 Marc Burger 20/03/2007

Maximal Representations

Σ_g oriented, compact, $(g \geq 2)$.

$G = \mathrm{Is}(X)^\circ$, X = hermitian symmetric n.c.t.

Associate an invariant to $\rho : \pi_1(\Sigma_g) \to G$.

<u>Examples</u> (1) $X = \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$

$$G = PU(1,1)$$

(2) $X = \mathcal{D}_{p,q} := \{ Z \in M_{p,q}(\mathbb{C}) : \mathrm{Id} - Z^* Z \gg 0 \}$

$G = SU(p,q)$ (forme $\sum_{i=1}^{p} |z_i|^2 - \sum |z_{p+j}|^2$)

with action by homographies:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} Z = (AZ + B)(CZ + D)^{-1}$$

In general: X = complex manifold with Riemannian metric (complete) such that $\forall x \in X$ $\exists\, s_x \in \mathrm{Aut}(X)$ with $T_x(s_x) = -\mathrm{Id}$.

$G = Is(X)^\circ$; let $(\ ,\ )_x$ = R.m.

$J_x \in End(T_x X)$ c.s.

<u>Then:</u> $\omega_x(X, Y) := g_x(X, JY)$

defines a G-invariant 2-form on X:

$$\omega_X \in \Omega^2(X)^G.$$

<u>Fact:</u> $X = \mathbb{C}^n \times \underbrace{X_c}_{\text{sect.} \geq 0} \times \underbrace{X_{nc}}_{\text{s.k.} \leq 0}$

~~Norm. Metric.~~

<u>Assume:</u> X is of n.c. type.

<u>Lemma</u> (Cartan) $d\omega = 0$.

[~~Proof: ... is G-inv. ...~~

Given now $\rho : \pi_1(\Sigma_1) \to G$

$\exists\ f : \tilde{\Sigma}_1 \to X$ smooth, ρ-equivariant.

[Indeed look at $(\tilde{\Sigma}_1 \times_\rho X) \to \Sigma_1$ choose local (sections, partition of unity + glue together

$$\tau(\rho) := \frac{1}{2\pi} \int_{\Sigma} f^{*}(\omega_X)$$

<u>Theorem.</u> Assume that min. hol. sect. $\kappa = -1$.

Then (1) $|\tau(\rho)| \leq |\chi(\Sigma)| \cdot \mathrm{Rank}\, X$

(2) $\rho \longrightarrow \tau(\rho)$ is constant.

(3) $\tau(\rho) \in \ell_X \cdot \mathbb{Z}$ $\quad (\ell_X \in \mathbb{Q}^{\times})$.

<u>Ex:</u> $\rho : \pi_1(\Sigma) \to PU(1,1)$ hyperbolization

<u>Then:</u> $\tau(\rho) = 2g - 2$ (Gauss – Bonnet).

<u>Thm.</u> (Goldman) . . . . .

<u>def:</u> ρ is <u>maximal</u> if $\tau(\rho) = |\chi(\Sigma)| \cdot \mathrm{Rank}\, X$

<u>Examples:</u> $\rho_1, \ldots, \rho_r : \pi_1(\Sigma) \longrightarrow SU(1,1)$

are hyperbolizations: $\pi_1(\Sigma) \longrightarrow SU(1,1)^r$

is maximal.

<u>Theorem</u> (B-I-W) If $\rho : \pi_1(\Sigma_1) \to G$ is maximal then it is discrete ~~with~~ and injective.

<u>Cohomologies.</u>

1.) <u>Continuous cohomology</u>

L = loc. compact group.

$C(L^k) = \{ f : L^k \to \mathbb{R},$ continuous $\}$

with L-action.

$d : C(L^k) \to C(L^{k+1})$

$$df(g_0, \dots, g_k) = \sum_{j=0}^{k} (-1)^j f(g_0, \dots, \hat{g}_j, \dots, g_k)$$

<u>Then:</u> $H^\bullet_c(L, \mathbb{R})$ is the cohomology of

$$0 \to C(L, \mathbb{R})^L \to C(L^2, \mathbb{R})^L \to \cdots$$

<u>e.g</u> $H^0(L, \mathbb{R}) \cong \mathbb{R}$.

Important feature: $\pi : L_1 \to L_2$ is a continuous homomorphism then

$$C(L_2^{k+1}, \mathbb{R}) \xrightarrow{\pi^{(k)}} C(L_1^{k+1}, \mathbb{R})$$

and hence $\pi_*^{(\cdot)} : H^\bullet(L_2, \mathbb{R}) \to H^\bullet(L_1, \mathbb{R})$.

Now let X = Sym. of n.c. type and

$$G = \mathrm{Is}(X)^\circ .$$

<u>Thm.</u> (Van-Est) $H_c^\bullet(G, \mathbb{R}) \simeq \Omega^\bullet(X)^G$

In our case: X hermitian symmetric and $\omega_X \in \Omega^2(X)^G$, let

$$c_X(g_1, g_2, g_3) = \frac{1}{2\pi} \int_{\Delta(g_1 x, g_2 x, g_3 x)} \omega_X$$

Then c_X is G-invariant cocycle and hence defines $\kappa_X \in H_c^2(G, \mathbb{R})$.

Now let $\rho : \pi_1(\Sigma) \to G$ be a homom.

Then $\rho^*(\kappa_X) \in H^2(\pi_1(\Sigma), \mathbb{R}) \simeq H^2_{sing}(\Sigma, \mathbb{R})$

<u>Fact:</u> $\tau(\rho) = \langle \rho^*(\kappa_X), [\Sigma] \rangle$.

2. Continuous bounded cohomology

Start again with L general and

$$C_b(L^k, \mathbb{R}) = \{ f \in C(L^k, \mathbb{R}) : \|f\|_\infty < +\infty \}.$$

Then $C_b(L^\cdot, \mathbb{R}) \subset C(L^\cdot, \mathbb{R})$

and $H^\cdot_{bc}(L, \mathbb{R})$ is the cohomology of

$$0 \to C_b(L, \mathbb{R})^L \to \cdots \to C_b(L^k, \mathbb{R}) \to \cdots.$$

As before any cont. $\pi : L_1 \to L_2$ induces

$$\pi^\cdot_{*,b} : H^\cdot_{bc}(L_2, \mathbb{R}) \to H^\cdot_{bc}(L_1, \mathbb{R}).$$

<u>Special Feature:</u> (1)

$$\begin{array}{ccc} \operatorname{Ker} d & \subset & C_b(L^{k+1}, \mathbb{R}) \\ \cup & & \\ \operatorname{Im} d & \subset & C_b(L^{k+1}, \mathbb{R}) \end{array}$$

so that $H^\cdot_{bc}(L, \mathbb{R})$ inherits a semi-norm $\| \ \|$.

(2) Comparison map: $H^\bullet_{bc}(L,\mathbb{R}) \to H^\bullet_c(L,\mathbb{R})$
contains lots of infor about L.

Example: ~~What is Ker~~

(1) $H^0_{bc}(L,\mathbb{R}) \simeq \mathbb{R}$; $H^1_{bc}(L,\mathbb{R}) = 0$.

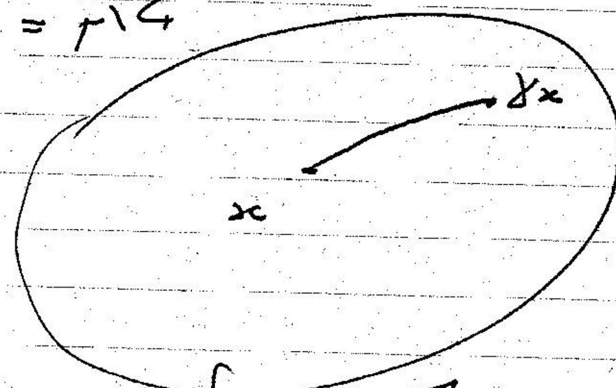
(2) $\mathrm{Ker}\left(H^2_{bc}(L,\mathbb{R}) \to H^2_c(L,\mathbb{R})\right)$

$$\simeq \; QH(L,\mathbb{R}) \big/ C_h(L,\mathbb{R})$$

where $f \in QH(L,\mathbb{R})$ means that

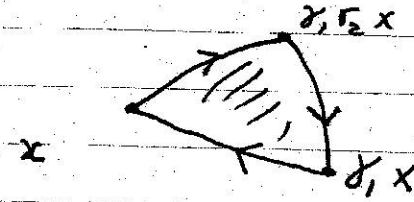
$$\sup_{a,b} |f(ab) - f(a) - f(b)| < +\infty.$$

For ex. Σ = hyp. surface and $\alpha \in \Omega^1_{\text{bounded}}(\tilde{\Sigma})^\Gamma$
$\Gamma = \pi_1 \Sigma$



let $f(\gamma) := \int_{[x,\gamma x]} \alpha$. Then:

$$f(\gamma_1\gamma_2) - f(\gamma_1) - f(\gamma_2) = \int d\alpha$$



and hence

$$\sup_{\gamma_1,\gamma_2} \left| f(\gamma_1\gamma_2) - f(\gamma_1) - f(\gamma_2) \right| \le \|d\alpha\|_\infty \cdot \pi .$$

Assume: Area$(\Sigma) < +\infty$. Then:

<u>Thm.</u> $H^2_{bc}(\Gamma, \mathbb{R})$ is infinite dimensional Banach space.

Back to X hermitian symmetric

<u>Thm</u> (C–ϕ; D–T) $\left| \frac{1}{2\pi} \int_\Delta \omega_X \right| \le \pi \cdot \mathrm{Rank}\, X$

~~Cor.~~

Thus C_X is bounded and hence

defines $\kappa_{b,X} \in H^2_{bc}(G, \mathbb{R})$.

What about $H^2_{bc}(G,\mathbb{R})$, G conn. s.s. l.c.

Like $G = SL(n,\mathbb{R})$

<u>Thm</u>. $H^2_{bc}(G,\mathbb{R}) \hookrightarrow H^2_c(G,\mathbb{R})$.

[<u>Inject.</u>: let $f: G \to \mathbb{R}$ be a cont. quasihom. Let $g^k n g^{-k} \to e$. Then

$|f(n) - f(g^k n g^{-k})| \leq 3C'$

and $\lim f(g^k n g^{-k}) \to f(e)$.

Now write $G = N_1 \cdot N_2 \cdots N_r$

unipotent subgroups].

In fact the above is an <u>isom</u>.

X herm. sym:

<u>Thm</u>. $\left| \frac{1}{2\pi} \int_\Delta \omega_X \right| \leq \pi \cdot \operatorname{Rank} X$

$\Rightarrow c_X$ defines $\kappa_{b,X} \in H^2_{bc}(G,\mathbb{R})$.

<u>Cor</u>: $\|\kappa_{b,X}\| = \pi \cdot \operatorname{Rank} X$.

Thus if now $\Gamma = \pi_1(\Sigma)$, Σ with or without boundary, one has the bounded class

$$\rho^*(\kappa_{b,X}) \in H^2_b(\Gamma, \mathbb{R})$$

and the latter space is ∞-dimensional in all cases.

Also: $\|\rho^*(\kappa_{b,X})\| \leq \|\kappa_{b,X}\| = \pi \cdot \mathrm{Rank}\, X$.

In fact a remarkable feature:

<u>Thm.</u> (Gromov) $H^{\bullet}_{bc}(\Sigma, \mathbb{R}) \cong H^{\bullet}_b(\pi_1(\Sigma), \mathbb{R})$

<u>Corollary</u> (Milnor–Wood).