Energy function (discussion group)

WENTWORTH 19/3/7

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S = closed Riemann surface

$u: S \to N$ $\leftarrow$ non positively curved.

(i) N cpct., fixing a homotopy class of maps

(ii) $\rho: \pi_1(S) \to Iso(\tilde{N})$, ρ-equivariant maps $u: \tilde{S} \to \tilde{N}$

Under certain situations, $\exists!$ energy minimizer ($\equiv$ harmonic map, thanks to convexity).

Heat flow: $\frac{\partial u}{\partial t} = -\tau(u) = -\mathrm{tr}\nabla du = 0$ $\quad u: S \to N$ depending on $t \in [0,\infty)$

$e(u) = |du|^2$

$$\left(\frac{\partial}{\partial t} + \Delta\right) e(u) = |\nabla du|^2 + Ric_S(du, du) - Riem_N(du, du, du, du).$$

Bochner

So if $Riem_N \leq 0$, get $\left(\frac{\partial}{\partial t} + \Delta\right) e(u) \geq -C\, e(u)$

$\Rightarrow$ hence $e(u)$ is uniformly pointwise bounded

For harmonic maps, Bochner says:

$$\Delta e(u) = |\nabla du|^2 + Ric_S(du, du) - Riem_N(du, du, du, du)$$

so for example, if $curv.(N) < 0$, $curv(S) > 0$, harmonic maps have to be constant.

Given ρ, get map $E: Teich(S) \longrightarrow \mathbb{R}^+$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad \overset{\shortparallel}{T(S)}$

tangent space of Teichmüller space:

$$T_\sigma T(S) = \text{harmonic Beltrami differential}$$

$$\mu(z) = \mu_{\bar{z}}^{z}\, d\bar{z} \otimes dz^{-1}$$

If $ds^2 = \lambda|dz|^2$ is a metric on S then

$\lambda \cdot \mu_{\bar z}^{z} = \bar\Phi$. Then μ harmonic $\Leftrightarrow$ Φ quadratic holomorphic differential

$\bar\partial : TS \longrightarrow \Omega^{0,1}(TS)$ $\quad \bar\partial^* \mu = 0$

Certain μ are $\mu = \bar\partial v$, v vector field
These Beltrami differentials are trivial (their harmonic representatives are 0).

$T_\sigma^* T(S) =$ holomorphic quadr. differentials

$$\langle \mu, \Phi \rangle = \int_S \mu_{\bar z}^{z} \, \Phi_{zz} \, |dz|^2$$ natural pairing between Beltrami diff.'s and hol. quadr. diff's.

Proposition σ_t family in $T(S)$, $\dot\sigma = \frac{d}{dt}\sigma_t\big|_{t=0} = \mu$

Then
$$\frac{d}{dt}\Big|_{t=0} E(\sigma_t) = -\int_S \mu_{\bar z}^{z} \Phi_{zz} |dz|^2 = -\langle \mu, \Phi \rangle,$$

where Φ is the Hopf differential,
$$\Phi = (u^* ds_N^2)^{2,0} = \left\langle \frac{\partial u}{\partial z}, \frac{\partial u}{\partial z} \right\rangle_N$$

this is holomorphic because of harmonicity.

Corollary If E has a critical point on $T(S)$, then that point is a conformal harmonic map

Indeed, $\langle \mu, \Phi \rangle = 0 \Leftrightarrow \Phi = 0 \Leftrightarrow u$ is conformal
$\forall \mu$

$\Downarrow$

u is a branched minimal immersion

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WENTWORTH

If N is Kähler, when is a harmonic map $S \to N$ holomorphic? If N has positive curvature, then minimal energy maps are holomorphic. (But in this situation don't have Eells–Sampson existence theorem.)

When is E proper? Not always.

Theorem (Tromba) $N = (S, \sigma_1)$ hyperbolic metric

$u: (S, \sigma) \to (S, \sigma_1)$ homotopic to a diffeo

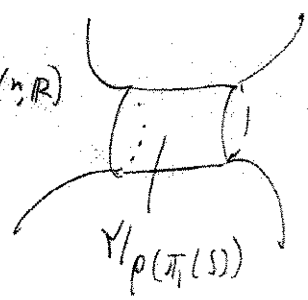
Then E is proper and has a unique minimum

$\Rightarrow$ get Nielsen realization conjecture: any finite subgroup of mapping class group is the automorphism group of a Riemann surface.

Theorem (Bill, Goldman, W.) ρ convex cocompact $\Rightarrow$ E proper

Convex cocompact: N nonpositively curved metric space ρ discrete embedding, & $\exists$ closed convex $Y \subset N$ such that $\rho(\pi_1(Y))$ preserves Y and $Y/\rho(\pi_1(S))$ is compact.

Thm (Labourie) ρ in Hitchin component of $SL(n,\mathbb{R})$ then energy is proper. If $n=3$, minimum is unique.



$$(\theta, t) \mapsto (n + \theta, t e^{n})$$

$\rho: \pi_1 \ell \to \mathrm{Isom}(N)$

What's the use of all this?

Bochner formula: $u : S \to N$

$$\Delta e = |\nabla du|^2 + Ric_S(du, du) - Riem_N(du, du, du, du)$$

If u is conformal and the sectional curvature of N is ≤ -1.
Assume for example $Ric_S = -1$. Then

$$\Delta e \geq -e + e^2$$

at a maximum of e, $\Delta e \leq 0 \Rightarrow e \leq 1$.
hence get uniform bound

<u>Implies compactness!</u>: f.ex. the following is compact:

$$\left\{ \begin{array}{l} \text{convex cocompact} \\ \quad \rho \;\; \pi_1(S) \to \mathrm{Isom}(N), \\ \text{sect'l curv of } N \leq -1, \\ \underbrace{|\rho(\gamma)|}_{\text{translation length.}} \geq \varepsilon > 0 \end{array} \right\} \Big/ \text{mapping class group}$$