respect to this perturbation.
- (8) In constraint satisfaction problems such as k-SAT, what are the useful response/correlation functions? Do they show singular, or power-law, behavior at the transition, analogous to spatial correlations and response functions in lattice systems such as the Ising model?
- (9) The second moment method in k-SAT gives a highly accurate lower bound for the critical density, when the approach of (Achlioptas and Moore: Proc. FOCS 2002) is refined by (Achlioptas and Peres: Proc. STOC 2003). This refinement can be thought of as adding a finite (not infinitesimal) external field which discourages literals from being true. Is there any physical meaning to this value of the external field? For instance, does it minimize fluctuations in, say, the free energy at zero temperature?
- (10) We believe (and, in some cases, have proved) that below the satisfiability/unsatisfiability transition in k-SAT there is a "clustering" transition, in which the solutions clump together in isolated groups. It has been conjectured that this clustering phenomenon makes it hard for search algorithms to find solutions. Can we find algorithms which, in simulation, find solutions in, say, linear time, for values of k and densities at which clustering has been rigorously established?
- (11) (Following up on the previous question) Can we prove that clustering occurs at some density below the satisfiability transition for k = 3, 4, 5? So far we only have proofs of clustering for larger values of k.
- (12) Prove an order/disorder phase transition for the equilibrium statistical mechanics model of hard spheres (or hard disks). Simulation evidence of such a transition is

generally believed to be convincing in both cases, though the nature of the transition is controversial for disks.

- (13) How should we add friction to jamming?
- (14) Consider the Ising model on the *d*-dimensional lattice. Fix  $\beta$  (and perhaps the external field strength *h*) and consider Glauber dynamics. The cutoff phenomenon in Markov chains refers to the existence of a time at which the total variation distance from the stationary (Gibbs) distribution jumps from 1 to 0 in the limit where the size of the lattice goes to infinity. We ask whether this sharp cutoff exists if and only if there is a unique Gibbs state in the infinite system.
- (15) Give a good definition of the thermodynamic limit of k-SAT. That is, clearly define the mathematical object that corresponds to random formulas  $\mathcal{F}_k(n, m = rn)$  with constant density r in the limit  $n \to \infty$ .
- (16) Consider the Ising model on a finite box in  $\mathbb{Z}^d$ . Set  $\beta > \beta_c$  and no external field. With fixed boundary conditions where all the boundary sites are set to +1, it is known that Glauber dynamics can take a long time to mix if the initial state consists of a large droplet of -1's. But with a "warm start," where the initial condition consists of all +1's or is random, is the mixing time optimal, i.e.,  $O(n \log n)$  where n is the number of sites in the box?