SUPERCHARACTERS AND COMBINATORIAL HOPF ALGEBRAS

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Following are brief statements of some problems raised during the AIM Workshop *Supercharacters and combinatorial Hopf algebras*, May 17-21, 2010.

1. More and beyond SC theory for U_n

Problem 1.1. *Let G be a finite group. Let X be a partition of Irr(G) (such that it forms a supercharacter theory) . Let $\sigma_X = \sum_{\chi \in X} \chi(1)\chi$. When is it true that the product of two of these σ's is a positive integer combination of σ's?*

Remark. This is not always true. It is always true for maximal and minimal supercharacter theory of S_n. Hopefully it is true for the four supercharacter theory of S_n (see *Supercharacter theories of cyclic p-groups* by A. Hendrickson). Is it true for the natural characters associated to algebra groups?

Problem 1.2. *Can one construct a nested supercharacter theory that would realize the Hopf algebra NCQSym (indexed by ordered partitions)?*

Remark. Nested means a projective system of groups. See *Algebraic structures on Grothendieck groups of a tower of algebras*, by H. Li.

Problem 1.3.

In the original supercharacter theory of Andre, the supercharacters/superclasses associated to single boxes above the identity are irreducible. Is Andre's supercharacter theory the coarsest containing this?. If not, what is the coarsest supercharacter theory containing the irreducibles?

Problem 1.4.

Characterize the Andre/Yan supercharacter theory -along the lines of the previous problem-.

Problem 1.5. *Consider all the automorphisms of the unitriangular group. Do these automorphisms give additional nice cumpling (coarsification) to the theory?*

Remark.

- For example, if the torus acts by conjugation, we get to ignore the labels on the set partitions.
- See *Automorphisms of certain unipotent groups* by John A. Gibbs.

Problem 1.6.
For U_n every supercharacter is a product of supercharacters that happen to be irreducible. This is not true for every pattern groups. When is it true (even for algebra groups)?

Problem 1.7.
Make sense out of superinduction *and* restriction *more generally than for algebra groups.*

Remark. To start with, trying the supercharacter theory induced by automorphisms and characteristic subgroups. Look at the article *Induction for association schemes*. Johnson & Smith (1986).

Problem 1.8. *Find natural supercharacter theory for Sylow p-subgroups of S_{p^n}. Are they nested? Does the restricition of a supercharacter for S_{p^n} to $S_{p^{n-1}}$ split into supercharacters?*

Problem 1.9. *Describe irreducible characters of maximal degree of pattern groups (as supercharacters) using dimension vectors and polynomial equations.*

Remark. Check paper by Dixmier.

Problem 1.10. *Try to carry over all of the U_n work to the Borel subgroup B_n of GL_n. Is there a similar (module to a) Hopf algebra?*

Problem 1.11. *In what sense is describing the conjugacy classes of U_n difficult? Is the number of conjugacy classes difficult?*

Problem 1.12. *Is superclass theory for U_n tame?. That is, is there a tame algebra whose category of modules is equivalent to the category of supercharacters of U_n?.*

Problem 1.13. *Consider the two presentations for NCSym given below:*

$$M_\mu = \sum_{\nabla\omega=\mu} \omega \tag{1.1}$$

where $w \in A^\star$ and $A = \{a_1, a_2, \cdots\}$ non commuting.

$$U_\mu = \sum_{\sigma\in S_n,\ \lambda(\sigma)=\mu} x_{1\sigma(1)} x_{2\sigma 2} \cdots \tag{1.2}$$

where $\{x_{ij}\}$ are commutative variables such that $x_{ij}x_{lj} = 0$ if $i \neq j$ or $x_{ij}x_{ik} = 0$ if $j \neq k$.

Is it possible to describe the Hopf isomorphism $SC(2)^\star \longrightarrow \Pi^\star$ with Hopf and internal comultiplication?

Problem 1.14. *To get a better understanding of the supercharacter basis, give a basis in NC-Sym that is nicely related to supercharacters. For example, see remark about p-basis pp.20 in* Supercharacters, symmetric functions in non commuting variables, and related Hopf algebras.

Problem 1.15. *Is there any relation between supercharacter theory of U_n and properties of GL_n?. Is there an analogous problem where this has been worked out?.*

Problem 1.16. *Tung Le and Kay Magaard have a way of refining supercharacters for U_n. Is there an analog for algebra groups?*

Remark. See *Supercharacters and pattern subgroups in the upper triangular groups* by T. Le.

Problem 1.17. *Is it true that there are four infinite families of supercharacter theories for symmetric groups and 6 or 7 exceptions? Are those four families nested?. See* Supercharacter theories of cyclic p-groups *by A. Hendrickson.*

Problem 1.18. *Can the four infinite families of supercharacter theories of S_n be realized as isomorphic to a Hopf quotient of Sym?. Can we define multiplication and comultiplication in one of the families to obtain Hopf algebra structure?.*