Group 3 Narasimhan-Seshadri Theorem [redacted]

Theorem of Narasimhan - Seshadri:

A holomorphic vector bundle $E \to X$ over a compact Riemann surface X is stable iff it admits a metric with a compatible connection whose curvature is a constant multiple of the identity.

If $\deg E = 0$ ~~then this connection arises~~ and the determinant of E is trivial then this connection arises from an irreducible representation $\pi_1(X) \to SU(n)$ (where n is the rank of E).

① What is a stable holomorphic bundle?
② What is a connection compatible with a metric?
③ How does a connection arise from a representation $\pi_1(X) \to SU(n)$?

① Holomorphic vector bundle: Transition functions are holomorphic maps
$$g_{\alpha\beta} : U_\alpha \times \mathbb{C}^n \to U_\beta \times \mathbb{C}^n$$

Equivalently (Nirenberg - Newlander): There is an operator
$$d_A'' : \Omega^0(E) \to \Omega^{0,1}(E) \quad \text{such that}$$
(i) $d_A''(fs) = (d''f)s + f(d_A''s) \qquad f \in \Omega^0(X),\ s \in \Omega^0(E)$

(ii) $(d_A'')^2 = 0$

$F \subset E$ is a <u>holomorphic sub-bundle</u> $\Leftrightarrow$ $d_A''(\Omega^0(F)) \subset \Omega^{0,1}(F)$

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(E, d_A'') is <u>stable</u> iff $\frac{\deg F}{\operatorname{rank} F} < \frac{\deg E}{\operatorname{rank} E}$ for every holomorphic sub-bundle $F \subset E$.

(2) Connection compatible with a metric:

Put a Hermitian metric h on the fibres of E

Connection: $d_A : \Omega^0(E) \to \Omega^1(E)$

$$d_A(fs) = f\, d_A s + (df)\, s \qquad f \in \Omega^0(X) \quad s \in \Omega^0(E)$$

d_A is compatible with h if

$$\cancel{d\langle s_1, s_2\rangle = \langle d_A s_1, s_2\rangle + \langle}$$

$$d h(s_1, s_2) = h(d_A s_1, s_2) + h(s_1, d_A s_2)$$

Locally: $d_A = d + A$ where A is a matrix-valued 1-form.
"Compatible" means that $A^* = -A$, where $(\cdot)^*$ is Hermitian transpose with respect to h.

Holomorphic structure $\longrightarrow$ compatible connection:

$d_A'' = d'' + A''$ locally.
Let $d_A = d + A'' - (A'')^*$ be the associated connection.

(3) How does a connection arise from a representation $\rho: \pi_1(X) \to SU(n)$?

$\tilde{X}$ universal cover of X

$\pi_1(X)$ acts on $\tilde{X}$ by deck transformations

$\pi_1(X)$ acts on $\mathbb{C}^n$ via the representation $\rho: \pi_1(X) \to SU(n)$

Form the quotient $E_\rho = \tilde{X} \times \mathbb{C}^n / \pi_1(X) =: \tilde{X} \times_\rho \mathbb{C}^n$

Claim: E_ρ has a <u>flat structure</u> (transition functions constant)

$g_{\alpha\beta}: U_\alpha \cap U_\beta \to SU(n)$ constant.

Flat structure $\Rightarrow$ flat connection on E_ρ

$$d_A: \Omega^0(E) \to \Omega^1(E)$$

such that $d_A^2 = 0$ (flat $\Leftrightarrow$ curvature zero)

Transition functions in $SU(n)$ $\Rightarrow$ compatible with the metric coming from $\mathbb{C}^n$ on the fibres

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In summary: Narasimhan - Seshadri says that:

Stable holomorphic bundle with $\deg E = 0$, trivial determinant

$\Updownarrow$

There exists a metric and a connection which arise from an irreducible representation $\rho: \pi_1(X) \to SU(n)$