Fock Goncharov coordinates $\quad S$ non-closed

$H = \{ \rho : \pi_1(S) = \Gamma \to Sl(n,\mathbb{R}) : \rho$ Hitchin component $\}$

Positive

$H \subset Rep(\Gamma, Sl(n,\mathbb{C}))$

Enlarge the deformations? Fact: ρ positive $\Rightarrow \rho(c_i)$ have positive real eigenvalues distinct

$F_i = \{$ sum of $\lambda_1, \dots, \lambda_i$ eigenspaces $\} \longleftarrow$

$\lambda_1 < \dots < \lambda_n$

$\mathcal{X}_{n,S} = \{ (\rho, F_{c_1}, \dots, F_{c_m}) : \rho : \Gamma \to Sl(n,\mathbb{C})$

$\rho(c_i) \cdot F_{c_i} = F_{c_i} \} / Sl(n,\mathbb{C})$

$\downarrow \pi$

$Rep(\Gamma, Sl(n,\mathbb{C}))$

fiber? $\pi^{-1}(\rho) \simeq \left(Sl(n,\mathbb{C}) / B \right)^{\rho(c_1)} \times \dots \times \left(Sl(n,\mathbb{C}) / B \right)^{\rho(c_m)}$

generically products of Weyl groups ---

Pick a complete hyperbolic metric h on S: $S = \mathbb{H}^2/\Gamma$ finite vol

$\mathcal{F}_\infty(S) = \{ p \in \partial\mathbb{H}^2 : p$ lifts a puncture $\}$

$$\mathcal{X}_{n,S} = \left\{ (\rho, f) : \rho : \Gamma \to SL(n,\mathbb{C}),\ f\ \Gamma\text{-equivariant} \atop f : \mathcal{F}_\infty(S) \to SL(n,\mathbb{C})/B \right\} \Big/ SL(n,\mathbb{C})$$

<u>Thm</u> (FG)

- $\mathcal{X}_{n,S}$ is rational $\exists\ i : (\mathbb{C}^*)^\ell \xrightarrow[\text{open}]{} \mathcal{X}_{n,S}$
- $\mathcal{X}_{n,S}$ is part of a <u>cluster ensemble</u>
- $\mathcal{X}_{n,S}$ Poisson in a natural way
- $i(\mathbb{R}^\ell_{>0}) = H$ (Hitchin component; independent of i)
- $\{x_i\}_{i=1}^{\ell}$ log canonical $\{x_i, x_j\} = a_{ij} x_i x_j$ $a_{ij} \in \mathbb{Z}$

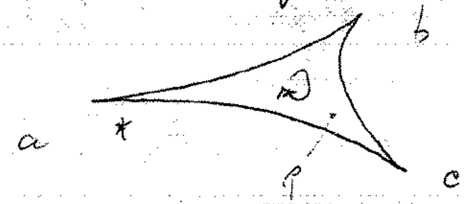
skew-sym.

$\downarrow$

canonical Poisson structure on H

$$\{\log x_i, \log x_j\} = a_{ij}$$

Pick an ideal triangulation of S



assume base point p is in the triangle.

$\exists\,(F_1, F_2, F_3) \in X^3 / SL(n,\mathbb{C})$ (even if the same puncture)

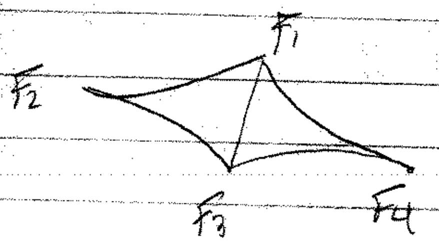
X = flag variety $\quad F_+$ preferred
F_- antipreferred

i.e. $\exists\, g$ s.t. $gF_1 = F_+$
$gF_2 = F_-$
$gF_3 = u \cdot F^+$ $\quad u$ lower triangular w/ 1's on diagonal

$$X^3 / SL(n,\mathbb{C}) \xrightarrow{\cong} N_- / H$$

H diagonal $\quad N_-$ lower unipotent

$$\mathcal{X}_{n,S} \xrightarrow[\text{rat'l map}]{\cancel{\text{birat}}} \prod_{t \text{ triangles}} \left(N_- / H \right)$$



Four flags: $(F_1\ F_2\ F_3)$

$(F_1\ F_3\ F_4)$

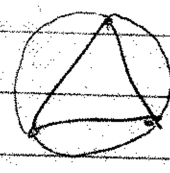
$$X^4/SL(n,\mathbb{C}) \xleftrightarrow[\text{birat'l}]{} X^3/SL(n,\mathbb{C}) \times X^3/SL(n,\mathbb{C}) \times H$$

Conclusion: $\mathcal{X}_{n,S} \xrightarrow[\Phi_e]{} H$

Thm: $\mathcal{X}_{n,S} \xrightarrow[\text{birat'l}]{} \prod_t \left(N_-\right)_{/t} \times \prod_{e \text{ edge}} H$

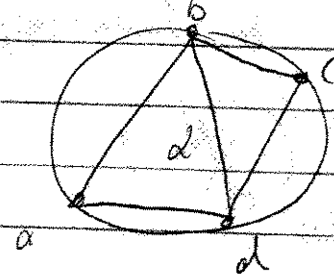
Special case: $SL(3,\mathbb{R})$

$SL(2,\mathbb{R})$



ideal triangle

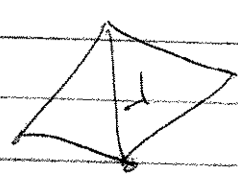
glue:



$\alpha = \beta(a,b,c,d)$

= cross ratio

Conversely:



Now do the same in $gl(3,\mathbb{C})$

(l_2, P_2)

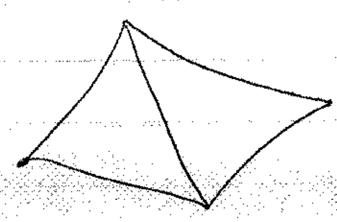
$F_1 = (l_1, P_1)$ (l_3, P_3)

$T(F_1 F_2 F_3) = \dfrac{(e_1|P_2)(e_2|P_3)(e_3|P_1)}{(e_1|P_3)(e_2|P_1)(e_3|P_2)}$

$0 \subset l_1 \subset P_1 \subset \mathbb{C}^3$

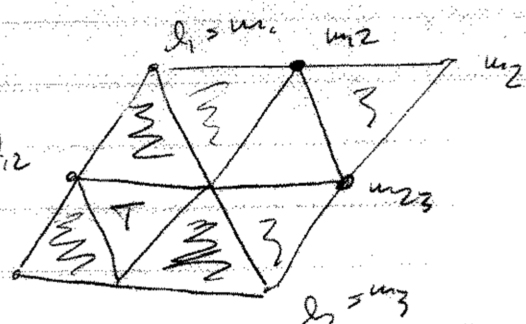
Only invariant ---

glue:



<u>Defn</u>: $F_1 F_2 F_3 \succ 0$ if $T(F_1 F_2 F_3) > 0$

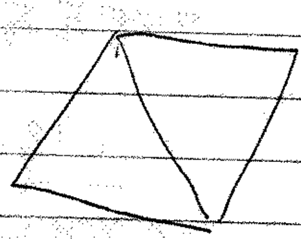
$P_1 \cap P_2 = l_{12}$



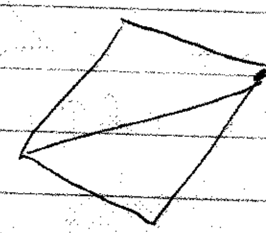
All 4 lines on the same plane

Defn: A config. of 4 flags is > 0

if $X > 0$, $Y > 0$, $\alpha > 0$, $\beta > 0$



positive $\Rightarrow$



is positive