(A)

entifie ainsi les
V-p de Hecke avec celle de

$$F_p$$

fonction L locale
s'identifie avec celle d'Artin

<u>ue</u>

K_p
u vecteur invariant dans π_p

ecteur propre de T_p.

$$(u) = a_p \cdot u \quad \text{pour } SL(2,\mathbb{R})$$

$$(\pi_p) = a_p \cdot p^{-\frac{k-1}{2}}$$

tion pour $GL(2,\mathbb{R})$

Introduction to harmonic mappings

I] The question

Let f_0, $(M,h) \to (N,g)$
Is there a « best map » f
homotopic to f_0 ?

<u>Best</u> : minimizing some quantity depending on f.

$$(*) \quad Vol(f) = \int_M «|\det Tf|» \, dvol_h$$

$$= « Vol(f^*g) »$$

natural but hard to deal with...

$$= \int_M \sqrt{\det(Tf^* Tf)} \, dvol_h$$

(B)

(**) Energy of f

$\hookrightarrow A \in \mathrm{End}(E,F)$

$$\|A\|^2 \overset{\text{euclidian}}{=} \mathrm{Tr}(AA^*) = \sum_i \|A(u_i)\|^2$$

$$\mathrm{Energy}(f) = E_{g,h}(f)$$

$$= \int_M \|Tf\|^2 \, d\mathrm{vol}_h .$$

marks (0) discrete versions

(i) $M = S^1$

$$\mathrm{Vol}(f) = \mathrm{length}(f) \text{ (ai) } = \int_{S^1} \|\dot f\| dt$$

$$E(f) = \mathrm{energy}(f) \text{ (aii) } = \int_{S^1} \|\dot f\|^2 dt$$

« Critical points » are geodesics $\neq$ (pas necessairement paramétrées)

and parametrised geodesics.

(ii) If $K_N \leq 0$, geodesics in a given homotopy class are minimizers, and more or less unique.

(iii) If $K \leq 0$, the energy is convex:

prop Let $\{f_t\}_{M \to N}^{t \in [0,1]}$ be a family of mappings. Assume that $\forall x \in M$: $t \to f_t(x)$ is a geodesic then

$$t \to E(f_t) \text{ is convex.}$$

- distance between two geodesics is convex

→ infinitesimal version,

+ integrate it

(iv) cultural remark

lot's of discrete versions of that.

II] Harmonic mappings

(B)

Def || f is harmonic
$\Updownarrow$
f is a critical point of the energy; $\forall$ all smooth variations f_t with $f_0 = f$ $\quad \frac{d}{dt} E(f_t) = 0$

Remark: (i) If $K_N \leq 0$ then harmonic mappings are minizers, and unique if $K<0$, more or less unique in general.

(ii) Euler lagrange equation

$$\underset{\text{geodesics}}{\gamma} \leftrightarrow \nabla_{\dot\gamma}\dot\gamma = 0$$

Prop | f is harmonic
$\Updownarrow$
$\mathrm{Tr}(\nabla Tf) = 0$

« $\sum_i [\nabla_{X_i}(Tf)](X_i) = 0$ »

Explanation of the formula:

$$f : M \to N$$

$$T_x f : T_x M \to T_{f(x)} N$$

$$Tf \in \Gamma(\mathrm{End}(TM, f^* TN))$$
$$\parallel$$
$$\Omega^1(M) \otimes f^* TN$$

one form on M with values in $f^* TN$

The bundle $TM^* \otimes f^* TN \to M$ has a connection

$$(\nabla_X A)(\underset{\substack{\cap\\ TM}}{u}) = \nabla^N_X (A(u)) - A(\nabla^M_X u)$$

$$\nabla Tf \in \Gamma(TM^* \otimes TM^* \otimes f^* TN)$$

(just means $(\nabla_{\underset{\substack{\cap\\ TM}}{X}} Tf)(\underset{\substack{\cap\\ TM}}{Y}) \in TN$)

we finally take the trace with respect

(B)

Examples :

(i) $M = \mathbb{R}^N$; $N = \mathbb{R}^P$.

$$Tf = df$$

$$\nabla Tf = D^2 f$$

we end up $\displaystyle\sum_i \frac{\partial^2 f}{\partial x_i \partial x_i} = 0$

(ii) $M = \mathbb{R}$, N

γ harmonic $\nabla_{\dot\gamma}\dot\gamma = 0$

Théorem (Eells-Sampson ≈ 62)

M, N and $K_N \leq 0$ and both compact, then there exist a harmonic mappings homotopic to a given mapping

Crash course on Gromov PP II

vague proof : minimize a nice convex function...

III | Surfaces : harmonicity and holomorphicity (M is a surface)

that if f harmonic, then $f = \mathrm{Re}(h)$ where h is holomorphic

(i) $E_{g,h}(f)$ only depends on the conformal class of g ← metric on S

$$E_{\lambda g, h}(f) = E_{g,h}(f).$$

Indeed : let S be a surface equipped with a complex structure J.

$f : S \to M$.

$$\underbrace{\langle Tf \wedge Tf \circ J \rangle}_{\in\, \Omega^2(S)}(\underset{\in TS}{X}, \underset{\in TS}{Y}) := h(Tf(X), Tf(JY)) - h(Tf(Y), Tf(JX))$$

$$E(f) = \int_S \underbrace{\langle Tf \wedge Tf \circ J \rangle}_{\text{conformal invariant}}.$$

(B)

ii) Rewrite the harmonic mapping equation

Complexify every thing.

$$f^*TN \rightsquigarrow (f^*TN)_{\mathbb{C}}$$

$$\begin{array}{ccc} Tf & \rightsquigarrow & (Tf)_{\mathbb{C}} \\ \cap & & \cap \\ \Omega^1(S)\otimes f^*TN & & \Omega^1(S)\otimes_{\mathbb{C}} (f^*TN)_{\mathbb{C}} \end{array}$$

« $\omega_{\mathbb{C}}(X) = \omega(X) - i\omega(JX)$ ».

Prop | f is harmonic

$\Updownarrow$

$(Tf)_{\mathbb{C}}$ is holomorphic 1-form.

$$\bar\partial Tf_{\mathbb{C}} = 0$$

$$\left[\nabla_{JX}(Tf)_{\mathbb{C}} - i\nabla_X (Tf)_{\mathbb{C}}\right] = 0$$

f harmonic mapping

$\Downarrow$?

holomorphic 1-form, ~~with~~ values in the (holomorphic) bundle $f^*TN_{\mathbb{C}}$ } Higgs field

Hitchin : one can go backwards

—

iii) harmonic mappings and minimal surfaces.

Prop | f harmonic

\+ f conformal $(\|T_sf(u)\| = \lambda_s\|u\|)$

$\Updownarrow$

f is minimal i.e ~~minize~~ is a critical point of Area(f)

(B)

(iii) harmonic mappings and holomorphic differentials.

(Chern-Weil theory)

Let P be a // symmetric tensor on N

(example is $P = h$ itself).

other examples later.

$\rightsquigarrow$ $P_{\mathbb{C}}$ ~ a // symmetric tensor on $(TN)_{\mathbb{C}}$.

prop | If f is harmonic then

$$P_{\mathbb{C}}(Tf_{\mathbb{C}}, \dots, Tf_{\mathbb{C}}) \in (T^*S)^{\otimes_{\mathbb{C}} n}$$

is holomorphic differential.

ex: $P = h$, the corresponding object is a quadratic differential the Hopf differential $H(f)$

ob $H(f) = 0 \Leftrightarrow f$ is conformal.

IV | Back to representations

$$\rho : \pi_1(S) \longrightarrow G = SL(n, \mathbb{R})$$

$G = \mathrm{Iso}(X)$; $SL(2, \mathbb{R})$

f $\tilde{S} \longrightarrow \tilde{N} = SL(n,\mathbb{R})/SO(n,\mathbb{R})$

space of metrics on $\mathbb{R}^n$

$n = 2$ hyperbolic space

f which are ρ-equivariant.

we want to generalize to case where

$N/\rho(\pi_1(S))$ is not compact

other interpretation

$$N \to (N \times \tilde{S})/\pi_1(S)$$
$$\downarrow$$
$$S$$

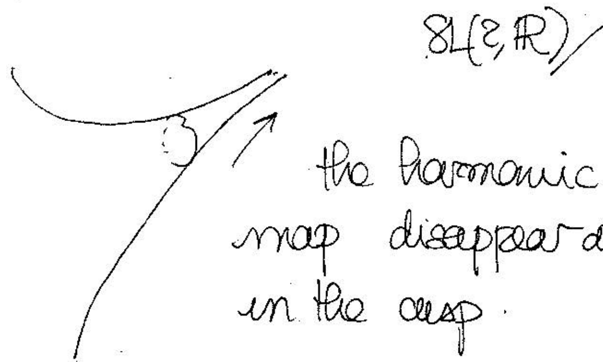
representations $SL(n,\mathbb{R})$ as sections

flat metric on

(B)

⚠ since $N/\rho(\pi_1(S))$ is not compact, we need to avoid some phenomenon

ex :



the harmonic map disappear at ∞^{ty} in the cusp.

Theorem (Corlette, Donaldson, L)

If $\rho(\pi_1(S))$ is good
(ex : $\overline{\rho(\pi_1(S))}^{Z} = SL(n,\mathbb{R})$
or $\rho(\pi_1(S))$ does not fix a point at ∞^{ty})
Then there exist a harmonic ρ-equivariant mapping

Hitchin fibration :

~ nice // tensor on N

$$N = SL(n,\mathbb{R})/SO(n,\mathbb{R}) = \{\text{metrics}\}$$

$$T_x N \simeq \{\text{Antisymmetric matrices} / g\}$$

$$P_n(A,\dots,A) = Tr(\underbrace{A \cdots A}_{n}).$$

$\mathrm{Rep}(\pi_1(S), SL(n,\mathbb{R}))\{\text{representations of } \pi_1(S) \to G\}$ $= \mathcal{H}(SL(n,\mathbb{R}))$

$$\downarrow$$

$$\{\text{harmonic mappings}\}$$

$$\downarrow$$

$$\{(P_2, \dots, P_n)\}$$

holomorphic differentials

$$= H^0(K^2) \oplus \cdots \oplus H^0(K^n)$$

$\mathcal{H}$

Thm (Hitchin 92)

There is a connected component such that $\mathcal{H}$ is a bijection

(B)

V] The Energy functional on Teichmüller space:

Let $\rho : \pi_1(S) \hookrightarrow G = SL(n, \mathbb{R})$
$SL(2, \mathbb{R})$.

Let J be a complex structure on S

$$E_\rho(J) = \inf_{f \in \Gamma_\rho(S)} \left(E_J(f) \right)$$

Then $E(J)$ only depends on the isotopy class of J.

Thm (Sacks-Uhlenbeck; Schoen-Yau)

J is a critical point of e

$\Updownarrow$

the corresponding mapping is conformal

Thm (L) ρ is Hitchin, then e_ρ is a proper mapping

(Schoen, $n = 2$).