Property of rapid decay.

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Abstract

This text contains questions and problems that were raised during the workshop on the Rapid Decay (RD) Property (January 22-27, 2006).

1 RD for cocompact lattices

1. Generalized Valette's conjecture: Let G be a locally compact compactly generated group and Γ be a cocompact lattice in G. If G has Property RD, then Γ has Property RD. The original Valette's conjecture concerns the case when G is a semi-simple Lie group. The conjecture has been proved [Ch1] for cocompact lattices in finite product of rank 1 simple Lie groups and [La1] for $G = SL(3, \mathbf{K})$ where $\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}$ and $\mathbf{O}$. It is open in general for simple groups of higher rank and in particular for $S0(2,3) = Sp(4, \mathbf{R})$. Also note that the converse to this conjecture is an easy result [Jo1].

2. A weaker conjecture: Assume that Γ_1 and Γ_2 are two cocompact lattices in G. Then if Γ_1 has property RD, then so does Γ_2. This conjecture can be generalized to groups admitting a measurable topological coupling. Recall that two finitely generated groups Γ_1 and Γ_2 admit a measurable topological coupling if there exists a locally compact topological $\Gamma_1 \times \Gamma_2$-space equipped with an invariant σ-compact Borel measure μ and such that the actions of Γ_1 and Γ_2 are proper and cocompact.

3. Note that non-cocompact lattices often do not have RD since they contain solvable subgroups with exponential growth.

4. Problem: find a variation of RD that is true for every lattice in connected semisimple Lie groups and that still have interesting consequences. Possible answer: find a notion of relative RD. For instance, $SL(n, \mathbf{Z})$ for $n \geq 3$ does not have Property RD because of solvable subgroups of exponential growth,

but maybe it would have RD "relatively" to a maximal solvable subgroup. Maybe solvable groups would have RD relatively to its exponentially distorted cyclic subgroups.

5. Let H be a cocompact normal subgroup of a locally compact, compactly generated group G with Property RD. Does H have RD? Note that the converse is easy to prove using induction of unitary representations. Indeed, if H has RD, then it is straightforward to see that the G-representation obtained by inducing to G the regular representation of H has RD and then to show that this implies that G has RD.

6. Another stability question: let

$$1 \to H \to \Gamma \to P \to 1$$

be a short exact sequence of finitely generated groups such that P has RD. Is it true that Γ has RD if and only if H has RD for the induced length?

7. Is every co-compact lattice in a semi-simple Lie group (**)-relatively hyperbolic in the sense of [DS2] with respect to quasi-flats?

2 RD for unitary representations

During the workshop, we emphasized the fact that RD can be formulated for any unitary representation $(\pi, \mathcal{H})$.

Definition 2.1. Let G be a locally compact group equipped with a length function L. We say that a unitary representation $(\pi, \mathcal{H})$ of G has Property RD for the length function L if there exists $s > 0$ and $C < \infty$ such that

$$\int_G \frac{\langle \pi(g)v, w \rangle}{1 + L(g)^s} d\mu(g) \leq C$$

for all unitary vectors $v, w \in \mathcal{H}$ and every $g \in G$.

1. Stability properties: first, note that λ_G has RD if and only if the group G has RD. If σ is weakly contained in π and if π has RD, then σ has RD. Conversely, if (π_i) is a decomposition of π into irreducible representations and if every π_i have RD with uniform constants s and C, then π has RD with the same constants.

2. Remark made during the workshop by M. G. Cowling: if Q is any separated quotient of G equipped with a quasi-G-invariant measure ν and if the representation of G on $L^2(Q,\nu)$ has RD, then G has RD. In particular, if P is an amenable closed subgroup of G, then G has RD if and only if its representation on $L^2(G/P)$ has RD.

3. It is immediate that the trivial representation of G has RD if and only if G has polynomial growth.

4. Does there exist a finitely generated group Γ without Property RD but having a unitary representation π with Property RD? (May be easy, using a group extension).

5. What groups have property RD for at least one unitary representation?

3 RD for length functions

1. First, observe that if G has RD for any length function L, then it has RD. Moreover, if Γ is finitely generated and has RD with respect to some length, then every subgroup has RD with respect to the induced length. In particular any cyclic subgroup of Γ is at most polynomially distorted.

2. An example: there exists a finitely group Γ which has RD for a length function that is exponentially distorted with respect to the word length. Namely, this is the case of the free group F_2 equipped with the length induced from its inclusion in the group $F_2 \rtimes \mathbf{Z}$ defined by the presentation $\langle a,b,c; a^c = a^2b, b^c = ab\rangle$.

4 Completely bounded property RD

Let Γ be a discrete group equipped with a length function L. Denote by

$$H^s_L(\Gamma) = \{f : \Gamma \to \mathbf{C},\ \sum_{\Gamma} \frac{|f(\gamma)|^2}{1+L(\gamma)^s} < \infty\}.$$

Recall that Γ has Property RD with respect to a length function l if and only if there exists $s>0$ such that $H^s_L(\Gamma) \hookrightarrow C^*_r(\Gamma)$ is a bounded operator.

Definition 4.1. Let X be a Banach space and let $(\|\cdot\|_n)_n$ be a sequence of norms on $M_n(X)$ such that

$$\|Diag(V,W)\|_{n+m} = \max(\|V\|_n, \|W\|_m), \quad \forall (V,W) \in M_n(X) \times M_m(X)$$

and

$$\|\alpha V \beta\|_m \leq \|\alpha\| \|V\|_n \|\beta\|, \quad \forall (\alpha, V, \beta) \in M_{m,n}(X) \times M_n(X) \times M_{n,m}(X)$$

where $\| \cdot \|$ denotes the usual operator norm. A completely bounded map $T : X \to Y$ between two Banach spaces X and Y equipped with such sequences of norms is completely bounded if there exists $K < \infty$ such that

$$\|TV\|_n \leq K \|V\|_n, \quad \forall V \in M_n(X)$$

where $T[\cdot] = [T(\cdot)]$.

Remark 4.2. C^*_Γ is naturally an operator space (so the $\| \cdot \|_n$ are the operator norms).

Problem: Make $H^s_L(\Gamma)$ an operator space in a natural way. For instance, prove that if Γ has polynomial growth, then it has "completely bounded RD" for the operator norms $\| \cdot \|_n$.

5 RD for groups acting on special metric spaces

1. Consider a building of type $\tilde{A}_2$ (e.g. Bruhat Tits building of $SL(3, \mathbf{Q}_p)$) Is the square root appearing in the expression $\frac{1}{2}(m+1)(n+1)(m+n+2)\sqrt{(\cdot)})$ (see [RRS]) necessary when bounding $\|f \star g\|_2$, g supported on words of shape (m, n)?

2. Does $\langle a, b, s, t; a^s = (ab)^2, b^t = (ab)^2 \rangle$ (D. Wise's non-Hopfian group) have Property RD? This is a CAT(0) group, so in particular has no distorted subgroup. However, this group does not act on any cube complex and is not relatively hyperbolic.

3. Assume that Γ acts on a simplicial tree. What are the conditions on the vertex (and edge) stabilizers for Γ to have RD? (Maybe solved, but unpublished by Steger and Talbi. The answer would be if and only if the stabilizers have RD for the induced length). For example, this would handle HNN extensions.

4. Can simplicial trees be replaced by δ-hyperbolic graphs with bounded degree?

6 RD and geometry

1. Is RD stable under quasi-isometry between finitely generated groups? If Γ and Λ have isometric Cayley graphs and if Γ has RD, does Λ have RD?

2. P. de la Harpe [Ha] has shown that hyperbolic groups have RD. This was extended to (**)-relatively hyperbolic groups with respect to RD subgroups by C. Drutu and M. Sapir [DS2]. Conversely, does RD have geometric consequences (such as for a space it acts on cocompactly)?

3. Define RD for a reasonable class of metric spaces such as graphs with bounded degree, or more generally, uniformly locally finite metric spaces.

4. Question (Lafforgue): Is RD equivalent to the existence of a polynomial P such that for all finite subspaces A, B, C of Γ such that $A \subset B(r)$,

$$|\{(a,b,c) \in A \times B \times C,\ abc = 1\}| \leq P(r)\sqrt{|A||B||C|}?$$

   Note that RD implies this property (apply RD to indicator functions).

5. Vague: is there a characterization of RD in terms of geometric properties such as some action on a boundary? Some action on a compact space?

6. Remark: the only known obstruction to RD for finitely generated groups is having an amenable subgroup of non-polynomial growth. Candidates for providing new counterexamples would be D. Wise's non-Hopfian groups, or co-compact lattices in semi-simple Lie groups(!).

7 Applications of RD

1. The Rieffel problem: let Γ be a discrete group equipped with a length function L. Let $D : l^2(\Gamma) \to l^2(\Gamma)$ be the unbounded operator defined by

$$D\delta_\gamma = L(\gamma)\delta_\gamma.$$

   Connes showed that $(l^2(\Gamma), D)$ induces a (possibly unbounded) metric on the state space of $C_r^*(\Gamma)$. Rieffel noticed that this construction for a metric generalizes the Monge-Kantorovic metric on probability measures. Motivated by Kantorovic's result Rieffel found natural to ask when the the metric contructed by Connes will give the weak*-topology on the state space. Rieffel proved that this will happen exactly when the set

$$\mathcal{L} := \{a \in C_r^*(\Gamma) : \mathrm{tr}(a) = 0, \|[D,a]\| \leq 1\}$$

is precompact in norm topology. With this characterization at hand an interesting question is then to ask for what discrete groups we have fulfilled the foregoing precompactness. It seems natural to investigate groups with rapid decay, since the precompactness of $\mathcal{L}$ has already been established for $\Gamma = \mathbb{Z}^d$ ([Ri]) and for Γ being a hyperbolic group ([OR]) (by completely different methods). Also it has been proved in [AC1] that RD implies that there exists a $k_0 \in \mathbb{N}$ such that for any natural number $k \geq k_0$ the set

$$\mathcal{L}_k := \{a \in C_r^*(\Gamma) : \mathrm{tr}(a) = 0, \|\underbrace{[D, [D, \ldots, [D, a] \ldots]]}_{k}\| \leq 1\}$$

is precompact in the norm topology.

2. From Roe's book [Ro]: for what discrete groups do we have

$$C_r^*\Gamma = C_u^*(|\Gamma|) \cap L(\Gamma)$$

where $C_u^*(|\Gamma|)$ is the uniform Roe's algebra of Γ and $L(\Gamma)$ is the Von Neumann Algebra of Γ? Imitating Haagerup's proof for free groups yield this for any Γ that has RD for a conditionally negative length function. Recall that L is a conditionally negative length fonction if $(\Gamma, \sqrt{L})$ isometrically embeds into a Hilbert space.

3. Conjecture of Kaplansky: let Γ be a torsion free finitely generated group, then $\mathbf{C}\Gamma$ has no 0-divisor. Problem: prove it when Γ has RD. This problem is motivated by the fact proved by Lafforgue [La2] that $RD+$ some (very general) geometric properties imply that the idempotents on $\mathbf{C}\Gamma$ are trivial.

4. Random walks: let Γ be a finitely generated group with Property RD and let ν be a finitely supported symmetric probability on Γ. Does there exist some constants $d = d(\nu)$ and $c = c(\nu)$ such that

$$\nu^{(2n)}(e) \sim cn^{-d}\rho^{2n}$$

where ρ is the spectral radius of the convolution operator associated to ν on $\ell^2(\Gamma)$?

5. Vague: is there a relation between RD and the entropy of ergodic actions of Γ on measure spaces?

8 Which one of these groups have RD?

1. *Out*(free group),
2. *Aut*(free group),
3. Braid groups B_n (B_3 has RD),
4. Mapping class group,
5. Artin groups (right angled groups have RD since they act freely on cube complexes).

Note that Coxeter groups have RD.

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