INTRODUCTION TO THE PROPERTY OF RAPID DECAY

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Abstract. We explain some facts related to the property of Rapid Decay, as a first reading for some participants to the AIM workshop on the property of Rapid Decay.

Disclaimer: This version is VERY preliminary... We probably left out some facts, so if you feel that you or someone you know hasn't been properly cited, we will be very happy to include further references.

A discrete group Γ is said to have *the property of Rapid Decay (property RD) with respect to a length function* ℓ if there exists a polynomial P such that for any $r \in \mathbb{R}_+$ and any f in the complex group algebra $\mathbb{C}\Gamma$ supported on elements of length shorter than r the following inequality holds:

$$\|f\|_* \leq P(r)\|f\|_2$$

where $\|f\|_*$ denotes the operator norm of f acting by left convolution on $\ell^2(\Gamma)$, and $\|f\|_2$ is the usual ℓ^2 norm. This short note is aimed at non specialists as a quick introduction to property RD. Comments and improvements will be more than welcome!

Contents

1. History

Property RD had a striking application in A. Connes and H. Moscovici's work proving the Novikov conjecture for Gromov hyperbolic groups [7] and is now relevant in the context of the Baum-Connes conjecture,

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mainly due to the work of V. Lafforgue in [16]. Further use of property RD can be found in the work of Nagnibeda and Grigorchuck [19], A. Nevo [20] and [21] or Antonescu and Christensen in [1].

First established for free groups by U. Haagerup in [10], property RD has been introduced and studied as such by P. Jolissaint in [13], who notably established it for groups of polynomial growth, and for classical hyperbolic groups. The extension to Gromov hyperbolic groups is due to P. de la Harpe in [11]. The first examples of higher rank groups with property RD have been given by J. Ramagge, G. Robertson and T. Steger in [24], where they established it for $\tilde{A}_2$ and $\tilde{A}_1 \times \tilde{A}_1$ groups. V. Lafforgue proved property RD for cocompact lattices in $SL_3(\mathbb{R})$ and $SL_3(\mathbb{C})$ in [15]. His result has been generalized by Chatterji in [2] to cocompact lattices in $SL_3(\mathbb{H})$ and $E_{6(-26)}$ as well as in a finite product of rank one Lie groups. It is well-known (see Section 2) that non-cocompact lattices in higher rank simple Lie groups do not have property RD, and it is a conjecture due to Valette that! all cocompact lattices in a semisimple Lie group should have property RD (see [27]). Nice partial results have been obtained in [25]. The situation is different in rank one, all lattices have property RD as proven in [4]. This result has recently been generalized in [9] where it is proven that any group that is hyperbolic relative to subgroups with property RD has property RD itself. Another class of discrete groups known to satisfy property RD is given by groups acting freely on CAT(0) cube complexes of finite dimension, see [4]. The notion of property RD is easily extended to locally compact groups. In [13], it is shown that property RD passes from a lattice to the ambient group (see Section 4 for the precise statement), but the converse is still open and would settle Valette's conjecture since semisimple groups have property RD (see [5]). In [12] it is shown that locally compact groups with pro! perty RD are unimodular, and in [5] we give a complete classi fication of connected groups with property RD. So far methods used for the topological cases and the discrete cases are very different.

2. Property RD and length functions

We will explain the basic notions related to property RD. Most of the results given in this section are either simple remarks or results contained in P. Jolissaint's paper [13].

Definition 2.1. Let Γ be a group, a *length function* on Γ is a map $\ell : \Gamma \to \mathbf{R}_+$ satisfying:

- $\ell(e) = 0$, where e denotes the neutral element in Γ,
- $\ell(\gamma) = \ell(\gamma^{-1})$ for any $\gamma \in \Gamma$,

- $\ell(\gamma\mu) \le \ell(\gamma) + \ell(\mu)$ for any $\gamma, \mu \in \Gamma$.

The map $d(\gamma,\mu) = \ell(\gamma^{-1}\mu)$ is a left Γ-invariant pseudo-distance on Γ. We will write $B_\ell(\gamma, r)$ for the ball of center $\gamma \in \Gamma$ and radius r with respect to the pseudo-distance ℓ, and simply $B(\gamma, r)$ when the context is clear.

If Γ is generated by some finite subset S, then the *algebraic word length* $L_S : \Gamma \to \mathbf{N}$ is a length function on Γ, where, for $\gamma \in \Gamma$, $L_S(\gamma)$ is the minimal length of γ as a word on the alphabet $S \cup S^{-1}$, namely,

$$L_S(\gamma) = \min\{n \in \mathbf{N} | \gamma = s_1 \dots s_n,\ s_i \in S \cup S^{-1}\}.$$

Let Γ act by isometries on a metric space (X, d). Pick a point $x_0 \in X$ and define $\ell(\gamma) = d(\gamma x_0, x_0)$, this is a length function on Γ. This last example is general in the sense that any length function ℓ comes from a metric on a space X with respect to which Γ acts by isometries. Indeed, if ℓ is a length function on Γ, define the subgroup H of Γ as,

$$H = \{\gamma \in \Gamma | \ell(\gamma) = 0\}$$

and then $X = \Gamma/H$. The map $d : X \times X \to \mathbf{R}_+$, $d(\gamma H, \mu H) = \ell(\mu^{-1}\gamma)$ is a well-defined Γ-invariant metric on X and $\ell(\gamma) = d(\gamma H, H)$ for any $\gamma \in \Gamma$.

Let $H < \Gamma$ be a subgroup of Γ and ℓ a length on Γ. The restriction of ℓ to H induces a length on H that we call *induced length.*

Definition 2.2. Denote by $\mathbf{C}\Gamma$ the set of functions $f : \Gamma \to \mathbf{C}$ with finite support, which is a ring for pointwise addition and convolution:

$$f * g(\gamma) = \sum_{\mu\in\Gamma} f(\mu)g(\mu^{-1}\gamma). \qquad (f, g \in \mathbf{C}\Gamma, \gamma \in \Gamma)$$

We denote by $\mathbf{R}^+\Gamma$ the subset of $\mathbf{C}\Gamma$ consisting of functions with target in $\mathbf{R}^+$. We shall consider the following completions of $\mathbf{C}\Gamma$:

(a) for $p \ge 1$, $\ell^p\Gamma = \overline{\mathbf{C}\Gamma}^{\|\ \|_p}$, where $\|f\|_p^p = \sum_{\gamma\in\Gamma} |f(\gamma)|^p$ (actually we will mainly be interested in the cases $p = 1, 2$),

(b) the *reduced C^*-algebra of* Γ, given by $C_r^*\Gamma = \overline{\mathbf{C}\Gamma}^{\|\ \|_*}$, where $\|f\|_* = \sup\{\|f*g\|_2 \,|\, \|g\|_2 = 1\}$ is the *operator norm* of $f \in \mathbf{C}\Gamma$,

(c) for $s > 0$, the s-Sobolev space $H_\ell^s(\Gamma) = \overline{\mathbf{C}\Gamma}^{\|\ \|_{\ell,s}}$, where $\|f\|_{\ell,s} = \sqrt{\sum_{\gamma\in\Gamma} |f(\gamma)|^2(1 + \ell(\gamma))^{2s}}$ is a *weighted ℓ^2 norm.*

(d) the (Fréchet) space of *rapidly decaying functions* $H_\ell^\infty(\Gamma) = \cap_{s\ge 0} H_\ell^s(\Gamma)$.

We leave as an easy exercise to check that for $f \in \mathbf{C}\Gamma$ we have $\|f\|_2 \le \|f\|_* \le \|f\|_1$

Definition 2.3 (P. Jolissaint, [13])**.** Let ℓ be a length function on Γ. We say that Γ has *property RD* (standing for *Rapid Decay*) with respect to ℓ (or that it satisfies the *Haagerup inequality*[1]), if there exists $C, s > 0$ such that, for each $f \in \mathbf{C}\Gamma$ one has

$$\|f\|_* \leq C\|f\|_{\ell,s}.$$

Proposition 2.4. *Let Γ be a discrete group, endowed with a length function ℓ. Then the following are equivalent:*

(1) *The group Γ has property RD with respect to ℓ (with Definition 2.3).*
(2) *There exists a polynomial P such that, for any $r > 0$ and any $f \in \mathbf{R}_+\Gamma$ so that f vanishes on elements of length greater than r, we have*
$$\|f\|_* \leq P(r)\|f\|_2.$$
(3) *There exists a polynomial P such that, for any $r > 0$ and any two functions $f, g \in \mathbf{R}_+\Gamma$ so that f vanishes on elements of length greater than r, we have*
$$\|f * g\|_2 \leq P(r)\|f\|_2\|g\|_2.$$
(4) *There exists a polynomial P such that, for any $r > 0$ and any $f, g, h \in \mathbf{R}_+\Gamma$ so that f vanishes on elements of length greater than r, we have*
$$f * g * h(e) \leq P(r)\|f\|_2\|g\|_2\|h\|_2.$$
(5) *The space of rapidly decaying functions $H^\infty_\ell(\Gamma)$ is contained in the reduced C^*-algebra $C^*(\Gamma)$ (the motivation for that definition is given below in the beginning of Section 3).*
(6) *Any subgroup H in Γ has property RD with respect to the induced length.*

The proof of the above proposition is left as an exercise (non-trivial but doable).

Definition 2.5. Let ℓ_1 and ℓ_2 be two length functions on a discrete group Γ. We say that ℓ_1 *dominates* ℓ_2 (and write $\ell_1 \geq \ell_2$) if there exists two integers C and k such that, for any $\gamma \in \Gamma$ one has:

$$\ell_2(\gamma) \leq C(1 + \ell_1(\gamma))^k.$$

We will say that ℓ_1 is *equivalent* to ℓ_2 if furthermore ℓ_2 dominates ℓ_1.

[1]We don't like this terminology as it is too close to the Haagerup property or a-T-menability, a very different property studied for instance in [6]

Remark 2.6. If Γ is a finitely generated discrete group, then the algebraic word length L_S associated to a finite generating set S of Γ dominates any other length ℓ on Γ. Indeed, for $\gamma \in \Gamma$, let $s_1 \dots s_n$ be a minimal word in the letters of S, then:

$$\ell(\gamma) = \ell(s_1 \dots s_n) \leq \sum_{i=1}^{n} \ell(s_i) \leq An = AL_S(\gamma)$$

where $A = \max_{s\in S}\{\ell(s)\}$ is finite since S is finite.

If a length ℓ_1 dominates another length ℓ_2 on a discrete group Γ and if Γ has property RD with respect to ℓ_2, then Γ has property RD with respect to ℓ_1 as well. Indeed, denote for $i = 1, 2$ and $r \in \mathbf{R}_+$ by $B_i(e, r)$ the ball of radius r for the length ℓ_i, centered at e and take $f \in \mathbf{C}\Gamma$ such that $S_f \subset B_1(e, r)$. This means that for $\gamma \in \Gamma$ such that $f(\gamma) \neq 0$, then $\ell_1(\gamma) \leq r$, and thus

$$\ell_2(\gamma) \leq C(1 + \ell_1(\gamma))^k \leq C(1 + r)^k,$$

which implies that $S_f \subset B_2(e, C(1 + r)^k)$ and applying point 2) of Proposition 2.4 we get that

$$\|f\|_* \leq P(C(1 + r)^k)\|f\|_2.$$

Thus Γ has property RD with respect to the length ℓ_1, and the polynomial is given by $Q(r) = P(C(1 + r)^k)$.

In particular, this shows that changing the set of generators on a finitely generated group will not change the degree of the polynomial involved, provided the considered generating sets are finite, and thus we can talk about word length without really caring about the chosen generating set.

This remark says in particular that a finitely generated group Γ has property RD with respect to the word length as soon as it has property RD for any other length, and thus explains why we might be sloppy regarding the length functions involved as soon as we deal with finitely generated groups.

Example 2.7 (P. Jolissaint, [13])**.** For a discrete group Γ, the map $\ell_0 : \Gamma \to \mathbf{R}_+$ defined by $\ell_0(\gamma) = 0$ for any $\gamma \in \Gamma$ is a length function, and Γ has property RD with respect to ℓ_0 if and only if Γ is finite. Indeed, if Γ has property RD with respect to ℓ_0, then there exists a constant C such that for any $f, g \in \mathbf{C}\Gamma$ then $\|f * g\|_2 \leq C\|f\|_2\|g\|_2$, which implies that $\ell^2\Gamma$ is an algebra. This can happen if and only if Γ is finite, see [23]. The same statement holds if we just assume ℓ_0 to be bounded.

3. First examples and results

The first example that seem to have inspired Connes is explained nicely in [13], we recall it here. It's the case where $G = \mathbb{Z}$ (or $\mathbb{Z}^n$ but I will just discuss $\mathbb{Z}$ here). In this case it is fairly well-known that under Fourier transform $n \mapsto z^n$ the Hilbert space $\ell^2(\mathbb{Z})$ is isomorphic to $L^2(S^1)$ (with Lebesgue measure. Furthermore, one can check that $C^*_r(\mathbb{Z})$ is *-isometrically isomorphic to $\mathcal{C}(S^1)$ the continuous functions over the circle and that $H^\infty(\mathbb{Z})$ corresponds to $\mathcal{C}^\infty(S^1)$ the smooth functions over the circle. The convolution described above becomes pointwise multiplication and property RD translates as the fact that smooth functions are continuous (see Proposition 2.4 point (5)).

Recall that a discrete group Γ has *polynomial growth with respect to a length* ℓ if there exists a polynomial P such that the cardinality of the ball of radius r (denoted by $|B(e,r)|$) is bounded by $P(r)$. The following result gives the only known obstruction to property RD, namely the presence of an amenable subgroup of exponential growth.

Theorem 3.1 (P. Jolissaint [13])**.** *Let Γ be a discrete amenable group. Then Γ has property RD with respect to a length function ℓ if and only if Γ is of polynomial growth with respect to ℓ. Moreover, the growth will be bounded by P^2, if P is the polynomial of Proposition 2.4 point 2).*

Proof. Let Γ be a discrete group endowed with a length function ℓ with respect to which Γ is of polynomial growth, then Γ has property RD with respect to ℓ. Indeed, take $f \in \mathbf{C}\Gamma$ such that $\mathrm{supp}(f) = S_f \subset B(e,r)$, then:

$$\|f\|_* \leq \|f\|_1 = \sum_{\gamma\in S_f} |f(\gamma)| \leq \sqrt{|S_f|}\sqrt{\sum_{\gamma\in S_f} |f(\gamma)|^2} = \sqrt{|S_f|}\|f\|_2,$$

the last inequality being just the Cauchy-Schwartz inequality. If Γ is of polynomial growth, then $|S_f| \leq |B(e,r)| \leq P(r)$ and thus $\|f\|_* \leq \sqrt{P(r)}\|f\|_2$. Conversely, according to Leptin [17] a group Γ is amenable if and only if, for any $f \in \mathbf{R}_+\Gamma$ one has $\|f\|_1 = \|f\|_*$. For f the characteristic function of a ball of radius r, then $\|f\|_* = \|f\|_1 = |B(e,r)| = \sqrt{|B(e,r)|}\|f\|_2$, and thus Proposition 2.4 forces Γ to be of polynomial growth with respect to ℓ. More precisely $\sqrt{|B(e,r)|} \leq P(r)$. □

One can see that it is enough to know radial rapid decay to deduce polynomial growth with amenability. Radial RD has been studied by Valette in [28] for groups acting on buildings. Unfortunately his results don't seem to extend to the full property RD. According to A.

Lubotzky, S. Mozes and M. S. Raghunathan in [18] there exists an infinite cyclic subgroup growing exponentially with respect to the word length in any non cocompact lattice in higher rank (exponentially distorted copy of $\mathbf{Z}$), and hence Theorem 3.1 combined with Proposition 2.4 point 5) shows that non cocompact lattices in higher rank cannot have property RD. It is part of a conjecture due to A. Valette (see [27]) that cocompact lattices in semisimple Lie groups should have property RD. Theorem 3.1 has the following consequence.

Corollary 3.2. *If a group Γ has property RD with respect to a length ℓ, then for any $w \in \Gamma$ of infinite order there exists two constants $K \geq 1$ and $t > 0$ such that for any $n \in \mathbf{N}$*

$$\ell(w^n) \geq Kn^t.$$

In fact, $t = 1/s$, where $s > 0$ is the degree of the polynomial needed in Proposition 2.4 points 2),3) and 4).

Proof. Take $w \in \Gamma$ of infinite order. The group $\langle w \rangle$ being amenable with property RD, it has polynomial growth according to Theorem 3.1, and more precisely, if we denote by $B_\ell^{\langle w \rangle}(n)$ a ball of radius n in $\langle w \rangle$ with respect to the length ℓ, then $|B_\ell^{\langle w \rangle}(n)| \leq C(1+n)^s$ for all $n \in \mathbf{N}$. Let $t = 1/s$ and $K = 1/C^t$, since

$$\{k \in \mathbf{N} \text{ such that } Kk^t \leq n+1\} = \{k \in \mathbf{N} \text{ such that } k \leq C(1+n)^s\}$$

if we assume that there exists $k \in \mathbf{N}$ such that $\ell(w^k) < Kk^t$ for some $k \in \mathbf{N}$ we get

$$\{k \in \mathbf{N} \text{ such that } \ell(w^k) \leq n\} \supset \{k \in \mathbf{N} \text{ such that } Kk^t \leq n\}$$

and hence

$$\begin{aligned} |B_\ell^{\langle w \rangle}(n)| &= 2|\{k \in \mathbf{N} \text{ such that } \ell(w^k) \leq n\}| - 1 \\ &\leq 2|\{k \in \mathbf{N} \text{ such that } k \leq C(1+n)^s\}| - 1 > C(1+n)^s \end{aligned}$$

which is a contradiction. $\square$

4. Locally compact groups

Here we are in the context of compactly generated groups, namely groups for which there is a compact set K, with nonempty interior and such that $G = \cup_{n \in \mathbb{N}} K^n$. A definition of property RD in this context reads as follows.

Definition 4.1. A group G has property RD with respect to a length function L if there is a polynomial P such that, for any continuous, compactly supported function $f \in \mathcal{C}_c(G)$ with support contained in a ball of radius r for the length function L, then

$$\|f\|_* \leq P(r)\|f\|_2.$$

As in the discrete case, one thinks of $f \in \mathcal{C}_c(G)$ acting on the Hilbert space $L^2(G)$ via left convolution:

$$f * \xi(x) = \int_G f(y)\xi(y^{-1}x)dy$$

where dy is a Haar measure on G. Then the operator norm $\|f\|_*$ is the norm of the linear operator $L^2(G) \to L^2(G), \xi \mapsto f * \xi$, namely

$$\|f\|_* = \sup\{\|f * \xi\|_2 | \|\xi\|_2 = 1\}$$

. A notable difference with the discrete case is that property RD passes to open subgroups only. For instance, $SL_3(\mathbb{R})$ has RD (see [5] or [15] combined with [13] theorem recalled below) but contain closed subgroups like $SL_n(\mathbb{Z})$ that don't have RD. And even asking a compact quotient doesn't help in general, as $SL_3(\mathbb{R})$ is a compact extension of AN which doesn't have RD because it is not unimodular (or soluble with exponential growth if you prefer). One can see as an exercise that the analogue of Jolissaint's Theorem 3.1 holds in the locally compact case.

Theorem 4.2 (Jolissaint)**.** *Let G be a locally compact, compactly generated group, and let $\Gamma < G$ be a discrete cocompact subgroup. If Γ has RD, then so does G.*

5. Open questions

(1) Is the converse of Theorem 4.2 true?
(2) Is property RD invariant under quasi-isometries?
(3) Is there a dynamical characterization of property RD?
(4) Are there groups that don't have RD and don't contain amenable subgroups with polynomial growth?
(5) Do CAT(0) groups have RD (ie, groups that act properly discontinuously by isometries and cocompactly on a CAT(0) metric space)?
(6) (doable) If G acts by isometries on a locally finite hyperbolic space, and if the stabilizers of vertices have RD with respect to the induced length and for a uniform polynomial, then G has property RD as well.
(7) Do $\mathrm{Out}(\mathbf{F}_n)$ or mapping class groups have property RD?

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