These are problems raised in the AIM workshop " Numerical invariants of singularities and higher-dimensional algebraic varieties ". The problem sessions were moderated by Julius Ross and Tommaso de Fernex. The list was compiled by Dano Kim. We refer to the introductory notes [6] for the definition of minimal log discrepancies and the log canonical threshold and more background. Varieties are over complex numbers except when it is specified to be over a field of characteristic p.

1. (Alexeev)

   Give a definition of log canonical , log terminal etc for pairs (X, Y) where X is a normal variety which is not necessarily $\mathbf{Q}$-Gorenstein and Y is a formal sum of subschemes of X.

2. (Mustaţă)

   When $f \in \mathbf{C}[[x_1, \cdots, x_n]]$ is giving a holomorphic function near 0, the log-canonical threshold $lct(f)$ is defined the usual way ([1] , [3]) . It is rational since it is computed by a log-resolution. In general when f is just a formal power series, results of [1] make it possible to define $lct(f)$ as a limit of lct(truncations of f). Is $lct(f)$ again a rational number?

3. (Mustaţă)

   We have the ACC conjecture for log canonical thresholds (Conjecture 2.5 of [6], see also [3]). When we consider only smooth varieties, the conjecture is equivalent to the following question :

   Fix n. $\forall c \in \mathbf{R}$, does there exist $k = k(c)$ such that

   $\{f \in \mathbf{C}[[x_1, \cdots, x_n]] | lct(f) \geq c\}$ depends on k-th truncations of f ?

   A weaker question is :

   Does there exist $k = k(f)$ such that $lct(f + g) \geq lct(f)$ for all g with $ord(g) \geq k$?

4. (Mustaţă)

   Let X be a $\mathbf{Q}$-Gorenstein variety. Then a conjecture of Shokurov (Conjecture 2.2 in [6]) says $mld(P; X) \leq dim(X)$ for all $P \in X$. As a weaker form of this question, is there any universal upper bound of $mld(P; X)$ one can prove (fixing $dim(X)$) ?

5. (Schwede)

   Let X be a variety with log-canonical singularities. Does X have Du Bois singularities? (See [5] for some nice historical discussion about the question and some progress, the introduction of [4] for some discussion of the question and [2],Chapter 12.)

6. (Ishii)

   (1) What kind of exceptional divisor over X (with mild singularities) gives a minimal log discrepancy which is nonnegative for some (X, D) ?

   (2) What about over a surface X ? Let E be an exceptional divisor of the minimal resolution $X' \to X$ of X. Does E compute some $mld \geq 0$?

7. (Alexeev, Mustaţă)

   Give one (non-trivial) application of motivic integration or arc spaces to termination of flips. What about under the assumption that the varieties in a sequence of flips $X_1 \to X_2 \to \cdots$ are all smooth ?

8. (Schwede)

   Let Y be a smooth projective variety and $X \subset Y$ an irreducible subvariety of codimension r. We assume that X is normal and $\mathbf{Q}$-Gorenstein. If X is locally complete intersection (l.c.i.) , then we have the equivalence : (Y, I^r) log-canonical $\leftrightarrow$ X log-canonical (where I is the ideal sheaf of X on Y). This is not true if X is not l.c.i.

   Question. $(Y, I^{(r)})$ log-canonical $\leftrightarrow$ X log-canonical ? ($I^{(r)}$ is the r-th symbolic power of I.)

9. (Mustaţă)

   Is there adjunction formula for multiplier ideals under restriction to subvarieties that are not defined by a regular sequence ?

10. (de Fernex)

    Let X be a smooth variety and $B \subset X$ a closed proper subscheme. We assume that there exists a prime divisor E over X, with center P, such that $a_E(X, cB) \leq 0$ for some $c > 0$.

    Fix an integer e such that $1 \leq e < dim X$.

    Does there exist a smooth subvariety $Y \subset X$ of codimension e ($P \subset Y$, $Y \not\subseteq B$) and a divisor F over Y with its center $c_Y(F) = P$ such that the log discrepancy $a_F(Y, cB|_Y - eP) \leq 0$?

11. (Shokurov)

    Fix n and consider projective varieties X with an exact canonical singularity of dimension n. Is the index of K_X at the singularity bounded?

    What about the case $dim X = 3$?

    More generally, for a fixed minimal log discrepancy of $(X, 0)$, is the index of K_X at any point with such an mld, bounded ?

12. (Shokurov)

    Consider projective varieties X of dimension n with log-canonical singularities which satisfy $K_X \equiv 0$. Then is the index of K_X bounded?

    This is open for dimension 4.

The following four problems are concerned with positive characteristic issues.

13. (Mustaţă)

    Let k be a field of characteristic p.

    (1) For $f \in k[[x_1, \cdots, x_n]]$, is the F-pure threshold $fpt(f)$ a rational number ?

    (2) Let $c = fpt(f)$. Is $\tau(f^c)$ a radical ideal ?

    (3) If the zero set of $\tau(f^c)$ is zero-dimensional , does there exist $N > 1$ such that $\forall g \in k[[x_1, \cdots, x_n]]$ of order $\geq N$, $fpt(f) = fpt(f + g)$?

14. Does the set $\{fpt(f) | f \in k[[x_1, \cdots, x_n]]\}$ have the ACC property?

15. (Takagi)

    Let $(R, \mathfrak{m})$ be a regular local ring of characteristic p. Suppose that ideals I, $\mathfrak{a}$ and $\mathfrak{b}$ of R satisfy the following conditions : $\mathfrak{a} \subset I$, $\mathcal{J}(I^{-\epsilon} \cdot \mathfrak{a}^s) \subset \mathfrak{m}$ and $\mathcal{J}(\mathfrak{b}^t) \subset \mathfrak{m}$.

    Then $\mathcal{J}(I^{-\epsilon} \cdot \mathfrak{a}^s \cdot \mathfrak{b}^t) \subset \mathcal{J}(I^{-\epsilon} \cdot \mathfrak{a}^s) \cdot \mathcal{J}(\mathfrak{b}^t)$?

16. (Takagi)

    Let $(R, \mathfrak{m})$ be a regular local ring of characteristic p and $\mathfrak{p} \subset R$ a prime ideal of height 2. If $SpecR/\mathfrak{p}$ has only log canonical singularities, then $\mathfrak{p}^{(3)} \subset \mathfrak{p}^2$?

17. (Cheltsov)

    Let X be a smooth Fano variety with $PicX = \mathbf{Z}$.

    Question Do we have the following equivalence (*)?

    (*) X is birationally superrigid $\longleftrightarrow$ (1) $\forall$ linear system with no fixed components $M \subset |-mK_X|$, $(X, \frac{1}{m}M)$ is canonical for all $m > 0$. (Cheltsov explains that the implication from left to right follows from MMP.)

    Now let's consider another condition: (2) $\forall$ divisor $D \in |-mK_X|$, $(X, \frac{1}{m}D)$ is canonical (or plt) for all $m > 0$.

    Cheltsov says, if X satisfies (1) and (2) , then $X \times \mathbf{P}^1$ is "almost birationally superrigid" in the sense that the only Mori fiber structures on it are the two projections. More precisely, every birational map from $X \times \mathbf{P}^1$ to another Mori fiber space is an isomorphism.

    In the other direction, there is the following:

    Theorem (Pukhlikov , [8])

    If $X_N \subset \mathbf{P}^N$ is sufficiently general ($N \geq 6$), then X satisfies the conditions (1) and (2).

    Question. What if $N = 4$ or $N = 5$?

18. (Cheltsov)

    Let $X_n \subset \mathbf{P}^4$ denote a hypersurface of degree n with a finite number of isolated ODP(ordinary double point)s.

    Question. When is X_n factorial ($Cl(X_n) = \mathbf{Z}$) ?

    A theorem of Clemens says:

    X_n is factorial $\leftrightarrow$ $SingX_n$ imposes independent linear conditions on homogeneous forms on $\mathbf{P}^4$ of degree n.

    There is the following example of ($Cl(X_n) \neq \mathbf{Z}$):

    X_n given by $xf_{n-1} + yg_{n-1} = 0$ where f_{n-1} and g_{n-1} are generic homogeneous polynomials of degree $n-1$ in x, y, z, w, v. Then $ClX_n = \mathbf{Z} \oplus \mathbf{Z}$.

    Question. If $|SingX_n| < (n-1)^2$, then $Cl(X_n) = \mathbf{Z}$?

    Known facts. 1) $|SingX_n| \leq \frac{2}{3}(n-1)^2 \rightarrow X_n$ is factorial.

    2) $S \subset X_n$ is a smooth subvariety of codimension 1. $\rightarrow S$ is Cartier.

    Question. Let $\varphi : \mathbf{P}^4 - - \rightarrow \mathbf{P}^2$ be a generic projection. Then is it true that at most $k(n-1)$ points of the set $\varphi(SingX_n)$ lie on a curve of degree k in $\mathbf{P}^2$?

19. (Siu)

Let $\pi : X \to \Delta$ be a holomorphic family of compact Kähler manifolds over the open unit 1-disk Δ with fiber X_t. Then is the dimension of $H^0(X_t, mK_{X_t})$ independent of $t \in \Delta$?

This is the Kähler case of Siu's famous invariance of plurigenera theorem for projective smooth varieties ([7]). One of the difficulties in using similar ideas from the algebraic case is that we cannot use an auxiliary ample line bundle as in [7].

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