FEYNMAN DIAGRAMS AND VASSILIEV INVARIANTS

RICCARDO LONGONI

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Abstract. We give an introduction to the technique invented by Richard P. Feynman in order to compute certain "infinite dimensional" integrals. This technique, called "perturbative expansion", relies on particular combinatorial structures, known as "Feynman diagrams". After discussing the finite dimensional case, we will apply the perturbative expansion to the Chern–Simons–Witten Quantum Field Theory, and use the relevant Feynman diagrams in order to construct the Vassiliev knot invariants.

Contents

1. Perturbative expansion: the finite dimensional case

1.1. Action and partition function. After the discover of the so-called "path integral" formulation of Quantum Mechanics, made by Richard P. Feynman in a famous paper published in 1948, physicists had the problem to compute an integral similar to the following:

$$Z = \int_{\mathbb{R}^n} \mathrm{d}\xi \, e^{\frac{\mathtt{i}}{\hbar} S(\xi)}, \tag{1.1}$$

where $\xi = \{\xi_1, \dots \xi_n\} \in \mathbb{R}^n$, $S(\xi)$ is a polynomial, and $\mathrm{d}\xi = \prod_i \mathrm{d}\xi_i$. It is not difficult to show that the main contributions to Z arise near the critical points. Therefore the main interest was in finding an approximate solution to this integral around a critical point, say $\xi_0 = 0$, of S.

One can observe that the above integral diverges unless $\mathtt{i}$ is a real negative number. In this case in fact one can use the so-called "perturbative expansion" technique, namely one can expand S as

$$S(\xi) = S(0) + \frac{1}{2}\xi_i A_{ij} \xi_j + \text{ cubic term in } \xi + \dots$$

(here and in the sequel, sum over repeated indices is understood), integrate the quadratic part of S as a usual gaussian integral, and treat remaining part as a perturbation. We will describe this techniques in the following of this Section. Moreover, we will notice that one can continue Z as a function of $\mathtt{i}$ to the whole complex plane minus the positive axis, and in particular this makes sense of Z (and of all the quantities that we will define shortly) when $\mathtt{i}$ is the unit imaginary number.

In order to simplify or computations we will assume $S(0) = 0$ and we will write $\xi^t A \xi = \xi_i A_{ij} \xi_j$.

We now define the following integrals:

$$Z = \int_{\mathbb{R}^n} \mathrm{d}\xi \, e^{\frac{\mathrm{i}}{\hbar} S(\xi)}$$
$$Z_0 = \int_{\mathbb{R}^n} \mathrm{d}\xi \, e^{\frac{\mathrm{i}}{2\hbar} \xi^t A \xi}$$
$$Z[J] = \int_{\mathbb{R}^n} \mathrm{d}\xi \, e^{\frac{\mathrm{i}}{\hbar} S(\xi) + J^t \xi}$$
$$Z_0[J] = \int_{\mathbb{R}^n} \mathrm{d}\xi \, e^{\frac{\mathrm{i}}{2\hbar} \xi A \xi^t + J^t \xi}$$
$$\langle \xi_{i_1}, \ldots, \xi_{i_m} \rangle = \frac{1}{Z} \int_{\mathbb{R}^n} \mathrm{d}\xi \, e^{\frac{\mathrm{i}}{\hbar} S(\xi)} \xi_{i_1} \ldots \xi_{i_m}$$
$$\langle \xi_{i_1}, \ldots, \xi_{i_m} \rangle_0 = \frac{1}{Z_0} \int_{\mathbb{R}^n} \mathrm{d}\xi \, e^{\frac{\mathrm{i}}{2\hbar} \xi A \xi^t} \xi_{i_1} \ldots \xi_{i_m}$$

Here $J = \{J_1, \ldots, J_n\}$ is a vector in $\mathbb{R}^n$. We can easily obtain $\langle \xi_{i_1}, \ldots, \xi_{i_m} \rangle$ and $\langle \xi_{i_1}, \ldots, \xi_{i_m} \rangle_0$ from $Z[J]$ or $Z_0[J]$ respectively. In fact, for any function $S(x)$ and any polynomial function $f(x)$, $x \in \mathbb{R}$, we have

$$\int_{\mathbb{R}} \mathrm{d}x \, e^{\frac{\mathrm{i}}{\hbar} S(x)} f(x) = f\left(\frac{\partial}{\partial J}\right) \int_{\mathbb{R}} e^{\frac{\mathrm{i}}{\hbar} S(x) + Jx} \Big|_{J=0} .$$

Clearly the formula can be generalized to functions on $\mathbb{R}^n$, yielding

$$\begin{aligned} \text{(1.2)} \quad \langle \xi_{i_1}, \ldots, \xi_{i_m} \rangle &= \frac{1}{Z} \int_{\mathbb{R}^n} \mathrm{d}\xi \, e^{\frac{\mathrm{i}}{\hbar} S(\xi)} \xi_{i_1} \ldots \xi_{i_m} \\ &= \frac{1}{Z} \frac{\partial^m}{\partial J_{i_1} \cdots \partial J_{i_m}} \int_{\mathbb{R}^n} \mathrm{d}\xi \, e^{\frac{\mathrm{i}}{\hbar} S(\xi) + J^t \xi} \Big|_{J=0} = \frac{\partial^m}{\partial J_{i_1} \cdots \partial J_{i_m}} \frac{Z[J]}{Z} \Big|_{J=0} \end{aligned}$$

Analogously

$$\langle \xi_{i_1}, \ldots, \xi_{i_m} \rangle_0 = \frac{\partial^m}{\partial J_{i_1} \cdots \partial J_{i_m}} \frac{Z_0[J]}{Z_0} \Big|_{J=0} \tag{1.3}$$

Remark 1.1. In the physics literature, $S(\xi)$ is called the "action", $\frac{1}{2} \xi^t A \xi$ is the "quadratic part of the action", Z is the "partition function", and $\langle i_1, \ldots, i_m \rangle$ is the "m-point Green function".

1.2. **Computation of $Z_0[J]$.** We can compute $Z_0[J]$, and hence $Z_0 = Z_0[0]$, because it is essentially a gaussian integral.

Proposition 1.2. *Suppose that there exist an invertible matrix R such that $A = R^t R$, then*

$$Z_0[J] = \sqrt{\frac{(2\pi\hbar)^n}{(-\mathtt{i})^n \det A}} \; e^{-\frac{\hbar}{2\mathrm{i}} J^t A^{-1} J} = Z_0 \; e^{-\frac{\hbar}{2\mathrm{i}} J^t A^{-1} J}.$$

Proof. We remark that the existence of an invertible matrix R such that $A = R^t R$ means that A is symmetric positive definite. Let us define $q = R\xi$ and $\tilde{J} = (R^{-1})^t J$. Then

$$Z_0[J] = \int_{\mathbb{R}^n} \mathrm{d}\xi \, e^{\frac{\mathtt{i}}{2\hbar}\xi^t A \xi + J^t \xi} = \frac{1}{\det R} \int_{\mathbb{R}^n} \mathrm{d}q \, e^{\frac{\mathtt{i}}{2\hbar} q^t q + \tilde{J}^t q} =$$
$$= \sqrt{\frac{1}{\det A}} \prod_{d=1}^{n} \int_{\mathbb{R}} \mathrm{d}x \, e^{\frac{\mathtt{i}}{2\hbar}x^2 + x\tilde{J}_d}$$

where in the last equality we have denoted by x the components q_i of q. Notice that we have n gaussian integrals equal except for the coefficient $\tilde{J}_d$. Recall the formula for the gaussian integral

$$\int_{\mathbb{R}} \mathrm{d}x \, e^{-\frac{\alpha}{2}x^2} = \sqrt{\frac{2\pi}{\alpha}}$$

Then

$$\int_{\mathbb{R}} \mathrm{d}x \, e^{-\frac{-\mathtt{i}}{2\hbar}x^2 + x\tilde{J}_d} = \int_{\mathbb{R}} \mathrm{d}x \, e^{-\frac{-\mathtt{i}}{2\hbar}(x + \frac{\hbar}{\mathtt{i}}\tilde{J}_d)^2} e^{-\frac{\hbar}{2\mathtt{i}}(\tilde{J}_d)^2} = \sqrt{\frac{2\pi\hbar}{-\mathtt{i}}} \; e^{-\frac{\hbar}{2\mathtt{i}}(\tilde{J}_d)^2}$$

Observe now that

$$\sum_d (\tilde{J}_d)^2 = \tilde{J}^t \tilde{J} = J^t R^{-1} (R^{-1})^t J = J^t A^{-1} J$$

and hence

$$Z_0[J] = \sqrt{\frac{(2\pi\hbar)^n}{(-\mathtt{i})^n \det A}} \; e^{-\frac{\hbar}{2\mathtt{i}}\sum_d (\tilde{J}_d)^2} = Z_0 \; e^{-\frac{\hbar}{2\mathtt{i}} J^t A^{-1} J}.$$

□

1.3. **Wick's Theorem and the computation of $\langle \xi_{1_1}, \dots, \xi_{i_m} \rangle_0$.** We can now compute $\langle \xi_{1_1}, \dots, \xi_{i_m} \rangle_0$ thanks to equation (1.3) and Proposition 1.2

$$\langle \xi_{i_1}, \dots, \xi_{i_m} \rangle_0 = \frac{\partial^m}{\partial J_{i_1} \cdots \partial J_{i_m}} \frac{Z_0[J]}{Z_0} \Big|_{J=0} = \frac{\partial^m}{\partial J_{i_1} \cdots \partial J_{i_m}} \; e^{-\frac{\hbar}{2\mathtt{i}} J^t A^{-1} J} \Big|_{J=0}$$

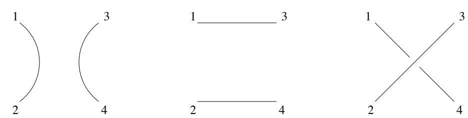
Observe now that the only nonzero contributions arise when pairs of derivatives are made on the exponential (a single derivative would multiply the exponential by J, which is then set to zero). This fact goes under the name of Wick's theorem and the precise statement is that

$$\frac{\partial^m}{\partial J_{i_1} \cdots \partial J_{i_m}} \; e^{-\frac{\hbar}{2\mathtt{i}} J^t A^{-1} J} \Big|_{J=0} =$$
$$= \left(-\frac{\hbar}{\mathtt{i}}\right)^{m/2} \sum_{\text{pairings } \sigma} A^{-1}_{i_{\sigma(1)}, i_{\sigma(2)}} A^{-1}_{i_{\sigma(3)}, i_{\sigma(4)}} \cdots A^{-1}_{i_{\sigma(m-1)}, i_{\sigma(m)}}$$

where the sum is taken over all possible pairings σ of $1, \ldots, m$, namely over all permutations σ of $1, \ldots, m$ such that $\sigma(2j-1) < \sigma(2j)$, $j = 1, \ldots, m/2$ and $\sigma(1) < \sigma(3) < \cdots < \sigma(m-3) < \sigma(m-1)$

There is a graph interpretation of Wick's theorem: we draw m points on the plane, corresponding to $i_1, \ldots, i_m$, and we draw a line for every pair $(i_{\sigma(1)}, i_{\sigma(2)}), \ldots$. We do this for all possible pairings. To any of this diagram we associate the corresponding quantity using the rule that to the line from $i_{\sigma(1)}$ to $i_{\sigma(2)}$ we associate $A^{-1}_{i_{\sigma(1)}, i_{\sigma(2)}}$, then we sum over all diagrams.

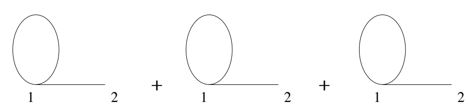
Example: $\langle \xi_1, \xi_2, \xi_3, \xi_4 \rangle_0$. There are three possible diagrams



Hence

$$\langle \xi_1, \xi_2, \xi_3, \xi_4 \rangle_0 = \frac{\hbar^2}{\mathtt{i}^2}(A^{-1}_{12}A^{-1}_{34} + A^{-1}_{13}A^{-1}_{24} + A^{-1}_{14}A^{-1}_{23})$$

Example: $\langle \xi_1, \xi_1, \xi_1, \xi_2 \rangle_0$. Again there are three possible diagrams, which are actually equal:



Hence

$$\langle \xi_1, \xi_1, \xi_1, \xi_2 \rangle_0 = \frac{3\hbar^2}{\mathtt{i}^2} A^{-1}_{11} A^{-1}_{12}$$

Example: $\langle \xi_i, \xi_j \rangle_0 = -\frac{\hbar}{\mathtt{i}} A^{-1}_{ij}$. This object, represented by a straight line, is often called "propagator".

1.4. **Computation of Z and $\langle \xi_{i_1}, \ldots, \xi_{i_m} \rangle$.** In general we have to consider the action $S(\xi) = \frac{1}{2}\xi^t A \xi - U(\xi) = \frac{1}{2}\xi^t A\xi + \sum_k U_k(\xi)$ where $U_k(\xi)$ are homogeneous polynomial of degree k in ξ and the sum is finite. The key formula for the computation of Z is the following

$$Z = \int_{\mathbb{R}^n} \mathrm{d}\xi \, e^{\frac{\mathtt{i}}{2\hbar}\xi^t A \xi + \frac{\mathtt{i}}{\hbar}U(\xi)} = $$
$$= e^{\frac{\mathtt{i}}{\hbar}U\left(\frac{\partial}{\partial J}\right)} \int_{\mathbb{R}^n} \mathrm{d}\xi \, e^{\frac{\mathtt{i}}{2\hbar}\xi^t A\xi + J^t\xi}\Big|_{J=0} =$$
$$= Z_0 \, e^{\frac{\mathtt{i}}{\hbar}U\left(\frac{\partial}{\partial J}\right)} \, e^{-\frac{\hbar}{2\mathtt{i}} J^t A^{-1} J}\Big|_{J=0}$$

1.5. **Example: interaction of degree three.** Suppose now that $S(\xi)$ has the form

$$S(\xi) = \frac{1}{2}\xi_i A_{ij}\xi_j + \frac{\lambda_{ijk}}{3!}\xi_i\xi_j\xi_k.$$

where A_{ij} is a symmetric positive definite matrix and λ_{ijk} is a symmetric tensor, i.e., we have $\lambda_{\pi(i)\pi(j)\pi(k)} = \lambda_{ijk}$ for every permutation π.

It is convenient to perform the change of variable $\xi \to \sqrt{\frac{1}{\hbar}}\,\xi$, and then expand the exponential of the cubic term in power series:

$$\begin{aligned} Z = \hbar^{n/2}\int_{\mathbb{R}^n} \mathrm{d}\xi\; e^{\frac{\mathrm{i}}{2}\xi_i A_{ij}\xi_j} e^{\mathrm{i}\sqrt{\hbar}\;\frac{\lambda_{ijk}}{3!}\;\xi_i\xi_j\xi_k} = \\ = \hbar^{n/2}\int_{\mathbb{R}^n} \mathrm{d}\xi\; e^{\frac{\mathrm{i}}{2}\xi_i A_{ij}\xi_j} \sum_{m=0}^{\infty} \frac{\mathrm{i}^m}{m!}\hbar^{m/2}\left(\frac{\lambda_{ijk}}{3!}\xi_i\xi_j\xi_k\right)^m = \\ = \sum_{m=0}^{\infty} \hbar^{n/2}\int_{\mathbb{R}^n} \mathrm{d}\xi\; e^{\frac{\mathrm{i}}{2}\xi_i A_{ij}\xi_j} \frac{\mathrm{i}^m}{m!}\hbar^{m/2}\left(\frac{\lambda_{ijk}}{3!}\xi_i\xi_j\xi_k\right)^m. \end{aligned}$$

Then we know how to compute each summand

$$\begin{aligned} \hbar^{n/2}\int_{\mathbb{R}^n} \mathrm{d}\xi\; e^{\frac{\mathrm{i}}{2}\xi_i A_{ij}\xi_j} \frac{\mathrm{i}^m}{m!}\left(\sqrt{\hbar}\frac{\lambda_{ijk}}{3!}\xi_i\xi_j\xi_k\right)^m = \\ = \hbar^{n/2}\frac{\mathrm{i}^m\hbar^{m/2}}{m!}\left[\left(\frac{\lambda_{ijk}}{3!}\frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j}\frac{\partial}{\partial J_k}\right)^m \int_{\mathbb{R}^n} \mathrm{d}\xi\; e^{\frac{\mathrm{i}}{2}\xi_i A_{ij}\xi_j + J_i\xi_i}\right]_{J=0} = \\ = Z_0\,\frac{\mathrm{i}^m\hbar^{m/2}}{m!}\left[\left(\frac{\lambda_{ijk}}{3!}\frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j}\frac{\partial}{\partial J_k}\right)^m e^{-\frac{1}{2\mathrm{i}}J_i A^{-1}_{ij} J_j}\right]_{J=0}. \end{aligned} \tag{1.4}$$

At this point we have performed the integration w.r.t. the variables ξ and we only have to do the derivative w.r.t. J and compute the resulting expression for $J = 0$. We notice again that the only non trivial contribution are the ones that doesn't leave any J that multiplies the exponential.

1.6. **Feynman graphs.** If we graphically represent every $(\frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j}\frac{\partial}{\partial J_k})$ by a vertex with three incoming edges, and every A^{-1}_{ij} by a line, then the nontrivial contributions to the integral (1.4) are parametrized by all possible diagrams obtained connecting the m trivalent vertices with some propagators, in such a way that every edge is paired with another edge. This in fact means that every derivative (which is represented by an incoming edge) matches with some J (which is represented as one of the end point of a line) and no derivative or J is not paired.

1.6.1. *Example:* $m = 2$. The term of $\frac{Z}{Z_0}$ proportional to $-\hbar/\mathtt{i}$ is given by

$$-\frac{\mathtt{i}^3}{2}\left[\frac{\lambda_{ijk}}{3!}\frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j}\frac{\partial}{\partial J_k}\frac{\lambda_{lmn}}{3!}\frac{\partial}{\partial J_l}\frac{\partial}{\partial J_m}\frac{\partial}{\partial J_n}\, e^{-\frac{1}{2\mathtt{i}}A^{-1}_{\alpha\beta}J_\alpha J_\beta}\right]_{J=0} =$$

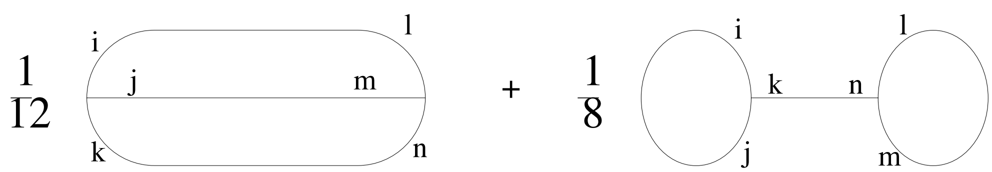
$$\left(\text{in order to simplify the notations, we define } \partial_i \triangleq \frac{\partial}{\partial J_i}\right)$$

$$= -\frac{\mathtt{i}^3}{2}\left(\frac{\lambda_{ijk}}{3!}\,\overset{1}{\partial_i}\overset{2}{\partial_j}\overset{3}{\partial_k}\right)\left(\frac{\lambda_{lmn}}{3!}\,\overset{1}{\partial_l}\overset{2}{\partial_m}\overset{3}{\partial_n}\right)e^{-\frac{1}{2\mathtt{i}}A^{-1}_{\alpha\beta}J_\alpha J_\beta}\Bigg|_{J=0} +$$

$$-\frac{\mathtt{i}^3}{2}\left(\frac{\lambda_{ijk}}{3!}\,\overset{1}{\partial_i}\overset{1}{\partial_j}\overset{2}{\partial_k}\right)\left(\frac{\lambda_{lmn}}{3!}\,\overset{2}{\partial_l}\overset{3}{\partial_m}\overset{3}{\partial_n}\right)e^{-\frac{1}{2\mathtt{i}}A^{-1}_{\alpha\beta}J_\alpha J_\beta}\Bigg|_{J=0} =$$

$$= \frac{1}{12}\left(\lambda_{ijk}\lambda_{lmn}A^{-1}_{il}A^{-1}_{jm}A^{-1}_{kn}\right) + \frac{1}{8}\left(\lambda_{ijk}\lambda_{lmn}A^{-1}_{ij}A^{-1}_{kn}A^{-1}_{lm}\right)$$

which has a diagrammatic representation as:



Notice that the two coefficients $\frac{1}{12}$ and $\frac{1}{8}$ are exactly the inverse of the number of automorphism of the graph. We will give below a proof of this fact at every order m.

1.6.2. *Partition function.* It is not difficult to figure out the general formula for the partition function. The result is the following

$$\frac{Z}{Z_0} = \sum_{m=0}^{\infty}\sum_{\Gamma\in\mathcal{D}_m}\frac{(-\mathtt{i})^{-m/2}\hbar^{m/2}}{|\mathrm{Aut}\Gamma|}\sum_{\text{labels}}\left(\prod_{\text{edges}}A^{-1}_{\text{edge}}\right)\left(\prod_{\text{vertices}}\lambda_{\text{vertex}}\right) \tag{1.5}$$

where the second sum is over the space $\mathcal{D}_m$ of all trivalent graph with m vertices.

To prove eq. (1.5) we only have to show that the numerical coefficient appearing in that formula is the inverse of $|\mathrm{Aut}\ \Gamma|$, where $|\mathrm{Aut}\ \Gamma|$ is the number of automorphisms of Γ. We remark that by an automorphism of a graph we mean a permutation of the half edges of the graph, respecting the adjacencies.

We first have to compute how many diagrams isomorphic to Γ are created while computing the derivatives w.r.t. the J's. Let Ξ be the set of all graphs obtained connecting m trivalent vertices in all possible ways, where m is the number of vertices of Γ. Let G be the semi-direct product of Σ_m and $(\Sigma_3)^m$ (Σ_k being the symmetric group of k elements). The group G acts on Ξ as follows: Σ_m permutes the vertices, while the ith copy of Σ_3 permutes the half edges meeting at the ith vertex.

The number of graphs isomorphic to Γ is therefore given by

$$\frac{|G|}{\mathrm{Stab}_G(\Gamma)} = \frac{m!\,(3!)^m}{\mathrm{Aut}(\Gamma)}. \tag{1.6}$$

On the other side, we have a factor $\frac{1}{m!}$ coming from the exponential, and a $\frac{1}{(3!)}$ for each vertex considered, due to the fact that we have divided λ_{ijk} by 3! in the non-quadratic part of $S(\xi)$. Multiplying eq. (1.6) by all these factors, we get $\frac{1}{|\mathrm{Aut}\ \Gamma|}$.

1.6.3. *m-point Green function.* We begin with the 2-point Green function

$$\begin{aligned}\langle \xi_{i_1}, \xi_{i_2}\rangle = \frac{1}{Z}\int_{\mathbb{R}^n} \mathrm{d}\xi\ e^{\frac{\mathtt{i}}{2}\xi^t A\xi + \mathtt{i}\sqrt{\hbar}\ \frac{\lambda_{ijk}}{3!}\xi_i\xi_j\xi_k}\ \xi_{i_1}\xi_{i_2} = \\ = \frac{Z_0}{Z}\frac{\partial}{\partial J_{i_1}}\frac{\partial}{\partial J_{i_2}} e^{\mathtt{i}\sqrt{\hbar}\ \frac{\lambda_{ijk}}{3!}\frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j}\frac{\partial}{\partial J_k}}\ e^{-\frac{1}{2\mathtt{i}}J^tA^{-1}J}\Big|_{J=0} = \\ = \frac{Z_0}{Z}\sum_{m=0}^{\infty}\sum_{\Gamma\in\mathcal{D}_{2,m}} \frac{(-\hbar/\mathtt{i})^{m/2}}{|\mathrm{Aut}\ \Gamma|}\sum_{\text{labels}}\left(\prod_{\text{edges}} A^{-1}_{\text{edge}}\right)\left(\prod_{\text{vertices}}\lambda_{\text{vertex}}\right)\end{aligned}$$

where the second sum runs over the space $\mathcal{D}_{2,m}$ of graphs with two univalent vertices, labelled by i_1 and i_2, and m trivalent vertices.

Now we have to consider the term $\frac{Z_0}{Z}$. We claim that it cancels exactly those (sub)diagrams in $\mathcal{D}_{2,m}$ which are not path connected to any of the univalent vertices. Denote by $\mathcal{C}_{2,*}$ the set of diagrams with two univalent vertices and any number of trivalent vertices, such that all connected components include at least one univalent vertex, then we can write

$$\langle \xi_{i_1}, \xi_{i_2}\rangle = \sum_{\Gamma\in\mathcal{C}_{2,*}} \frac{(-\hbar/\mathtt{i})^{t(\Gamma)/2}}{|\mathrm{Aut}\ \Gamma|}\sum_{\text{labels}}\left(\prod_{\text{edges}} A^{-1}_{\text{edge}}\right)\left(\prod_{\text{vertices}}\lambda_{\text{vertex}}\right) \tag{1.7}$$

where $t(\Gamma)$ is the number of trivalent vertices of the diagram Γ.

Clearly a similar formula gives the m-point Green function when instead of $\mathcal{C}_{2,*}$ we consider $\mathcal{C}_{m,*}$:

$$\langle \xi_{i_1}, \dots \xi_{i_m}\rangle = \sum_{\Gamma\in\mathcal{C}_{m,*}} \frac{(-\hbar/\mathtt{i})^{t(\Gamma)/2}}{|\mathrm{Aut}\ \Gamma|}\sum_{\text{labels}}\left(\prod_{\text{edges}} A^{-1}_{\text{edge}}\right)\left(\prod_{\text{vertices}}\lambda_{\text{vertex}}\right). \tag{1.8}$$

Here $t(\Gamma)$ is again the number of trivalent vertices of the diagram Γ, and the diagrams in $\mathcal{C}_{m,*}$ have m univalent vertices labelled by $i, \dots, i_m$ and any number of trivalent vertices.

1.6.4. *Feynman rules.* We can rephrase equations (1.5), (1.7) and (1.8) by saying that each of the contributions to the Green and partition functions are parameterized by certain uni-trivalent diagrams. Moreover Green and partition functions can be recovered from the diagrams using a set of rules, called *Feynman rules*, which can be described as follows

(1) consider a diagram in $\mathcal{C}_{m,*}$ or $\mathcal{D}_*$ and put the appropriate labels on its edges and vertices;
(2) associate a "propagator" A^{-1}_{ij} to every edge and a "vertex interaction" λ_{ijk} to every vertex of the diagram;
(3) sum over all the labels;
(4) multiply the result for the proper number of $-\hbar/\mathtt{i}$ and divide by the number automorphism of the graph;
(5) sum over all graph in $\mathcal{C}_{m,*}$ or $\mathcal{D}_*$.

Remark 1.3. We have made our computation only in the case of an "interaction of degree three", namely considering $U(\xi) = \frac{\lambda_{ijk}}{3!}\xi_i\xi_j\xi_k$. More generally we can consider polynomial of any valence p. In that case formulas (1.5) and (1.7) hold, except that instead of considering diagrams with trivalent vertices we have to consider diagrams with vertices of valence p.

1.7. **Gauge theories.**

There are cases in which the quadratic part of the action is degenerate, and hence we cannot in principle apply the algorithm developed above. There are however specific examples, the so called gauge theories, which are tractable.

Suppose there is an l dimensional Lie group G acting freely on the vectors $\xi \in \mathbb{R}^n$ such that the action is invariant under this action, namely $S(g\xi) = S(\xi)$ for every $g \in G$. Then clearly the matrix A of the quadratic part of the action is not invertible. The reason is that S is constant on the orbits of the group G, and the idea is to reduce the integral over all the space $\mathbb{R}^n$ to an integral over the quotient space $\mathbb{R}^n/G$, where hopefully the induced action is non degenerate.

The first step is to find a section of the fiber bundle $\mathbb{R}^n \to \mathbb{R}^n/G$. This can be done by taking a smooth function $F\colon \mathbb{R}^n \to \mathbb{R}^l$ which has one and only one zero on each orbit.

Suppose moreover that the action of G respects the measure $\mathrm{d}\xi$, and the integral over the total space $\mathbb{R}^n$ can be factored as the integral over $\mathbb{R}^n/G$ times the volume of the orbit. Then we can write

$$Z = \mathrm{Vol}(G)\int_{\mathbb{R}^n} \mathrm{d}\xi e^{\frac{\mathtt{i}}{2\hbar}\xi_i A_{ij}\xi_j + \frac{\mathtt{i}}{\hbar}U(\xi)}\delta(F(\xi))\det\left(\frac{\partial F}{\partial y}\right)$$

where y are local coordinates of the orbits of G near the section $F(\xi)=0$ and $\det(\frac{\partial F}{\partial y})$ is the Jacobian of the change of coordinates from $\mathbb{R}^n/G$ to the section $F(\xi)=0$.

In order to be able to apply the perturbative expansion algorithm again, we have to express the Dirac delta and the determinant as the exponentials of some function. In the first case we have the usual expression for the Fourier transform of

$\delta(x)$, namely

$$\delta(F(\xi)) = \frac{1}{(2\pi\hbar)^l}\int_{\mathbb{R}^l} \mathrm{d}\zeta\; e^{\frac{\mathtt{i}}{\hbar}\zeta^t F(\xi)} \tag{1.9}$$

In the second case we have to introduce two anti-commuting variables $c \in \mathbb{R}^l_a$ and $\bar{c} \in \mathbb{R}^l_a$, plus some integration rules (see Appendix A) such that

$$\det\Lambda = \left(\frac{\mathtt{i}}{\hbar}\right)^l \int_{\mathbb{R}^l_a\times\mathbb{R}^l_a} \mathrm{d}c\,\mathrm{d}\bar{c}\; e^{\frac{\mathtt{i}}{\hbar}\bar{c}^t\Lambda c} \tag{1.10}$$

The vectors c and $\bar{c}$ are often called *ghosts*. At the end, the partition function is again expressed, up to a constant factor, as an integral of an exponential

$$Z \sim \int_{\mathbb{R}^n/G\times\mathbb{R}^l\times\mathbb{R}^l_a\times\mathbb{R}^l_a} \mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}c\,\mathrm{d}\bar{c}\; e^{\frac{\mathtt{i}}{\hbar}\left(\frac{1}{2}\xi^t A\xi + U(\xi) + \zeta^t F(\xi) + \bar{c}^t\frac{\partial F}{\partial y}c\right)}$$

The function which appear inside the exponential (divided by $\mathtt{i}/\hbar$) is sometimes called the "total action" and is the sum of the three contributions, namely the initial action plus the gauge fixing term (1.9) and the ghost part (1.10):

$$S_{\text{tot}} = S(\xi) + S_{\text{g.f.}}(\xi,\zeta) + S_{\text{ghost}}(c,\bar{c},\xi)$$

We can now perform the perturbative expansion by inverting the quadratic part of the above action $\mathcal{S}(\xi,\zeta,\bar{c},c)$. If we define the new variable

$$X = \begin{pmatrix}\xi\\ \zeta\\ \bar{c}\\ c\end{pmatrix}$$

then the matrix associated to the quadratic part of the total action is

$$\Lambda = \begin{pmatrix} A & \frac{1}{2}\left(\frac{\partial F}{\partial \xi}(0)\right)^t & & \\ \frac{1}{2}\frac{\partial F}{\partial \xi}(0) & 0 & & 0 \\ & & 0 & \frac{1}{2}\left(\frac{\partial F}{\partial y}(0)\right)^t \\ & 0 & -\frac{1}{2}\frac{\partial F}{\partial y}(0) & 0 \end{pmatrix} = \begin{pmatrix}\Lambda_c & 0\\ 0 & \Lambda_a\end{pmatrix}$$

Notice that Λ is a real matrix. It has been divided into four blocks: the upper-left block Λ_c corresponds to the commutative variables and is symmetric while the lower-right block Λ_a, corresponding to the ghosts, is antisymmetric. We suppose that the function F has been taken in such a way that both Λ_c and Λ_a are invertible. The gaussian integral of the quadratic part of the total action, can be computed using the formulas in Appendix A, and the result is as

$$\int e^{\frac{\mathtt{i}}{\hbar}X^t\Lambda X}\mathrm{d}^{n+l,2l}X = \left(\frac{2\pi\hbar}{-\mathtt{i}}\right)^{(n+l)/2}\left(\frac{\mathtt{i}}{\hbar}\right)^l (\det\Lambda_c)^{-1/2}(\det\Lambda_a)^{1/2}.$$

Therefore there will be a propagator and vertices for the commuting variables (called "field propagator" and "field vertices") from Λ_c, a propagator for the anti-commuting variables (called "ghost propagator") from the quadratic part of λ_a and vertices connecting field and ghost propagators (called "ghost-field vertices") from the remaining part of λ_a. In a sense, we may say that the field propagator and vertices gives a correction to the propagator and vertices of the original action.

Since we already know how to compute the "commutative" propagators, we only present here the formula for the ghost propagator. If we denote by X_a the vector of the anti-commuting variables of X, then

$$\langle X_{a\,i} X_{a\,j} \rangle = (\Lambda_a)_{ij}.$$

A proof of this formula is given in Appendix A.

Remark 1.4. Let us consider the space $\mathcal{F}$ of polynomials in the variables ζ, c, $\bar{c}$ with coefficients in $\mathcal{C}^\infty(\mathbb{R}^n)$, and define a grading (the "ghost number" gh) on $\mathcal{F}$ by setting $gh(c) = 1 = gh(\bar{c})$ and zero otherwise. We claim that it is possible to define an odd derivation Q on $\mathcal{F}$ (the so-called "BRST operator") such that

$$Q^2 = 0$$

and

$$\int \mathrm{d}\xi\, \mathrm{d}\zeta\, \mathrm{d}c\, \mathrm{d}\bar{c}\; Q(L) = 0 \quad \text{for every } L \in \mathcal{F}$$

Suppose now that the action S depends on some parameter $\lambda \in \mathbb{R}^m$ and denote by δ_λ the infinitesimal variation w.r.t. λ. If we want to prove the independence of Z from λ, then a sufficient condition is given by $Q(S_{Tot}) = 0$ and $\delta_\lambda S_{Tot} = Q(L)$ for some $L \in \mathcal{F}$. In fact, we have

$$\begin{aligned} \delta_\lambda Z = \int \mathrm{d}\xi\, \mathrm{d}\zeta\, \mathrm{d}c\, \mathrm{d}\bar{c}\; \delta_\lambda(e^{\frac{i}{\hbar}S_{Tot}}) = \int \mathrm{d}\xi\, \mathrm{d}\zeta\, \mathrm{d}c\, \mathrm{d}\bar{c}\; e^{\frac{i}{\hbar}S_{Tot}} Q(L) = \\ = \int \mathrm{d}\xi\, \mathrm{d}\zeta\, \mathrm{d}c\, \mathrm{d}\bar{c}\; Q(e^{\frac{i}{\hbar}S_{Tot}} L) = 0 \end{aligned} \tag{1.11}$$

Similarly, if $\mathcal{O}$ is a function of ξ, a sufficient condition for $\langle \mathcal{O} \rangle = \int \mathrm{d}\xi\, \mathrm{d}\zeta\, \mathrm{d}c\, \mathrm{d}\bar{c}\; e^{\frac{i}{\hbar}S_{Tot}} \mathcal{O}$ to be independent from λ is that $Q(\mathcal{O}) = 0$ and $\delta_\lambda \mathcal{O} = Q(K)$ for some $K \in \mathcal{F}$.

2. Infinite dimensional case

In the infinite dimensional case we will replace the integration over $\mathbb{R}^n$ with an integration over some infinite dimensional manifold. However in almost all cases it is impossible to define a measure on that spaces, and we will "define" these integrals as their "perturbative expansion" computed using a set of Feynman rules that one can easily write following what we have done in the finite dimensional case.

2.1. **Dictionary finite vs. infinite dimensional case.** Let M be a d dimensional manifold. For many purposes we may think $M = \mathbb{R}^d$. We have the following dictionary between the finite dimensional case and the infinite dimensional case, namely we pass from vectors ξ_i to maps $\phi(x)$, from sum to integrals, and so on.

Finite dimensional $i \in \{1, \dots, n\}$	Infinite dimensional $x \in M$
$\xi\colon \{1, \dots, n\} \to \mathbb{R}$	$\phi\colon M \to \mathbb{R}$
ξ_i	$\phi(x)$
$\sum_i$	$\int_M$
A_{ij}	$K(x, y)$
J_i	$J(x)$
$\xi^t A \xi = \sum_{i,j} \xi_i A_{ij} \xi_j$	$\phi K \phi = \int_M \mathrm{d}x \mathrm{d}y \phi(x) K(x, y) \phi(y)$
$J^t \xi = \sum_i J_i \xi_i$	$J\phi = \int_M \mathrm{d}x J(x) \phi(x)$
$Z[J] = \int \mathrm{d}\xi e^{\frac{i}{2\hbar}\xi^t A \xi + i J^t \xi}$	$Z[J] = \int D\phi e^{\frac{i}{2\hbar}\phi K \phi + iJ\phi}$
A^{-1} $\sum_k A^{-1}_{ik} A_{kj} = \delta_{ij}$	$G = K^{-1}$ $\int \mathrm{d}z G(x, z) K(z, y) = \delta(x - y)$
$\hbar A^{-1}_{ij} = \left(-i\frac{\partial}{\partial J_i}\right)\left(-i\frac{\partial}{\partial J_j}\right) Z_0[J]\Big\vert_{J=0}$	$\hbar G(x, y) = \left(-i\frac{\delta}{\delta J(x)}\right)\left(-i\frac{\delta}{\delta J(y)}\right) Z_0[J]\Big\vert_{J=0}$

Some remarks are in order for $\frac{\delta}{\delta J(x)}$. It is a linear operator on the space of maps which depends on $J(x)$, namely it acts linearly on elements of the form $F(J(x))$. We require that $\frac{\delta}{\delta J(x)}$ is a derivation and that $\frac{\delta}{\delta J(x)} J(y) = \delta(x-y)$. This three properties uniquely determine the operator $\frac{\delta}{\delta J(x)}$. For example one can compute the following derivative:

$$\frac{\delta}{\delta J(x)}\, e^{\int \mathrm{d}y\, J(y)\, \phi(y)} = \phi(x)\, e^{\int \mathrm{d}y\, J(y)\, \phi(y)}$$

2.2. **Partition function and Green functions.** Integration over the space of maps is not well define, but in analogy with the finite dimensional case we write

$$
\begin{aligned}
Z &= \int D\phi\, e^{\frac{i}{2\hbar}\phi K\phi + \frac{i}{\hbar}U(\phi)} \\
Z_0 &= \int D\phi\, e^{\frac{i}{2\hbar}\phi K\phi} \\
Z[J] &= \int D\phi\, e^{\frac{i}{2\hbar}\phi K\phi + \frac{i}{\hbar}U(\phi) + iJ\phi} \\
Z_0[J] &= \int D\phi\, e^{\frac{i}{2\hbar}\phi K\phi + iJ\phi} \\
\langle \phi(x_1)\cdots\phi(x_m)\rangle &= \frac{1}{Z}\int D\phi\, e^{\frac{i}{2\hbar}\phi K\phi + \frac{i}{\hbar}U(\phi)}\phi(x_1)\cdots\phi(x_m) \\
\langle \phi(x_1)\cdots\phi(x_m)\rangle_0 &= \frac{1}{Z_0}\int D\phi\, e^{\frac{i}{2\hbar}\phi K\phi}\phi(x_1)\cdots\phi(x_m) \\
&= \left(-i\frac{\delta}{\delta J(x_1)}\right)\cdots\left(-i\frac{\delta}{\delta J(x_m)}\right)\left.\frac{Z_0[J]}{Z_0}\right|_{J=0}
\end{aligned}
$$

By analogy with the finite dimensional case we can write the perturbative expansion of the objects listed above. More explicitly we want to apply the "dictionary" to equations (1.5) and (1.7) in order to get meaningful formulas for the partition function and the Green functions in the infinite dimensional case.

2.3. **Example: ϕ^4 quantum field theory.** Take an action of the form

$$
\begin{aligned}
S(\phi) &= \int_M \mathrm{d}x\,\mathrm{d}y\,\phi(x)K(x,y)\phi(y) + \frac{1}{4!}\int_M \mathrm{d}x\,\phi^4(x) = \\
&= \int_M \mathrm{d}x\,\mathrm{d}y\,\phi(x)K(x,y)\phi(y) + \\
&+ \frac{1}{4!}\int_M \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w\,\delta(x-y)\delta(y-z)\delta(z-w)\,\phi(x)\phi(y)\phi(z)\phi(w).
\end{aligned}
\tag{2.1}
$$

Then Z will be by an obvious adaption of formula (1.5), namely a sum over all possible diagrams with vertices of valence four of certain numbers obtained by assigning the function $G(x,y) = K^{-1}(x,y)$ to every edge and one integration over M to each vertex (note that the factor $U(x,y,z,w) = \delta(x-y)\delta(y-z)\delta(z-w)$ identifies all the labels at each vertex). In the case of the m point Green function, we simply have to consider diagrams with m univalent vertices and any number of four-valent vertices and use again the obvious adaption of formula (1.8).

3. Chern–Simons Quantum Field Theory

3.1. **Chern–Simons action.** First we establish some conventions. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. By $\mathrm{Ad}\colon G\times G\to G$ we mean the adjoint action of G on itself, by $\mathrm{Ad}\colon G\times\mathfrak{g}\to\mathfrak{g}$ the adjoint action of G on $\mathfrak{g}$ and by $\mathrm{ad}\colon\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ the Lie bracket in $\mathfrak{g}$. Then we define $\mathrm{Ad}P := P\times_{\mathrm{Ad}} G$ and $\mathrm{ad}P := P\times_{\mathrm{Ad}}\mathfrak{g}$

Let G be a Lie group such that there exist an ad-invariant inner product Tr on $\mathfrak{g}$ (e.g. a semi-simple Lie group and the Killing form) and let P be a G-principal bundle over a smooth 3-manifold M. Here and in the following we will only consider the case where M is compact without boundary or $M=\mathbb{R}^3$.

Let α_0 be a fixed flat connection on $P\to M$, and α be another connection. Then $A:=\alpha-\alpha_0$ is a basic form, namely $A\in\Omega^1(M,\mathrm{ad}P)$.

We now define the following functional S over the space $\mathcal{A}\triangleq\Omega^1(M,\mathrm{ad}P)$

$$S(A) := \frac{1}{4\pi}\int_M \mathrm{Tr}\left(A\wedge \mathrm{d}_{\alpha_0}A+\frac{2}{3}A\wedge A\wedge A\right) \tag{3.1}$$

where $\mathrm{Tr}(A_1\wedge A_2\wedge A_3)=\frac{1}{2}\mathrm{Tr}(A_1\wedge[A_2,A_3])=\frac{1}{2}\mathrm{Tr}([A_1,A_2]\wedge A_3)$.

In order to give a geometrical interpretation for S we need to recall some facts about secondary characteristic classes. If $f\colon\mathfrak{g}^{\times k}\to\mathbb{R}$ is a symmetric, multilinear and ad-invariant function and F_α denotes the curvature of α, then it is possible to show that the $2k$-form $f(F_\alpha,\dots F_\alpha)$ can be projected to a closed $2k$-form on M, whose cohomology class is independent from α. Moreover, if we set $F_t := t\mathrm{d}\alpha+\frac{1}{2}t^2[\alpha,\alpha]$, then $T_f(\alpha):=\int_0^1 kf(A,F_t,\dots,F_t)$ is a $2k-1$ form such that $\mathrm{d}T_f(\alpha)=f(F_\alpha,\dots F_\alpha)$. The form $T_f(\alpha)$ is called the *Chern–Simons* form. Notice that, in general, $T_f(\alpha)$ can only be defined on P, and hence the cohomology class of $f(F_\alpha,\dots F_\alpha)$ on M may not be trivial. Choosing $f(F_\alpha,F_\alpha)=\mathrm{Tr}(F_\alpha,F_\alpha)$, we have

$$T_{\mathrm{Tr}}(\alpha)=\mathrm{Tr}\left(\alpha\wedge\mathrm{d}\alpha+\frac{2}{3}\alpha\wedge\alpha\wedge\alpha\right)$$

From now on, we will consider the case in which P is a trivial bundle [1]. Then $T_{\mathrm{Tr}}(\alpha)$ becomes a 3-form on M, and we can define

$$CS(\alpha):=\int_M\mathrm{Tr}\left(\alpha\wedge\mathrm{d}\alpha+\frac{2}{3}\alpha\wedge\alpha\wedge\alpha\right).$$

[1]We remark that if G is a compact simply connected Lie group, then every principal G-bundle over a 3-manifold is trivial. In fact $\pi_1(G)=\pi_2(G)=0$, together with the fact that the total space EG of the classifying bundle is contractible, implies $\pi_1(BG)=pi_2(BG)=\pi_3(BG)=0$. This means that every map from a 3-manifold M to BG is homotopic to the constant map, and hence every principal G-bundle on M is trivial

The relation with $S(A)$ is as follows

$$(3.2)\quad CS(\alpha)-CS(\alpha_0)=CS(\alpha_0+A)-CS(\alpha_0)=$$
$$=\int_M \mathrm{Tr}\left((\alpha_0+A)\wedge \mathrm{d}(\alpha_0+A)+\frac{2}{3}(\alpha_0+A)\wedge(\alpha_0+A)\wedge(\alpha_0+A)\right)-CS(\alpha_0)=$$
$$=\int_M \mathrm{Tr}\left(\alpha_0\wedge \mathrm{d}\alpha_0+\frac{2}{3}\alpha_0\wedge\alpha_0\wedge\alpha_0\right)+$$
$$+\text{ a term linear in } A \text{ vanishing because } \alpha_0 \text{ is a flat connection. } +$$
$$+\int_M \mathrm{Tr}\left(A\wedge \mathrm{d}A+\frac{2}{3}A\wedge A\wedge A+A\wedge[\alpha_0,A]\right)-CS(\alpha_0)=$$
$$=\int_M \mathrm{Tr}\left(A\wedge \mathrm{d}_{\alpha_0}A+\frac{2}{3}A\wedge A\wedge A\right)=4\pi\, S(A).$$

Therefore, $S(A)$ represent the expansion of the "action" $\frac{1}{4\pi}CS(\alpha)$ around the critical point α_0.

We define the group of *gauge transformations* $\mathcal{G}$ to be the group of automorphisms of P that induce the identity on M. The elements of $\mathcal{G}$ can be identified with the maps $g\colon P\to G$ such that $g(ph)=h^{-1}g(p)h$, for every $p\in P$ and $h\in G$, as follows: given such a map g, we define an $f\in\mathcal{G}$ by setting $f(p)=pg(p)$ (and vice versa). As a consequence, the group $\mathcal{G}$ is isomorphic to $\Omega^0(M,\mathrm{Ad}P)$. The gauge group $\mathcal{G}$ acts on a connection α as follows

$$\alpha\mapsto \mathrm{Ad}_{g^{-1}}\alpha+g^{-1}\mathrm{d}g.$$

Infinitesimally the action is given by

$$\alpha\mapsto\alpha+\mathrm{d}c+[\alpha,c]\qquad \text{for } c\in\Omega^0(M,\mathrm{ad}P).$$

Notice that $\Omega^0(M,\mathrm{ad}P)$ can be considered as the Lie algebra of the group $\mathcal{G}=\Omega^0(M,\mathrm{Ad}P)$. It is not difficult to see that $S(A)$ is invariant under infinitesimal gauge transformations

$$(3.3)\quad CS(\alpha+\mathrm{d}c+[\alpha,c])-CS(\alpha)=\int_M \mathrm{Tr}\left((\mathrm{d}c+[\alpha,c])\wedge\mathrm{d}\alpha+\alpha\wedge\mathrm{d}[\alpha,c]+\right.$$
$$\left.+2(\mathrm{d}c+[\alpha,c])\wedge\alpha\wedge\alpha\right)=$$
$$=\int_M \mathrm{Tr}\left([\alpha,c]\wedge\mathrm{d}\alpha+\alpha\wedge[\mathrm{d}\alpha,c]+\alpha\wedge[\alpha,\mathrm{d}c]+2\mathrm{d}c\wedge\alpha\wedge\alpha+2[\alpha,c]\wedge\alpha\wedge\alpha\right)=$$
$$=\int_M \mathrm{Tr}\left(c\wedge[\alpha,[\alpha,\alpha]]\right)=0.$$

In general $CS(\alpha)$ depends on the gauge class of α, but it is possible to prove that the quantity $e^{\frac{i}{\hbar}CS(\alpha)}$ is gauge invariant when $1/\hbar$ is an integer number.

The group $\mathcal{G}$ also acts on an element $A = \alpha - \alpha_0 \in \mathcal{A}$ acting as above on α and leaving α_0 invariant. Hence, the quantity $e^{\frac{i}{\hbar}S(A)}$ is also gauge invariant.

Since P is trivial, there is a canonical choice of a flat connection. Consider in fact the canonical projection $\pi_2\colon M \times G \to G$ and let θ be the canonical 1-form on G, i.e., the left-invariant 1-form on G with values in $\mathfrak{g}$ determined by the condition $\theta(\xi) = \xi$ for $\xi \in T_eG \simeq \mathfrak{g}$. Maurer–Cartan equation for θ implies that $\alpha_0 := \pi_2^*\theta$ is a flat connection on P. α_0 is called the *canonical flat connection.* If we now pick a section σ of P, then the pull back of α_0 to M is the zero form, α may be identified with A and $S(A)$ can be written as $\frac{1}{4\pi}\int_M \mathrm{Tr}\left(A \wedge \mathrm{d}A + \frac{2}{3}A \wedge A \wedge A\right)$.

Remark 3.1. Notice that the action is independent from the metric, but it depends on the choice of a connection A. Therefore, if there were a measure $\mathrm{D}A$ on the space of fields $\mathcal{A}$, then the average of the metric-independent quantity $e^{\frac{i}{\hbar}S(A)}$ over $\mathcal{A}$:

$$Z = \int_{\mathcal{A}} \mathrm{D}A\, e^{\frac{i}{\hbar}S(A)} \tag{3.4}$$

would be an invariant of the 3-manifold M.

3.2. **Wilson loop observable.** The Wilson loop is a functional on the space $\mathcal{A}$ which depends on a choice of a knot K (i.e., an embedding of S^1 into M), and is defined as the trace, in some representation R of G, of the holonomy of the loop K w.r.t. a connection α. Since we have identified the connection α with the form A, we will use the symbol $\mathrm{Tr}_R \mathrm{Hol}_K(A)$. Notice that thanks to the cyclic property of Tr, the Wilson loop does not depend on the point used to compute the holonomy, and does not change when we perform a gauge transformation.

Using again the triviality of P we can write the Wilson loop as follows

$$\begin{aligned} \mathrm{Tr}_R \mathrm{Hol}_K(A) &= \mathrm{Tr}_R \mathcal{P}\exp\left(\int_K A\right) = \\ &= \mathrm{Tr}_R\Bigg(1 - \int_0^1 \mathrm{d}t_1 A(\dot{K}(t_1)) + \int_0^1 \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2\ A(\dot{K}(t_1))A(\dot{K}(t_2)) + \cdots \\ &\quad + (-1)^n \int_0^1 \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n\ A(\dot{K}(t_1))A(\dot{K}(t_2)) \cdots A(\dot{K}(t_n)) + \cdots\Bigg) \end{aligned} \tag{3.5}$$

where $\dot{K}(t)$ is the tangent vector to the knot K at the point $K(t)$. Here we used the same symbol R for the representation $R\colon G \to \mathrm{Aut}(V)$ on some space V, and the induced representation on the Lie algebra $R\colon \mathfrak{g} \to \mathrm{End}(V)$.

If one assumes the existence of a measure on the space $\mathcal{A}$ (with the usual properties of measure on $\mathbb{R}^n$) and if $\mathrm{Tr}_R \mathrm{Hol}_K(A)$ is restricted to a function on the space of imbedded knot in $\mathbb{R}^3$, then the average of $e^{\frac{i}{\hbar}S(A)}\, \mathrm{Tr}_R \mathrm{Hol}_K(A)$ over $\mathcal{A}$

$$W(K) = \int_{\mathcal{A}} \mathrm{D}A\, e^{\frac{i}{\hbar}S(A)}\, \mathrm{Tr}_R \mathrm{Hol}_K(A) \tag{3.6}$$

turns out to be an object depending only on the isotopy class of K. In other words, $W(K)$ is a *knot invariant*. For a "proof" of this fact we refer e.g. to [3] (see also the next subsection).

In particular, for $M = \mathbb{R}^3$ and $G = SU(N)$, Witten [19] was able to relate $W(K)$ with a suitable reparametrization of the HOMFLY polynomial of the knot K. More precisely, if we set $q = e^{\frac{2\pi i}{N+1/\hbar}}$, then the following skein relation holds:

$$-q^{N/2}W(L_+) + (q^{1/2} - q^{-1/2})W(L_0) + q^{-N/2}W(L_-) = 0$$

with the normalization

$$W(\text{unknot}) = \frac{q^{N/2} - q^{-N/2}}{q^{1/2} - q^{-1/2}}$$

and L_+, L_- L_0 as in the usual skein relations.

The problem is that $W(K)$ is not well defined because a measure on $\mathcal{A}$ does not exists. The idea then is to treat it as a finite dimensional integral and compute it "formally" by means of the perturbative expansion technique of section 1.

3.3. **Gauge fixing and BRST operator.** Since the action $S(A)$ has a symmetry, before performing the perturbative expansion, we have to apply the algorithm of gauge fixing explained in subsection 1.7.

We first have to find a function F on $\mathcal{A}$ such that $F^{-1}(0)$ determines a section of the bundle $\mathcal{A} \to \mathcal{A}/\mathcal{G}$. Then we have to introduce the variable $\zeta(x)$ (more precisely a $\mathfrak{g}$-valued 3-form on M) and the "ghosts" $c(x)$ and $\bar{c}(x)$ ($\mathfrak{g}$-valued functions on M), and add the gauge fixing and the ghosts terms to the action. Finally one writes the total action

$$S_{Tot}(A, \zeta, c, \bar{c}) = S(A) + S_{\text{gf}}(A, \zeta) + S_{\text{ghost}}(c, \bar{c}, A))$$

There are several possible choices for the gauge fixing: in the following subsection we will show that the condition $\mathrm{d}^*(A) = 0$, where $\mathrm{d}^* = *\mathrm{d}*$, is a good choice, in the sense that allows us to invert the quadratic part of the action. In this case $S_{\text{gf}}(A, \zeta)$ and $S_{\text{ghost}}(c, \bar{c}, A)$ can be found e.g. in [3]. Another choice is the the so-called "axial gauge" [11], that works in the case $M = \mathbb{R}^3$ only, and give rise to the Kontsevich integral (see e.g. [3]).

The BRST operator Q can be introduced, just as in the finite dimensional case, and it is possible to show [3] that $Q(S_{Tot}) = 0$, $Q(\mathrm{Tr}\,\mathrm{Hol}_K(A)) = 0$ and that the exterior derivative of $e^{-\frac{1}{\hbar}S_{Tot}}\,\mathrm{Tr}\,\mathrm{Hol}_K(A)$ (thought as a function on the space of knots) is Q-exact. This implies that $W(K)$ is a closed function on $\mathrm{Imb}\,(S^1, \mathbb{R}^3)$, i.e., a knot invariant (see Rem 1.4).

3.4. **Perturbative expansion of the Chern-Simons Quantum Field Theory.** In this subsection we want to describe the perturbative expansion of $W(C) = \int_{\mathcal{A}} DA\, D\phi\, Dc\, D\bar{c}\, e^{\frac{i}{\hbar}S_{\text{tot}}(A)}\,\mathrm{Tr}\,\mathrm{Hol}_K(A)$ when M is a rational homology sphere and when $M = R^3$. Here we will forget the ghost contribution, because, as shown in [3], they do not give rise to significant contributions to the final result.

3.4.1. *Propagator.* If we pick a basis $\{T_a\}$ for $\mathfrak{g}$ we can write the quadratic part of the action as follows

$$\int_M \mathrm{Tr}\,(A \wedge \mathrm{d}A) = \mathrm{Tr}(T_a, T_b) \int_M A^a \wedge \mathrm{d}A^b = \mathrm{Tr}(T_a, T_b) \int_M A^a \wedge **\,\mathrm{d}A^b = \\ = \mathrm{Tr}(T_a, T_b)(A^a, *\mathrm{d}A^b)$$

where $*$ is the Hodge operator and where $(\phi, \psi) = \int_M \phi \wedge *\psi$ is the usual bilinear form on $\Omega^*(M)$.

Therefore there is a "Lie algebra" and a "differential geometric" part of the propagator. The "Lie algebra" part is given by the inverse to $t_{ab} := \mathrm{Tr}(T_a, T_b)$, i.e., by a matrix whose components t^{ab} satisfy the relations $t^{ab}t_{bc} = \delta^a_c$.

As for the "differential geometric" part, our task now is to find the integral kernel of $(*\mathrm{d})^{-1}$. Unfortunately this can be done only in some particular cases, namely when M is a rational homology sphere (i.e., $H_1(M) = 0$) or $M = \mathbb{R}^3$. Let us consider the r.h.s. case first.

Notice that $(*\mathrm{d})^2 = *\mathrm{d}*\mathrm{d} = \mathrm{d}^*\mathrm{d}$, is "half" of the Laplacian $\Box = \mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*$. Inverting the Laplacian will be the first step to determine the propagator. Recall that one can decompose $\Omega^*(M) = \mathrm{Im}\,\mathrm{d} \oplus \mathrm{Im}\,\mathrm{d}^* \oplus \mathcal{H}$ where $\mathcal{H} = \ker \Box$. Let us define the operator G on Ω^* as follows:

$$G|_{\mathrm{Im}\,\mathrm{d}} = \Box^{-1}, \quad G|_{\mathrm{Im}\,\mathrm{d}^*} = \Box^{-1}, \quad G|_{\mathcal{H}} = id.$$

Lemma 3.2. *The operator* $P = \mathrm{d}^* \circ G$ *satisfies the equation*

$$\mathrm{d}\,P + P\,\mathrm{d} = id - \pi_{\mathcal{H}} \tag{3.7}$$

where $\pi_{\mathcal{H}}$ *is the projection of* $\Omega^*(M)$ *on* $\mathcal{H}$.

Proof. We have to show that the equation holds on $\mathrm{Im}\,\mathrm{d} \oplus \mathrm{Im}\,\mathrm{d}^* \oplus \mathcal{H}$. On $\mathcal{H}$, both sides vanishes. If $\omega = \mathrm{d}\alpha$, then we have $\mathrm{d}\mathrm{d}^*G\mathrm{d}\alpha + \mathrm{d}^*G\mathrm{d}^2\alpha = \mathrm{d}\alpha$. If $\omega = \mathrm{d}^*\alpha$, then $\mathrm{d}\mathrm{d}^*G\mathrm{d}^*\alpha + \mathrm{d}^*G\mathrm{d}\mathrm{d}^*\alpha = \mathrm{d}^*\alpha$, because G commutes with d^*. □

Proposition 3.3. *If* $H^1(M) = 0$*, then* $Q\alpha = P(*\alpha)$ *is the inverse of* $*\mathrm{d}$ *on its image, i.e.,* $*\mathrm{d}Q\alpha = \alpha$ *when* $\alpha \in \mathrm{Im}(*\mathrm{d})$.

Proof. The equation $\mathrm{d}\,P + P\,\mathrm{d} = id - \pi_{\mathcal{H}}$ implies $*\mathrm{d}\,P + *P\,\mathrm{d} = * - *\pi_{\mathcal{H}}$. Let $\alpha \in \mathrm{Im}(*\mathrm{d}) = \ker(\mathrm{d}*)$. Then, the last equation applied to $*\alpha$ yields

$$*\mathrm{d}\,P*\alpha + *P\,\mathrm{d}*\alpha = **\alpha - *\pi_{\mathcal{H}}*\alpha = *\mathrm{d}\,Q\alpha + 0 = \alpha + 0.$$

□

An operator $P\colon \Omega^*(M) \to \Omega^{*-1}(M)$ satisfying the equation $\mathrm{d}\,P + P\,\mathrm{d} = id - S$ for some smooth operator S, is called a *parametrix.* We observe that this operator P is not unique since $P' := P + \mathrm{d} \circ Q - Q \circ \mathrm{d}$, is again a parametrix, for any linear operator $Q\colon \Omega^*(M) \to \Omega^{*-2}(M)$. We now want to recall the construction made in [7] of a differential form θ which is the kernel of a parametrix. The ambiguity in defining a parametrix will be reflected in an ambiguity in choosing θ.

Let us define $C_2^0(M) = M \times M \setminus \Delta$ where Δ is the diagonal. We claim there exist a compactification $C_2(M)$ of this space obtained by "blowing up" the diagonal, i.e., by inserting instead of Δ the sphere bundle of the normal bundle to Δ. Let us denote this bundle by $\partial C_2(M)$. Then we have the diagram

$$\begin{array}{ccc} \partial C_2(M) & \xrightarrow{i_\partial} & C_2(M) \\ {\scriptstyle \pi^\partial}\downarrow & & \downarrow{\scriptstyle \pi} \\ \Delta & \xrightarrow[i_\Delta]{} & M \times M \end{array}$$

We will return on this space later. We simply mention here that $C_2(M)$ is compact and the natural projections π_1, $\pi_2\colon M \times M \to M$ extend smoothly to $C_2(M)$.

One of the main results of [7] is the following

Theorem 3.4. *There exists $\theta \in \Omega^2(C_2(M))$ with the following properties*

$$\pi^\partial_* i^*_\partial \theta = -1 \tag{3.8}$$

$$\mathrm{d}\theta = \chi_\Delta \tag{3.9}$$

where χ_Δ is a representative of the Poincaré dual of Δ.

Proof. Since $\pi^\partial\colon \partial C_2(M) \to \Delta$ is a sphere bundle, then one can introduce a global angular form $\eta \in \Omega^2(\partial C_2(M))$ such that the restriction of η to each fiber is a (normalized) generator of the cohomology of the fiber, and that $\mathrm{d}\eta = -\pi^{\partial *}e$, where e is the Euler class. Since M is odd dimensional, e is trivial.

In order to extend η to $C_2(M)$, we consider a neighborhood U of Δ in $M \times M$ and consider $\tilde{U} = \pi^{-1}U$. Then $\tilde{U}$ is a neighborhood of $\partial C_2(M)$ in $C_2(M)$ of the form $\partial C_2(M) \times [0,1]$. Let us consider a real valued smooth function ρ on $\tilde{U}$ constant and equal to -1 near $\partial C_2(M)$, and constant and equal to 0 near the other boundary of $\tilde{U}$. Denote by $p\colon \tilde{U} \to \partial C_2(M)$ and consider the form $\tilde{\eta} := \rho p^*\eta$. Observe that $\mathrm{d}\tilde{\eta} = \mathrm{d}\rho p^*\eta$ is a representative of the Thom class of the normal bundle to Δ, and hence, extending $\tilde{\eta}$ by zero on the whole $C_2(M)$, $\mathrm{d}\eta$ is the pull back of a representative χ_Δ of the Poincaré dual of the diagonal, i.e., $\mathrm{d}\eta = \pi^*(\chi_\Delta + \mathrm{d}\alpha)$ for some $\alpha \in \Omega^2(M \times M)$. Finally, we set $\theta := \tilde{\eta} - \pi^*\alpha$. □

A consequence of the previous Theorem is the following

Proposition 3.5. *$P\alpha = \pi_{1*}(\pi_2^*\alpha \wedge \theta)$ is a parametrix for the operator $S\alpha = \pi_{2*}(\chi_\Delta \wedge \pi_1^*\alpha)$, i.e., satisfies the equation $\mathrm{d}\,P + P\,\mathrm{d} = id - S$.*

Proof. Follows from the previous Theorem and a generalization of the Stokes formula for $C_2(M)$

$$\mathrm{d}\pi_{2*} = -\pi_{2*}\mathrm{d} + \pi^\partial_* i^*_\partial.$$

Then

$$
\begin{aligned}
(3.10)\quad \mathrm{d}P\alpha + P\mathrm{d}\alpha &= -\mathrm{d}\pi_{2*}(\theta\wedge\pi_1^*\alpha) - \pi_{2*}(\theta\wedge\pi_1^*(\mathrm{d}\alpha)) = \\
&= \pi_{2*}(\mathrm{d}\theta\wedge\pi_1^*\alpha) + \pi_{2*}(\theta\wedge\pi_1^*(\mathrm{d}\alpha)) - \pi_*^\partial i_\partial^*(\theta\wedge\pi_1^*\alpha) - \pi_{2*}(\theta\wedge\pi_1^*(\mathrm{d}\alpha)) = \\
&= \pi_{2*}(\chi_\Delta\wedge\pi_1^*\alpha) - \pi_*^\partial i_\partial^*(\theta\wedge\pi_1^*\alpha) = \\
&= -S\alpha - \pi_*^\partial(i_\partial^*\theta\wedge\underbrace{i_\partial^*\pi_1^*}_{\pi^{\partial *}})\alpha = -S\alpha - \pi_*^\partial i_\partial^*\theta\wedge\alpha = (1-S)(\alpha)
\end{aligned}
$$

□

In the next Proposition, we show that the operator P considered in Proposition 3.5, satisfies equation (3.7).

Proposition 3.6. $S = \pi_{\mathcal{H}}$

Proof. If $\{\tau_i\}$ are harmonic forms which form a basis of $H^*(M)$ and $\{\omega_i\}$ its dual basis (that is, $\int_M \omega_i\wedge\tau_j = \delta_{ij}$), then

$$\chi_\Delta = \sum_i (-1)^{\deg\omega_i}\ \pi_1^*\omega_i\wedge\pi_2^*\tau_i$$

is a representative of the Poincaré dual of Δ. If the differential of θ is the pullback of this representative, then we can write $S\alpha = \sum_i \tau_i \int_M \omega_i\wedge\alpha$.

Suppose now that M has a Riemannian structure and that $\{\tau_i\}$ is orthonormal. Then the dual basis $\{\omega_i\}$ can be chosen by $\omega_i = (-1)^{\deg\tau_i(n-\deg\tau_i)} * \tau_i$ and we have $S\alpha = \sum_i \tau_i\langle\alpha,\tau_i\rangle = \pi_{\mathcal{H}}\alpha$. □

We finally consider the case $M=\mathbb{R}^3$. We first notice that the map

$$\begin{array}{rccc}\phi\colon & C_2^0(\mathbb{R}^3) & \to & S^2\\ & (x,y) & \mapsto & \frac{x-y}{|x-y|}\end{array}$$

can be smoothly extended to a function $\phi\colon C_2(\mathbb{R}^3)\to S^2$. If v is a normalized volume form on S^2, and $\theta := \phi^* v$, then

Proposition 3.7. $P(\alpha) = \pi_{2*}(\theta\wedge\pi_1^*\alpha)$ *satisfies the equation*

$$\mathrm{d}\,P + P\,\mathrm{d} = id$$

namely, it is a parametrix for the exterior derivative d *on* $\mathbb{R}^3$.

Proof. An application of the Stokes formula, just as in Prop. 3.5. □

Notice that, contrary to the r.h.s. case, here θ is a closed form, such that $\theta^2 = 0$. Sometimes, the form θ is called *tautological form* [8].

3.4.2. *Vertices.* Let us pick a basis $\{T_a\}$ for $\mathfrak{g}$ and write

$$\frac{2}{3}\int_M \mathrm{Tr}(A\wedge A\wedge A) = \frac{1}{3}\int_M A^a\wedge A^b\wedge A^c\, t_{abc} \tag{3.11}$$

and

$$\mathrm{Tr}_R \int_0^1 \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n A(\dot K(t_1))A(\dot K(t_2))\cdots A(\dot K(t_n)) = \tag{3.12}$$

$$= R^{\beta_n}_{a,\beta_1} R^{\beta_1}_{a,\beta_2}\cdots R^{\beta_{n-1}}_{a,\beta_n} \int_0^1 \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n A^{a_1}(\dot K(t_1))A^{a_2}(\dot K(t_2))\cdots A^{a_n}(\dot K(t_n))$$

where $t_{abc} = f^d_{ac}\, t_{dc}$, $[T_a, T_b] = f^c_{ab} T_c$, $t_{ab} = \mathrm{Tr}(T_a\, T_b)$ and R_a is the matrix associated to T_a by the representation R.

From (3.11) and (3.12) we can deduce the valence of the vertices in the Feynman graphs and the explicit expression for the object which correspond, in the infinite dimensional case, to the coefficient λ_{ijk}. We have two types of vertices:

- trivalent vertices corresponding to the integral (3.11), whose coefficient is

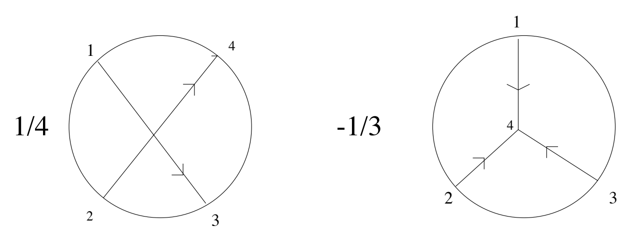
$$t_{abc}\int_M$$

- univalent vertices corresponding to the integral (3.12), whose coefficient is

$$R^{\alpha}_{a,\beta}\int_K$$

  where by $\int_K$ we mean integration on the knot K. Clearly we have to remember that all the points to be integrated on the knot have to be integrated in some order.

By convention we represent the knot K in the Feynman diagrams as a distinguished circle and we attach the univalent vertices to it. Hence, the typical Feynman graph of the Chern–Simons theory will be as in figure 3.4.2.



3.4.3. *Decorations on the diagrams.* The Feynman rules we have just described need some further clarification. In fact we see that in order for this assignment to be unique, we need to specify some decoration on the diagrams. For instance, if we consider the "Lie algebra part" of vertices and propagator, we see that $t_{ab} = t_{ba}$

and $t_{abc} = t_{cab}$, but $t_{abc} = -t_{bac}$. Therefore the diagrams should be decorated by a choice, up to even permutations, of a cyclic orientation at each trivalent vertex.

This of course renders unique the assignment of a Lie algebra factor to a diagram. We will show later that this decoration also works for the "differential-geometric part".

4. Knot invariants from Chern–Simons Quantum Field Theory

4.1. **Definition of the space of diagrams.** Following the above discussion, a diagram will be a *connected graphs* consisting of an *oriented circle* and many *edges* joining vertices which may lie either on the circle (*external vertices*) or off the circle (*internal vertices*). We also require that each vertex should be at least trivalent. If a vertex has valence M greater than three, we attach $M-3$ extra *half edges* to the vertex itself. These half edges are called "free", since they have only one endpoint attached to a vertex.

In a graph we define a *small loop* to be an edge whose end-points are the same vertex. We call a small loop external or internal according to the nature of the corresponding vertex. A *chord* is an edge whose endpoints are external vertices. An edge is called *regular* if it is not a chord.

We now have to assign a decoration on the graphs as follows: we number the free half edges and we choose a cyclic ordering of all the half edges (free or attached to some other half edge), merging at each vertex. Notice that if the graph is trivalent there are no free half edges and the only decoration is a choice of a cyclic ordering at each vertex.

We define $\mathcal{D}'$ to be the real vector space generated by these decorated graphs and $\mathcal{D} := \mathcal{D}'/\sim$ where we quotient by the following equivalence relations:

- $\Gamma \sim 0$ if two vertices in Γ are joined by more than one edge;
- $\Gamma \sim 0$, if Γ contains an internal small loop;
- for $\Gamma, \widehat{\Gamma} \in \mathcal{D}'_o$,

$$\Gamma \sim (-1)^{\pi+|\sigma|}\, \widehat{\Gamma}, \tag{4.1}$$

  where π is the order of the permutation of the labels of the free half edges and $|\sigma| = \sum_i |\sigma_i|$, $|\sigma_i|$ being the order of the permutations of the cyclic ordering of the half edges at the ith vertex.

The *order* of a graph Λ is defined as

$$\operatorname{ord}\Lambda = e - v_i, \tag{4.2}$$

where e is the number of edges and v_i is the number of internal vertices.

The *degree* of a graph Λ is defined as

$$\deg\Lambda = 2\,e - 3\,v_i - v_e, \tag{4.3}$$

where e is the number of edges, v_e is the number of external vertices and v_i is the number of internal vertices.

In practice the order of a graph is minus its Euler characteristic and the degree measures the defect of a graph from being trivalent. In the particular case when the graph is trivalent, its degree is zero and its order is half the total number of vertices.

We consider $\mathcal{D}$ as a *graded vector spaces* with respect to both the *order* and the *degree* and we denote by $\mathcal{D}^{k,m}$ the equivalence classes of decorated graphs of *order* k *and degree* m.

In the following we will need some definitions. A *chord* is an edge whose end-points are external. A *chord diagram* a diagram whose vertices are only external. A *short chord* is a chord whose end-points are two consecutive vertices on the oriented circle.

There exists an alternative decoration: label both internal and external vertices and assign an orientation (represented by an arrow) to each edge. Assume that the labelling of the external vertices is cyclic w.r.t. the orientation of the circle. Moreover, whenever there is an external small loop, fix an ordering of the two half-edges that form it (notice that this ordering is chosen independently from the edge orientation).

The space $\mathcal{D} = \mathcal{D}'/\sim$ can be recovered by replacing the last equivalence relation (4.1) as follows: for $\Gamma, \widehat{\Gamma} \in \mathcal{D}'$, set $\Gamma \sim (-1)^{\pi_1+\pi_2+l+s}\, \widehat{\Gamma}$, where π_1 is the order of the permutation of the internal vertices, π_2 is the order of the (cyclic) permutation of external vertices, l is the number of edges whose orientation has been reversed, and s is the number of external small loops on which the ordering of the half-edges has been reversed.

4.2. **Lie algebra weights and weight systems.** Having some precise definition of the space of diagrams, we describe in a more invariant way the "Lie algebra" part of the Feynman rules of the Chern-Simons Quantum Field Theory. Therefore in this subsection we will only consider trivalent diagrams.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra such that there exists an ad-invariant symmetric non degenerate bilinear from Tr on $\mathfrak{g}$. In particular we have the following objects

- the Lie bracket is a distinguished element $f \in \mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$ such that "antisymmetry" and "Jacobi identity" hold. Using Tr we can consider the corresponding element $t \in \mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}^*$. Notice that t is totally antisymmetric.
- the bilinear form can be seen as an element $\mathrm{Tr} \in \mathfrak{g}^* \otimes \mathfrak{g}^*$, and its inverse as $\mathrm{Tr}^{-1} \in \mathfrak{g} \otimes \mathfrak{g}$.
- a representation of $\mathfrak{g}$ is a tensor $R \in \mathfrak{g}^* \otimes V \otimes V^*$.

To every diagram Γ we can attach the weight $\tau(\Gamma)$ by placing a tensor t on every internal vertex, a tensor R on every external vertex (by convention we put the V^* component on the incoming arc and the V component on the outgoing arc), a tensor Tr^{-1} on every edge. Finally $\tau(\Gamma)$ is obtained by making the contractions between $\mathfrak{g}$ and $\mathfrak{g}^*$, and between V and V^*, according to the diagram.

Suppose now that we have three diagrams everywhere equal except in a neighborhood of some vertex where they look like the three diagrams of figure 1. Let us denote these diagrams by the letters S, T and U respectively.

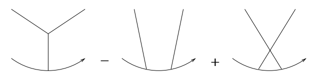


FIGURE 1. The STU relation

Also consider three diagrams everywhere equal except in a neighborhood of some vertex where they look like the three diagrams of figure 2, and denote these diagrams by the letters I, H and X respectively.

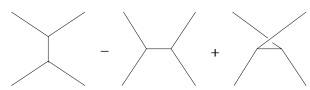


FIGURE 2. The IHX relation

Finally if A is a diagram, let us denote by $\bar{A}$ the diagram where the cyclic order at one vertex of A has been reversed. Then

Theorem 4.1. *(1) $\tau(\bar{A}) = -\tau(A)$;*
(2) $\tau(S) = \tau(T) + \tau(U)$;
(3) $\tau(I) = \tau(H) + \tau(X)$;

Proof. (1) antisymmetry of t;
(2) R is a representation, i.e., $R([a, b]) = R(a)R(b) - R(b)R(b)$ for every $a, b \in \mathfrak{g}$.
(3) Jacobi identity;

□

Another property we would like to have is the following:
(4) τ vanishes on diagrams containing short chords.

Unfortunately this is not strictly true, but it is always possible to "renormalize" τ in such a way that (4) holds. We do not want to discuss this point here and simply refer to [4] for a detailed discussion.

Definition 4.2. Any linear map $W: \mathcal{D}^{k,0} \to \mathbb{R}$ such that properties (1), (2), (3) and (4) hold, is called a *weight system of order k*. The space of weight systems of order k is denoted by $\mathcal{W}_k$.

In other words, if we denote by STU the subspace of $\mathcal{D}^{k,0}$ generated by the diagrams of figure 1, by IHX the subspace generated by the diagrams in figure 2 and by $1T$ the subspace generated by diagrams containing short chords, then a weight system of order k is a linear map $W: \mathcal{D}^{k,0}/STU, IHX, 1T \to \mathbb{R}$.

Remark 4.3. It is possible (but difficult) to prove that not every weight system comes from a Lie algebra via the above construction.

4.3. **Dual description: graph cohomology.** We introduce a coboundary operator $\delta\colon \mathcal{D}^{k,m} \to \mathcal{D}^{k,m+1}$ which acts linearly and, applied on a diagram Γ produces a linear combination of diagrams as follows: it collapse, one at a time, every regular edge and arc of Γ, leaving a free half edge in place of the collapsed edge or arc. If there are m free half edges, then we label the new free half edge with $m+1$. One can easily show that δ is well defined and that $\delta^2 = 0$. The cohomology of δ is the so-called *graph homology* and is denoted by $H^{k,m}(\mathcal{D})$.

Theorem 4.4. *If we restrict to trivalent diagrams, every weight system defines a graph cocycle. More precisely, if w is a weight system, then*

$$\Gamma = \sum_{\Gamma \in \mathcal{D}^{k,0}} \frac{1}{|\mathrm{Aut}(\Gamma)|} W(\Gamma)\, \Gamma \in \mathcal{D}^{k,0}$$

is a graph cocycle. Vice versa, every graph cocycle determines a weight system.

Proof. The adjoint operator to δ is an operator ∂ which acts linearly on $\mathcal{D}$, but of collapsing edges and arcs, ∂ decollapses the vertices in all possible ways. In particular it sends to zero every trivalent diagrams, and if applied to a diagram with all trivalent vertices except one tetravalent, produces three diagrams as in figure 2 or 1 (depending if the tetravalent vertex is internal or external resp.). Finally notice that no cocycle can be constructed using diagrams with short chords. Hence the 0th graph homology group at order k is given by $\mathcal{D}^{k,0}/STU, IHX, 1T$, and the weight systems are, by definition, the real valued linear functional on this space.

The only technical difficulties then arise from the symmetry factor $\frac{1}{|\mathrm{Aut}(\Gamma)|}$, for which we refer to [1]. ☐

4.4. **Configuration space integrals.** For any compact n-dimensional manifold M, we consider first the configuration space $C_q^0(M) \triangleq M^q \setminus \{\bigcup_S \Delta_S\}$, where S runs over the ordered subsets of the first q integers with $|S| \geq 2$, and Δ_S denotes the (multi)-diagonal labelled by S, namely, the subset of M^q defined by the equations $x_{j_1} = x_{j_2} = \cdots = x_{j_{|S|}}$, $j_i \in S$. Let us denote by $Bl(M^{|S|}, \Delta_S)$ the differential geometric blow up of the manifold $M^{|S|}$ along the diagonal Δ_S, obtained by replacing Δ_S with the sphere bundle of the normal bundle of Δ_S in $M^{|S|}$. Notice that there is an imbedding

$$C_q^0(M) \hookrightarrow M^q \times \prod_{|S| \geq 2} Bl(M^{|S|}, \Delta_S)$$

given on M^q by the inclusion, and on each S factor by the projection on $C_{|S|}^0(M)$ followed by the inclusion in $Bl(M^{|S|}, \Delta_S)$. We define $C_q(M)$ to be the closure of $C_q^0(M)$ inside the above space. One immediately sees that $C_q(M)$ is compact whenever M is compact.

This compactification $C_q(M)$ of $C_q^0(M)$ was introduced in [2] as a modification of the Fulton–MacPherson construction [12] (see also [8]). The main properties of this compactification is that $C_q(M)$ is a stratified space, and in particular a smooth manifold with corners whose boundaries (or *strata* correspond to the "collision" of at least two of the q points of M. More precisely the stratum of codimension 0 is $C^0_q(M)$, while the stratum of codimension 1 is given by the union over all subsets $S \subset \{1,\ldots,n\}$ of $C^0_{q-|S|+1}(M) \times_M P_{|S|}M$. Here $\times_M$ denotes the pull-back bundle and by P_kM we mean the space of equivalence classes of k distinct points in TM which lies on the same fiber, where the equivalence class is given by scaling and translation in the fiber. In particular P_kM is a fiber bundle over M whose fibers are diffeomorphic to the quotient of $C^0_k(\mathbb{R}^n)$ by the group of translation and rescaling. More generally, all the strata of $C_q(M)$ are naturally labelled by rooted trees with q leaves.

If we choose M to be S^1, then $C^0_q(S^1)$ is not connected. We choose a connected component by fixing an order of the points on S^1 (consistent with its orientation). We denote the closure of the connected component of $C^0_q(S^1)$ by the symbol C_q.

We need a suitable compactification of $C^0_q(\mathbb{R}^3)$. Since $\mathbb{R}^3$ is not compact, we cannot rely directly on the above construction. Instead, following [8], we identify $\mathbb{R}^3$ with $S^3 \setminus \{\infty\}$ and define $C_q(\mathbb{R}^3)$ as the fiber over $\infty \in S^3$ of $C_{q+1}(S^n) \to S^3$ (say, projecting to the last factor). This way, we also have a compactification (and corresponding boundary faces) at infinity. (For example, $C_1(\mathbb{R}^3)$ is the 3-dimensional ball.)

Finally, we define the space $C_{q,t}(\mathbb{R}^3)$ of $q+t$ distinct points in $\mathbb{R}^3$, the first q of which are constrained on an imbedding of S^1, as a pulled-back bundle as follows:

$$\begin{array}{ccc} C_{q,t}(\mathbb{R}^3) & \xrightarrow{\hat{ev}} & C_{q+t}(\mathbb{R}^3) \\ \downarrow p_1 & & \downarrow \\ C_q \times \operatorname{Imb}(S^1,\mathbb{R}^3) & \xrightarrow{ev} & C_q(\mathbb{R}^3) \end{array} \tag{4.4}$$

where the map $ev : C_q \times \operatorname{Imb}(S^1,\mathbb{R}^3) \to C_q(\mathbb{R}^3)$ is the evaluation map applied to q distinct points in S^1 and $\hat{ev}$ is its lift. The diagram is commutative by construction, and one can prove (see [8]) that the spaces $C_{q,t}(\mathbb{R}^3)$ are smooth manifolds with corners.

It is not difficult to check that the maps $\phi_{ij}\colon C^0_q(\mathbb{R}^3) \to S^2$,

$$\phi_{ij}(x_1,\ldots,x_q) \triangleq \frac{x_j - x_i}{|x_j - x_i|},$$

extend to smooth maps on $C_q(\mathbb{R}^3)$. As we have seen in subsection 3.4, the "differential geometric" part of the propagator, is given by the tautological forms

$$\theta_{ij} \triangleq \hat{ev}^*\phi^*_{ij}v \in \Omega^2\left(C_{q,t}(\mathbb{R}^3)\right),$$

where v is the standard top form on S^2.

In the language of configuration spaces and tautological forms, the "differential geometric" part of the Feynman rules of the Chern–Simons Quantum Field Theory are as follows: consider one of the Feynman diagrams considered in subsection 4.1 and replace every edge of the diagram by a tautological form (e.g. the edge between the vertices i and j is replaced by θ_{ij}). Then integrate (push-forward) the product of these forms along the map $C_{q,t}(\mathbb{R}^3) \to C_q \to \mathrm{Imb}\,(S^1, \mathbb{R}^3)$ where q is the number of external edges of the diagram and t the number of internal edges.

We claim that the Feynman rules, except for some anomalies which we do not discuss here, are a chain map

$$I\colon (\mathcal{D}^{*,0}, \delta) \to (\Omega^0(\,\mathrm{Imb}\,(S^1, \mathbb{R}^3)), \mathrm{d}). \tag{4.5}$$

This follows from the generalized Stokes formula, which says that if λ is some closed differential form on $C_{q,t}(\mathbb{R}^3)$ (e.g., a product of tautological forms) and p is the map $C_{q,t}(\mathbb{R}^3) \to \mathrm{Imb}\,(S^1, \mathbb{R}^3)$, then we have

$$d\, p_* \lambda = (-1)^{\deg p_*^{\partial} \lambda} \int_{\partial C_{q,t}(\mathbb{R}^3)} \lambda,$$

where $\partial C_{q,t}(\mathbb{R}^3)$ is the union of all the boundaries of codimension-1. Then, with some little extra work, one can see that the derivative of a push-forward of tautological forms made according to some diagram Γ is indeed sum of push-forward of tautological forms made according some new diagrams, which precisely arise as the coboundary δ of Γ. For precise statements and proof see [1, 8, 16].

The map (4.5) then induces a map in homology, still denoted by I,

$$H^{*,0}(\mathcal{D}) \to H^0(\,\mathrm{Imb}\,(S^1, \mathbb{R}^3))$$

Combining this fact with the observation that every weight system determines a graph cocycle (Thm. 4.4), we have

Proposition 4.5. *Every weight system W of order k gives rise to a knot invariant V_W determined by the formula*

$$V_W \triangleq \sum_{\Gamma \in \mathcal{D}^{k,0}} \frac{1}{|\mathrm{Aut}\,\Gamma|} I(\Gamma) W(\Gamma)$$

5. Vassiliev invariants

Let $\mathrm{Imb}\,(S^1, \mathbb{R}^3)$ be the space of imbeddings of S^1 into $\mathbb{R}^3$ and $\mathrm{Imm}_k(S^1, \mathbb{R}^3)$ the space of immersions of S^1 into $\mathbb{R}^3$ which are imbeddings except for k transversal double points.

A smooth function $f\colon \mathrm{Imb}\,(S^1, \mathbb{R}^3) \to \mathbb{R}$ is called a knot invariant if $df = 0$. One can extend every knot invariant to the space $\mathrm{Imm}_k(S^1, \mathbb{R}^3)$ using the following formula

$$f(\times) := f(\times_{+}) - f(\times_{-}).$$

A knot invariant F is called a Vassiliev invariant of order $\leq k$ if f is zero on $\mathrm{Imm}_{k+1}(S^1, \mathbb{R}^3)$. Let $\mathcal{V}_k$ be the space of Vassiliev invariants of order $\leq k$.

The following examples are due to Bar-Natan [3].

Example 5.1. The Conway polynomial in the variable z is defined by the skein relation $C(\times) - C(\times) = zC()()$. Therefore we have

$$C(\times) = C(\times) - C(\times) = zC()().$$

hence if we have an immersion with more than k double points, $C(\times\cdots\times)$ is divisible at least by z^{k+1} and hence the coefficient of z^k in $C(\times\cdots\times)$ vanishes. Therefore, the kth coefficient of the Conway polynomial is a Vassiliev invariant of order k.

Example 5.2. The HOMFLY polynomial in the variables q and N is defined by the skein relation

$$q^{N/2}P(\times) - q^{N/2}P(\times) = (q^{1/2} - q^{-1/2})P()().$$

Let us change the variable $q = e^x$ and expand P in powers of x. Then we get the relation

$$P(\times) - P(\times) = x \cdot (\text{something defined on immersions with one double point less}).$$

Hence, the kth coefficient of the HOMFLY polynomial is a Vassiliev invariant of order k.

In the remaining part of the subsection we will prove the following

Theorem 5.3. *Every Vassiliev invariant determine a weight system and vice versa:*

$$\mathcal{V}_k/\mathcal{V}_{k+1} \simeq \mathcal{W}_k.$$

5.1. **Chord diagrams and 4T relations.** Recall that a chord diagram is a diagram with external vertices only. Let $\mathcal{CD}^{k,m}$ be the subspace of $\mathcal{D}^{k,m}$ generated by chord diagrams.

Consider now the linear combinations of any four chord diagrams which are equal everywhere except near some vertices, where they look like figure 3.

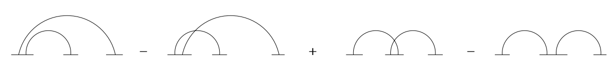


FIGURE 3

The subspace of $\mathcal{CD}^{k,0}$ generated by the diagram of figure 3 is denoted by the symbol $4T$ and is called the subspace of the $4T$ relations (at the order k).

We define a *c-weight system of order* k to be a linear map $\mathcal{CD}^{k,0}/4T, 1T \to \mathbb{R}$.

Proposition 5.4. $\mathcal{CD}^{k,0}/4T, 1T \simeq \mathcal{D}^{k,0}/STU, IHX, 1T$

Proof. We sketch the proof contained in [4]. Notice first that the inclusion $\mathcal{CD}^{k,0} \to \mathcal{D}^{k,0} \to \mathcal{D}^{k,0}/STU, IHX, 1T$ descend to the quotient $\mathcal{CD}^{k,0}/4T, 1T$. This fact is shown in figure 4.

The inverse function $D^{k,0}/STU, IHX, 1T \to \mathcal{CD}^{k,0}/4T, 1T$ is defined using recursively the STU relations in order to lower the number of internal vertices of the diagram, until one gets a chord diagram. Finally one has to check that this procedure is independent of the order in which the internal vertices are killed. ☐

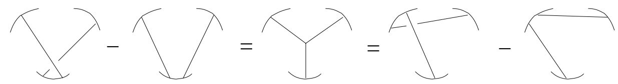


FIGURE 4

Corollary 5.5. *There is a one-to-one correspondence between weight systems and c-weight systems.*

5.2. **Weight systems from Vassiliev invariants.** We now want to show how a Vassiliev invariant V determines a weight system. We will first define a c-weight system, which in turn correspond uniquely to a weight system.

Let $D \in \mathcal{CD}^{k,0}$ and let K_D an immersion of the external circle of D into $\mathbb{R}^3$ made as follows: it is an imbedding everywhere except for the vertices joined by a chord, which are mapped to the same point. Given a Vassiliev invariant V of order k, define the c-weight system W_V as

$$W_V(D) \triangleq V(K_D).$$

First notice that $W_V(D)$ is well defined. In fact K_D has k double points and

$$0 = V(\underbrace{\times\cdots\times}_{k}\times) = V(\underbrace{\times\cdots\times}_{k}\times) - V(\underbrace{\times\cdots\times}_{k}\times).$$

Being V an invariant, immediately implies $W_V(1T) = 0$. Finally, the equation $W_V(4T) = 0$ can be established by a direct computation (see [4]).

5.3. **Vassiliev invariants from weight systems.** Given a weight system W of order k, we have seen that $V_W = \sum_{\Gamma \in \mathcal{D}^{k,0}} \frac{1}{|\mathrm{Aut}\,\Gamma|} I(\Gamma)W(\Gamma)$ is a knot invariant (see Proposition 4.5). Moreover, with some extra work, one can prove (see [1]) that $V_W(K^j) = 0$ if K^j is an singular knot with $j > k$ double points and that $V_W(K^D) = W(D)$. In other words V_W is a Vassiliev knot invariant whose associated weight system is W itself.

Vice versa, given a Vassiliev invariant V, one immediately sees that V and V_{W_V} differ by an invariant of type $k-1$. In fact

$$W_{V - V_{W_V}} = W_V - W_{V_{W_V}} = W_V - W_V = 0$$

This completes the proof of Thm. 5.3.

APPENDIX A. GRASSMANN VARIABLES AND BEREZINIAN INTEGRATION

The material in this Appendix comes from [10] p. 1-46.

A.1. **Definitions and main properties.** Let $\mathcal{G}^r$ be the Grassmann algebra with r generators, i.e., a (complex) vector space space endowed with an associative and bilinear product as follows: $\mathcal{G}^r$ is generated by ξ_a, $a = 1, \cdots, r$, with relations

$$\xi_a \xi_b + \xi_a \xi_b = 0.$$

Moreover we assume that there is an identity for the product. Hence $\mathcal{G}^r$ is a 2^r dimensional vector space generated by $1, \xi_a, \xi_a\xi_b(a < b), \ldots, \xi_1\xi_2\cdots\xi_r$, and a general element of $\mathcal{G}^r$ is

$$z = z_0 + z_a\xi_a + z_{ab}\xi_a\xi_b + \ldots + z_{a_1\cdots a_n}\xi_{a_1}\cdots\xi_{a_r} + \ldots$$

the z_*'s being complex numbers. Infinite dimensional Grassmann algebras are defined along similar lines. An element of a Grassmann algebra is called odd (resp. even) if its monomials contain an odd (resp. even) number of ξ's. So, every g can be split into its odd and even part $z = z_o + z_e$. The monomial which do not contain any ξ is called the body of z and is denoted by z_B. Purely odd Grassmann elements are called a-numbers, and they form a subset of $\mathcal{G}^\infty$ denoted by $\mathbb{C}_a$. Purely even Grassmann elements, also called a-numbers, form a sub-algebra which is denoted by $\mathbb{C}_c$.

A function $f\colon \mathbb{C}_a \to \mathcal{G}^\infty$ is called "analytic" if, for an infinitesimal displacement $\mathrm{d}v$ of an a-number v, there exist two coefficients $\frac{\bar{\mathrm{d}}}{\mathrm{d}v}f(v)$ and $f(v)\frac{\bar{\mathrm{d}}}{\mathrm{d}v}$ independent of $\mathrm{d}v$ such that $\mathrm{d}f(v) = \frac{\bar{\mathrm{d}}}{\mathrm{d}v}f(v) = f(v)\frac{\bar{\mathrm{d}}}{\mathrm{d}v}$. One can easily see that the only analytic functions are the linear functions $f(v) = a + bv$. The coefficients $\frac{\bar{\mathrm{d}}}{\mathrm{d}v}f(v)$ and $f(v)\frac{\bar{\mathrm{d}}}{\mathrm{d}v}$ are called left and right derivative respectively. The same definitions apply for the case of functions $f\colon \mathbb{C}_a \to \mathcal{G}^\infty$.

In order to reduce to "real" Grassmann numbers, one introduces a complex conjugation $*$ such that $(z_1 + z_2)^* = z_1^* + z_2^*$, $(z_1z_2)^* = z_1^*z_2^*$ for every $z_1, z_2 \in \mathcal{G}^\infty$. Then one set $(\xi^a)^* = \xi^a$. Real Grassmann numbers are those z such that $z^* = z$. The subset of real elements of $\mathbb{C}_a$ (resp. $\mathbb{C}_c$) is denoted by $\mathbb{R}_a$ (resp. $\mathbb{R}_c$). The symbol x will generally denote a real variable.

We want to define the integration of a differentiable function $f\colon \mathbb{R}_a \to \mathcal{G}^\infty$. Since they are only of the form $f(x) = a + bx$ we just have to make sense of $\int \mathrm{d}x$ and $\int x\mathrm{d}x$. If one sets

$$\int \mathrm{d}x = 0$$
$$\int x\mathrm{d}x = 1,$$

then one can easily show that the following identities hold

$$\int [f(x)+g(x)]\mathrm{d}x = \int f(x)\mathrm{d}x + \int g(x)\mathrm{d}x$$

$$\int af(x)\mathrm{d}x = a\int f(x)\mathrm{d}x \qquad a\in\mathcal{G}^\infty$$

$$\int f(x+a)\mathrm{d}x = \int f(x)\mathrm{d}x$$

$$\int f(x)\frac{\bar{\mathrm{d}}}{\mathrm{d}x}g(x)\mathrm{d}x = \int f(x)\frac{\bar{\mathrm{d}}}{\mathrm{d}x}g(x)\mathrm{d}x$$

$$\int f(x)e^x\mathrm{d}x = f(\frac{\partial}{\partial J})\int e^{x+Jx}\Big|_{J=0}.$$

The integral in $\mathbb{R}_a^n$ is defined accordingly using the "volume element" $\mathrm{d}x=\mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_l$, where $x_i\in\mathbb{R}_a$. More explicitly, the general form of a differentiable function $f\colon\mathbb{R}_a^l\to\mathcal{G}^\infty$ is

$$f(x)=\sum_{l=0}^{l} a_{\alpha_1\ldots\alpha_r}x_{\alpha_r}\ldots x_{\alpha_1}$$

and its integral is the coefficient of $x_l\ldots x_1$

$$\int f(x)\mathrm{d}x = a_{1\ldots l}$$

In a similar manner one defines the integral of functions defined in $\mathbb{R}^n\times\mathbb{R}_a^l$, using the volume element $\mathrm{d}x^{n,l}=\mathrm{d}x^1\ldots\mathrm{d}x^n\mathrm{d}x^{n+1}\ldots\mathrm{d}x^{n+l}$.

If we perform a change of basis $\bar{x}=Lx$ where

$$L=\begin{pmatrix}A & C\\ D & B\end{pmatrix}$$

is an invertible matrix formed by two sub-matrices of c-numbers A and B, and two sub-matrices of a-numbers C and D, then the volume element changes with the quantity $\det(A-CB^{-1}D)(\det B)^{-1}$.

Another useful formula we need in order to perform the perturbative expansion is the following

$$\int \mathrm{d}X f(X)e^{-\frac{1}{2}X^t\Lambda X} = f\left(\frac{\partial}{\partial J}\right)\int \mathrm{d}X e^{-\frac{1}{2}X^t\Lambda X+XJ^t}\Big|_{J=0}$$

which is obviously true for any polynomial f and for $J\in\mathbb{R}^{n+l}$.

A.2. **Gaussian integral.** Let X be a vector of $n+l$ Grassmann numbers, the first n of which belongs to $\mathbb{R}$ and the remaining belongs to $\mathbb{R}_a$. Let Λ be a real matrix of the form

$$L=\begin{pmatrix}A & 0\\ 0 & B\end{pmatrix}$$

where A is a $n \times n$ symmetric and invertible matrix, and B is a $l \times l$ (l even) antisymmetric matrix. We want to compute the integral

$$I = \int_{\mathbb{R}^n \times \mathbb{R}^l_a} e^{-\frac{1}{2} X^t \Lambda X}.$$

Let O_1 and O_2 be two orthogonal real matrices with determinant $+1$, such that

$$O_1 \, A \, O_1^{-1} = \begin{pmatrix} \lambda_1 & & & \\ & \cdot\cdot & & \\ & & \cdot\cdot & \\ & & & \lambda_n \end{pmatrix}$$

and

$$O_2 \, A \, O_2^{-1} = \begin{pmatrix} 0 & \mu_1 & & & & \\ -\mu_1 & 0 & & & & \\ & & \cdot\cdot & & & \\ & & & \cdot\cdot & & \\ & & & & 0 & \mu_{l/2} \\ & & & & -\mu_{l/2} & 0 \end{pmatrix}$$

next, set $\bar{X} = LX$ where

$$L = \begin{pmatrix} O_1 & 0 \\ 0 & O_2 \end{pmatrix}.$$

Then

$$\text{(A.1)} \quad I = \int_{\mathbb{R}^n \times \mathbb{R}^l_a} e^{-\frac{1}{2}(\lambda_1 \bar{X}_1^2 + \ldots \lambda_n \bar{X}_n^2) - (\mu_1 \bar{X}_{n+1} \bar{X}_{n+2} + \ldots \mu_{l/2} \bar{X}_{n+l-1} \bar{X}_{n+l})} \mathrm{d}\bar{X} =$$
$$= (2\pi)^{n/2} (\det A)^{-1/2} \mu_1 \ldots \mu_{l/2} = (2\pi)^{n/2} (\det A)^{-1/2} (\det B)^{1/2}$$

In addition if now $x \in \mathbb{R}^l_a$, $J \in \mathbb{R}^l$, B is a real antisymmetric invertible $l \times l$ matrix (l even) and O_2 as above, we can compute the following quantity

$$\text{(A.2)}\quad \frac{\partial}{\partial J_i}\frac{\partial}{\partial J_k}\int_{\mathbb{R}^l_a} e^{-\frac{1}{2}x^tBx+J^tx}\Big|_{J=0} = \quad (\text{setting } \bar{x}=O_2\,x \text{ and } \bar{J}=O_2 J)$$

$$= \frac{\partial}{\partial J_i}\frac{\partial}{\partial J_k}\int_{\mathbb{R}^l_a} e^{-\frac{1}{2}(\mu_1\bar{x}_1\bar{x}_2+\ldots+\mu_{l/2}\bar{x}_{l-1}\bar{x}_l)+\bar{J}_1\bar{x}_1+\ldots\bar{J}_n\bar{x}_n}\Big|_{J=0} =$$

$$= \frac{\partial}{\partial J_i}\frac{\partial}{\partial J_k}\int_{\mathbb{R}^l_a} e^{-\frac{1}{2}(\mu_1\bar{x}_1\bar{x}_2+\ldots+\mu_{l/2}\bar{x}_{l-1}\bar{x}_l)+\bar{J}_1\bar{x}_1+\ldots\bar{J}_n\bar{x}_n} e^{\mu_1^{-1}\bar{J}_1\bar{J}_2+\ldots+\mu_{l/2}^{-1}\bar{J}_{l-1}\bar{J}_l}$$

$$e^{-(\mu_1^{-1}\bar{J}_1\bar{J}_2+\ldots+\mu_{l/2}^{-1}\bar{J}_{l-1}\bar{J}_l)}\Big|_{J=0} =$$

$$= \frac{\partial}{\partial J_i}\frac{\partial}{\partial J_k}\int_{\mathbb{R}^l_a} e^{-\frac{1}{2}(\mu_1(\bar{x}_1+\mu_1^{-1}\bar{J}_2)(\bar{x}_2+\mu_1^{-1}\bar{J}_1)+\ldots+\mu_{l/2}(\bar{x}_{l-1}+\mu_{l/2}^{-1}\bar{J}_l)(\bar{x}_l+\mu_{l/2}^{-1}\bar{J}_{l-1}))}\Big|_{J=0} =$$

$$= \frac{\partial}{\partial J_i}\frac{\partial}{\partial J_k}\int_{\mathbb{R}^l_a} e^{-\frac{1}{2}(\mu_1\bar{x}_1\bar{x}_2+\ldots+\mu_{l/2}\bar{x}_{l-1}\bar{x}_l)}\Big|_{J=0} = \frac{\partial}{\partial J_i}\frac{\partial}{\partial J_k}\, e^{J^t\tilde{B}J}\Big|_{J=0} = B_{ik}$$

where $\tilde{B}$ is the matrix which coincide with B above the diagonal and it is zero elsewhere.

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Dipartimento di Matematica "G. Castelnuovo" — Università di Roma "La Sapienza", Piazzale Aldo Moro, 5 — I-00185 Roma — Italy
E-mail address: `longoni@mat.uniroma1.it`