BOUNDED COHOMOLOGY 3/20/07

1. Continuous Cohomology:

$H_c^\bullet(L, \mathbb{R})$ L = loc compact group

$C(L^k, \mathbb{R}) = \{ f : L^k \to \mathbb{R} \text{ : continuous function} \}$

L acts by precomposition

$$C(L^k, \mathbb{R})^L \xrightarrow{d} C(L^{k+1}, \mathbb{R})^L$$

$$df(x_0, \dots, x_k) = \sum_{i=0}^{k} (-1)^i f(x_0, \dots, \hat{x}_i, \dots x_k)$$

$H_c^\bullet(L, \mathbb{R})$ = cohomology of complex of L-invariants

$H_c^0(L, \mathbb{R}) = \mathbb{R}$, $H_c^1(L, \mathbb{R}) = \mathrm{Hom}(L, \mathbb{R})$

$\pi : L_1 \to L_2$ cont-hom induces $\pi^\bullet : H_c^\bullet(L_2, \mathbb{R}) \to H_c^\bullet(L_1, \mathbb{R})$

<u>Exercise</u>: If L is compact, $H^k(L, \mathbb{R}) = 0$ $k \geq 1$

~~Thm~~ X symmetric space of noncompact type

$G = \mathrm{Iso}(X)$

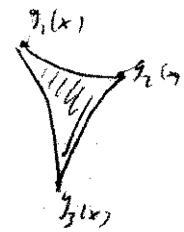
<u>Thm (Van Est)</u>: $H_c^\bullet(G,\mathbb{R}) = \Omega^\bullet(X)^G$

(There is a generalization due to Monod)

<u>X Hermitian symmetric:</u>

$\omega_X \in \Omega^2(X)^G \qquad x \in X$

$$c_x(g_1,g_2,g_3) = \frac{1}{2\pi}\int_{\Delta(g_1x,\, g_2x,\, g_3x)} \omega_X \in ZC(G^3,\mathbb{R})^G$$



closed.

$K_X \in H_c^2(G,\mathbb{R})$

"

$[c_x]$

$\rho : \pi_1(\Sigma) \longrightarrow G$

$\rho^*(K_X) \in H^2(\pi_1(\Sigma),\mathbb{R}) \simeq H^2(\Sigma,\mathbb{R})$

$H_2(\Sigma,\mathbb{R})$

$\tau(\rho) = \langle \rho^*(K_X), [\Sigma] \rangle$

<u>Extension:</u>

n-dim X $\qquad \Gamma < SO(n,1)$

$\rho : \Gamma \longrightarrow SO(n,1)$

$\mathrm{Vol}(\rho) = \langle\ ,\ \rangle_{\Gamma}^{H^n}$

$K \triangleleft G$ max compact subgroup

$BG \cong BK$

$G_u/K \longrightarrow BK$

$$H(BG) = H(BK) \longrightarrow H(G_u/K)$$

$$\Downarrow$$

$$\Omega(X)^G \cong H_c(G, \mathbb{R})$$

<u>Thm:</u> $H^2(BG) \longrightarrow H^2_c(G, \mathbb{R})$

$\qquad\qquad\qquad\qquad \downarrow$

$\qquad\qquad\qquad\qquad \kappa_X$

is in the image of the map.

and is integral

2. Continuous bounded Cohomology:

$C_b(L^k, \mathbb{R}) = \{ f : L^k \to \mathbb{R} : f \text{ bounded, continuous function } (\|f\|_\infty < +\infty) \}$

$0 \to C_b(L, \mathbb{R})^L \to C_b(L^2, \mathbb{R})^L \to \cdots$

$H^\bullet_{cb}(L, \mathbb{R})$.

$\ker d \overset{\text{closed}}{\subset} C_b(L^{k+1}, \mathbb{R})^L$

$\operatorname{Im} d \subset C_b(L^{k+1}, \mathbb{R})$

get a seminorm on $H^\bullet_{bc}(L, \mathbb{R})$

$H^\bullet_{bc}(L, \mathbb{R}) \longrightarrow H_c(L, \mathbb{R})$

Thm: $H^2_{bc}(L, \mathbb{R})$ is a Banach space

(1) $H^0_{bc}(L, \mathbb{R}) \cong \mathbb{R}$

(2) $H^1_{bc}(L, \mathbb{R}) = 0$

(3) $\ker\left(H^2_{bc}(L, \mathbb{R}) \to H^2_c(L, \mathbb{R}) \right) = QH(L, \mathbb{R}) / C_b(L, \mathbb{R})$

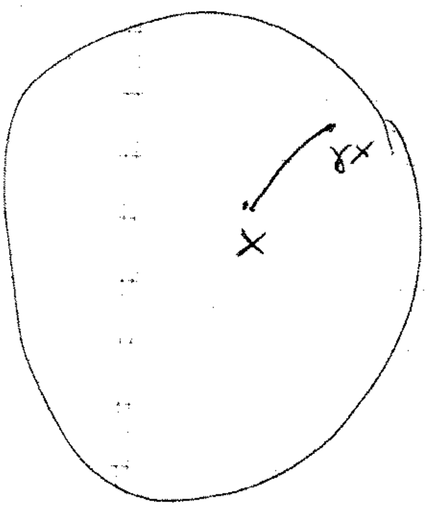
QH quasihomomorphism

$$\sup_{a,b} |f(ab)-f(a)-f(b)| < +\infty$$

<u>Thm:</u> $H^2_{bc}(\pi_1(\Sigma), \mathbb{R})$ is infinite dim.

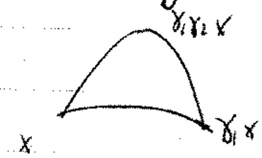
Σ hyperbolic surface $\Gamma \circlearrowright D$ $\quad \alpha \in \Omega^1(D)^\Gamma$

assume $\|d\alpha\|_\infty < +\infty$



$$f(\gamma) = \int_{[x,\gamma x]} \alpha$$

$$f_\alpha(\gamma_1\gamma_2) - f_\alpha(\gamma_1) - f_\alpha(\gamma_2) = \int d\alpha$$



$$|\ldots| \le \|d\alpha\|_\infty \cdot \pi$$

$\{f_\alpha\} \in H^2_b(\pi_1(\Sigma), \mathbb{R})$ inf-dim

~~Def~~: Conjecture:

$$H_b^k(G,\mathbb{R}) \longrightarrow H_c^k(G,\mathbb{R}) \quad \text{surjective}$$

if G is a Lie, semisimple finite ~~[illegible]~~ center

Thm (Gromov) G real, alg.

$$\begin{array}{ccc} & & H^\cdot_{bc}(G,\mathbb{R}) \\ & & \downarrow \\ H^\cdot(BG,\mathbb{R}) & \longrightarrow & H^\cdot_c(G,\mathbb{R}) \end{array}$$

$$\mathrm{Im}(\longrightarrow) \subset \mathrm{Im}(\downarrow)$$

Answer to conjecture: YES for $SL(n,\mathbb{C})$

Prop: $H^2_{bc}(G,\mathbb{R}) \xrightarrow{\sim} H^2_c(G,\mathbb{R})$ $\quad$ G semisimple finite center

Proof of injectivity:

$G = SL(2,\mathbb{R})$: $\quad n \in G$, $a \in G$ $\quad a^k n a^{-k} \to e$

$$|f(n) - f(\underset{\downarrow\, f(e)}{a^k n a^{-k}})| \leq 3C \Rightarrow f \text{ bd along } \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = N_+$$

$$\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = N_-$$

$$SL(2,\mathbb{R}) = N_+ N_- N_+ .$$

$\square$

$$\kappa_X \in H^2_c(G, \mathbb{R})$$

$$c_X(g_1, g_2, g_3) = \frac{1}{2\pi} \int_{\Delta(g_1 x, g_2 x, g_3 x)} \omega_X$$

Thm:

$$\left| \int_\Delta \omega_X \right| \leq \pi \operatorname{Rank}(X)$$

$$\Longrightarrow \kappa_X^b \in H^2_{bc}(G, \mathbb{R})$$

Corollary $\|\kappa_X^b\| = \pi \operatorname{Rk}(X)$

$$\rho : \pi_1(\Sigma) \longrightarrow G$$

$$\rho^* : H^2_{bc}(G, \mathbb{R}) \longrightarrow H^2_b(\pi_1(\Sigma), \mathbb{R}) \underset{\text{isometric}}{\simeq} H^2_b(\Sigma, \mathbb{R})$$

If $\partial \Sigma = \phi$ $\quad \tau(\rho) := \langle \rho^*(\kappa_X^b), [\Sigma]_{\ell^1} \rangle$

$$|\tau(\rho)| \leq \|\rho^*(\kappa_X^b)\| \, \|[\Sigma]_{\ell^1}\| \leq \|\kappa_X^b\| \|$$

$= \pi R k \cdot (4g-4)$

$$H_b^2(\Sigma, \partial\Sigma, \mathbb{R}) \xrightarrow{\cong} H_b^2(\Sigma, \mathbb{R})$$

$$\uparrow \qquad\qquad\qquad \downarrow$$

$$H_b^1(\partial\Sigma, \mathbb{R}) = 0 \qquad\qquad H_b^2(\partial\Sigma, \mathbb{R}) = 0$$

$$H_{bc}^{\bullet}(L, \mathbb{R}) = 0$$

L amenable

$$H_{bc}^2(G, \mathbb{R}) \to H_b^2(\pi_1(\Sigma), \mathbb{R}) \cong H_b^2(\Sigma, \mathbb{R})$$

$$\| \quad H_b^2(\Sigma, \partial\Sigma, \mathbb{R})$$

$$\downarrow \quad H^2(\Sigma, \partial\Sigma, \mathbb{R})$$

$$\tau(\rho) = \langle \quad , [\Sigma, \partial\Sigma] \rangle$$

Fact $n \geq 1$ $\pi_1(\Sigma)$ free on $2g+n-1$ generators

$\mathrm{Rep}(\ \ ) \cong G^{2g+n-1}$

Thm: $\rho \longrightarrow \tau(\rho)$ is continuous with

range $|\chi(\Sigma)|\,[-\pi\,\mathrm{Rk}(X), \pi\,\mathrm{Rk}(X)]$

~~Thm~~

If $\rho : \pi_1(\Sigma) \to G = Sp(V)$ is maximal

$\forall\, \rho(\gamma)$ has a fixed point in $\mathcal{L}(V)$