Introduction to harmonic mappings

I. The question

Is there a "best map" \( f \) homotopic to \( f_0 \) ?

**Best**: minimizing some quantity depending on \( f \)

\[ (*) \quad \text{Vol}(f) = \int_M \left| \det T_f \right| \, d\text{vol}_K \]

\[ \leq \text{Vol}(f^*g) \]

natural but hard to deal with

\[ = \int_M \sqrt{\det(T_f^*T_f)} \, d\text{vol}_K \]
Energy of $f$

$A \in \text{End}(E,F)$

Euclidean

$\|A\|^2 = \text{Tr}(AA^*) = \sum_i |A(u_i)|^2$

Energy $(f) = E_{g,h}(f)$

$= \int_M \|f_t\|^2 \text{dvol}_h$

Marks (i) discrete version

(ii) $M = S^1$

$\text{vol}(f) = \text{length}(f)(ai)$

$E(f) = \text{energy}(f)(ai) = \int_{S^1} \|f_t\|^2 dt$

« Critical points » are geodesics, and parametrised geodesics.

(iii) If $K_N \leq 0$, geodesics in a given homotopy class are minimizers, and more or less unique.
II Harmonic mappings

Def

\[ f \text{ is harmonic} \]

\[ \downarrow \]

\[ f \text{ is a critical point of the energy, } \forall \text{ all smooth variations } \]

\[ f_t \text{ with } f_0 = f \text{ and } \frac{d}{dt} E(f_t) = 0. \]

Remark

(i) If \( K_N \leq 0 \)

then harmonic mappings are minizers, and

unique if \( K < 0 \), more or less unique in general.

(ii) Euler Lagrange equation

\[ f \quad \Rightarrow \quad \nabla_{g} f = 0 \]

Prop

\[ f \text{ is harmonic} \]

\[ \downarrow \]

\[ \text{Tr}(\nabla Tf) = 0 \]

\[ \sum_i \left[ \nabla_{x_i} (f) \right] (X_i) = 0 \]

Explanation of the formula:

\[ f : M \to N \]

\[ T_x f : T_x M \to T_{f(x)} N \]

\[ Tf \in \Gamma\left( \text{End}(TM, f^*TN) \right) \]

\[ \Sigma^m(M) \otimes f^*TN \]

one form on \( M \) with values in \( f^*TN \)

The bundle \( TM^* \otimes f^*TN \to M \)

has a connection

\[ (\nabla_X A)(u) = \nabla_X^N (A(u)) - A(\nabla_X^M A^TM) \]

\[ \nabla Tf \in \Gamma\left( TM^* \otimes TM^* \otimes f^*TN \right) \]

(just means \( (\nabla_X f)(Y) \in TN \))

We finally take the trace with respect
Example:

(a) \( M = \mathbb{R}^n \), \( N = \mathbb{R}^p \)

\[ T_f = df \]
\[ \nabla T_f = D_f \]

we end up

\[ \sum_i \frac{\partial^2 f}{\partial x_i^2} = 0 \]

(b) \( M = \mathbb{R}, \quad N \)

\( \gamma \) harmonic \( \nabla \gamma \gamma = 0 \)

Théorème (Eells-Sampson \( \approx 62 \))

\( M, N \) and \( K_N < 0 \) and both compact, then there exist a harmonic mappings homotopic to a given mapping.

\[ \langle T_f \wedge T_f \rangle(x, y) := h(T_f(x), T_f(y)) \]

\[ J^2(S) \]

\[ E(f) = \int_S \langle T_f \wedge T_f \rangle \]

\( f : S \to M \)

III | Surface: harmonicity and holomorphicity (\( M \) is a surface)

that if \( f \) harmonic, then \( f = \text{Re}(h) \) where \( h \) is holomorphic

(1) \( E g, h \) \( (f) \) only depend on the conformal class of \( g \) metric on \( S \)

\[ E_g, h \ (f) = E_g, h \ (f). \]

Indeed: Let \( S \) be a surface equipped with a complex structure \( J \).

\[ E(f) = \int_S \langle T_f \wedge T_f \rangle \]

conformal invariant.
\[
\Delta (f) = -i \nabla^2 (f) = 0
\]

Prop: if \( f \) is harmonic, \( f \) is holomorphic 1-form.

\[
\nabla (f) = \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} = 0
\]

\[
\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}
\]

Substitute every instance of \( f \) with \( \bar{f} \).

Prop: holomorphic 1-form and minimal surface

\[
\text{Area}(f) + f \text{ harmonic} \Rightarrow f \text{ minimal surface}
\]

\[
\text{Hitchin: one can go backwards}
\]

Prop: holomorphic 1-form with values in \( \mathbb{C} \).

\[
\text{Hitchin: one can go backwards}
\]
(ii) Harmonic mappings and holomorphic differentials.

(Weil-Weil theory)

Let $P$ be a $\parallel$ symmetric tensor on $\mathbb{R}$.

Example is $P = \delta$ itself.

Other examples:

$P_c \sim \delta \parallel$ symmetric tensor on $(TN)_c$.

Proposition: If $f$ is harmonic then $P_c(Tf_c, \ldots, Tf_c) \in (T^*S)^\oplus$ is holomorphic differential.

Let $P = \delta$, the corresponding object is a quadratic differential.

The Hert differential $H(f)$.

\[ H(f) = 0 \iff f \text{ is conformal.} \]

IV Back to representations:

\[ G = \text{Iso}(\mathbb{R}^n) \rightarrow G = SU(n, \mathbb{R}) \]

\[ \pi_1(S) \rightarrow G = SL(n, \mathbb{R}) \]

\[ f : \hat{S} \rightarrow N = SU(n, \mathbb{R})/SO(n, \mathbb{R}) \]

$f$ which are $\pi$-equivariant.

We want to generalize to case where $N/\pi_1(S)$ is not compact.

Other interpretation:

\[ N \rightarrow (N \times \hat{S})/\pi_1(S) \]

As sections flat metric on $S$. Representation of $SU(n, \mathbb{R})$. Representation flat metric on.
\[ \Delta \text{ since } N/\rho(\pi_1(S)) \text{ is not compact, we need to avoid some phenomenon} \]

\[ \text{ex: } \xrightarrow{\text{the harmonic map disappears at } \infty} \]

\[ \text{in the cusp} \]

**Theorem (Corlette, Donaldson, L)**

- If \( \rho(\pi_1(S)) \) is good
  - \( \text{ex: } \rho(\pi_1(S)) = SL(n, \mathbb{R}) \)
  - or \( \rho(\pi_1(S)) \) does not fix a point at \( \infty \)

Then there exist a harmonic \( \rho \)-equivariant mapping

**Hitchin fibration**

\[ \frac{1}{2} \text{ since } \text{ tensor on } N \]

\[ N = SL(n, \mathbb{R})/SO(n, \mathbb{R}) = \text{ of metrics } \]

\[ T_N \cong \text{ of antisymmetric matrices } / g \]

\[ P(A, \ldots, A) = \text{Tr}(A \ldots A) \]

\[ \text{Rep}(\pi_1(S)) SL(n, \mathbb{R}) \{ \text{representations of } \pi_1(S) \to G \} \]

\[ \text{if } \]

\[ \text{Holomorphic differentials} \]

\[ H^0(K^2) \oplus \cdots \oplus H^0(K^n) \]

**Thm (Hitchin 92)**

There is a connected component such that \( \text{SL} \) is a Frobenius
The Energy functional on Teichmüller space:

Let \( \rho : \pi_1(S) \to G = SL(m, \mathbb{R}) \cup SL(2, \mathbb{R}) \).

Let \( J \) be a complex structure on \( S \).

\[ E_\rho(J) = \inf_{f \in \pi_1(S)} \left( E_\rho(f) \right) \]

Then \( E(J) \) only depends on the homotopy class of \( J \).

Thm (Sacks-Uhlenbeck; Schoen-Yau):

\( J \) is a critical point of \( \rho \)

\[ \uparrow \]

the corresponding mapping is conformal.