Parabolic Higgs bundles and representations

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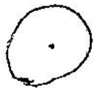
Unitary case

disc  $\pi_1(D\setminus\{0\}) \to U_n$

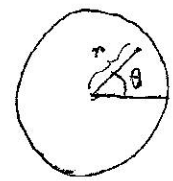
$$D = d + A\,d\theta \qquad A = i\begin{pmatrix}\alpha_1 & & 0\\ & \ddots & \\ 0 & & \alpha_n\end{pmatrix}$$

monodromy $= e^{-2\pi A}$ choice $0\le\alpha_1\le\dots\le\alpha_n<1$

Holomorphic bundle: $D^{0,1} = \bar\partial - \frac{1}{2}\begin{pmatrix}\alpha_1 & & 0\\ & \ddots & \\ 0 & & \alpha_n\end{pmatrix}\frac{d\bar z}{\bar z}$ ← holomorphic over $D\setminus\{0\}$

orthonormal basis (e_i), holomorphic sections $(s_i) = r^{\alpha_i} e_i$

$$\bar\partial r^{\alpha_i} = \frac{\alpha_i}{2} r^{\alpha_i} \frac{d\bar z}{\bar z}$$



(We can multiply by z^{n_i}.)

The sections (s_i) define an extension of the bundle over 0.

More intrinsically: ~~Ex~~ the holom. sections of E ~~outside~~ near 0 are those of $E|_{D\setminus\{0\}}$ which remain bounded.

Automorphisms of E: automorphism of $E|_{D\setminus\{0\}}$ which remain bounded

In basis (s_i), $g = (g_{ij})$ g bounded $\iff$ $r^{\alpha_i-\alpha_j} g_{ij}$ bounded

holomorphic outside 0

$\iff$ g holomorphic over D and if $\alpha_i < \alpha_j$ then $g_{ij}(0) = 0$.

Hence $g(0) \in \begin{pmatrix}\boxed{*} & & 0\\ & \boxed{*} & \\ * & & \boxed{*}\end{pmatrix}$ parabolic subgroup generated by the (α_i).

Parabolic holomorphic bundle

Σ riemann surface, marked points $x_i \in \Sigma$

def A parabolic holomorphic bundle is:

- a holomorphic bundle $E \to \Sigma$
- at each x_i, a parabolic subgroup $P_i \subset GL(E_{x_i})$, + a compatible set of weights (= strictly dominant character of P_i).

The curvature of $D = d + A\,d\theta$ is $F(D) = 2\pi A\,\delta_0\,dx \wedge dy$

If this is the behavior of a flat connection on E, then:

$$\deg E = \frac{i}{2\pi}\int \operatorname{Tr}(F(D)) = -\sum \alpha_i$$

<u>def</u> $\text{par-deg}\, E = \deg E + \sum \alpha_i$

<u>Theorem</u> (Mehta - Seshadri)

$$\left\{\begin{array}{l}\text{irreducible repr.}\\ \pi_1(\Sigma \setminus \{x_i\}) \to U_n\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{stable/}\overset{\text{holom}}{\text{parabolic}}\\ \text{bundles of par-deg} = 0\end{array}\right\}$$

at each x_i, monodromy $e^{-2\pi i\begin{pmatrix}\alpha_1 & 0\\ 0 & \ddots\ \alpha_n\end{pmatrix}} \longleftarrow \begin{pmatrix}\alpha_1 & 0\\ 0 & \ddots\ \alpha_n\end{pmatrix}$

<u>Other groups</u> - same theorem

- some problem with the logarithm

(<u>example</u>, for SU_2 $\begin{pmatrix}-\frac{1}{2} & 0\\ 0 & 1/2\end{pmatrix}$ $\begin{pmatrix}\alpha & 0\\ 0 & -\alpha\end{pmatrix}$ $0 \le \alpha < \frac{1}{2}$

<u>Complex case (Simpson 1990)</u>

<u>Local model</u>: $D = d + A\,d\theta$ $A \in \mathfrak{gl}_n\mathbb{C}$. suppose A semisimple.

take $n=1$ for simplicity.

$$D = d + i(\alpha + i\beta)\,d\theta$$
$$= \underbrace{d + i\alpha\,d\theta}_{\text{unitary}} - \underbrace{\beta\,d\theta}_{\text{hermitian}}$$

meromorphic, simple pole

$\longrightarrow$ same parabolic bundle as before + Higgs field $\varphi = (-\beta\,d\theta)^{1,0}$

$$= \frac{i}{2}\beta\frac{dz}{z}.$$

<u>Correspondence</u>

	local systems	Higgs
monodromy	$e^{-2\pi i\alpha + 2\pi\beta}$	parabolic weight α
		eigenvalue of Res_0: imaginary part $\mu = \beta/2$
	filtered local system weight γ $\longleftarrow$	real part $\lambda = -\frac{\gamma}{2}$

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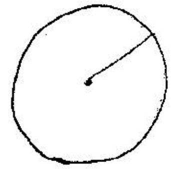
$h = |z|^{2\gamma}$ is also a harmonic metric.

Then $\varphi = \left(\frac{1}{2}\beta - \frac{\gamma}{2}\right)\frac{dz}{z}$

Geometrically



r^{γ_i}: the different growth order defines a flag in E_0.

$F_\alpha(E_0) = \{$sections growing at most as $r^\alpha\}$

Need to have picked a ray.

this extends to other groups. $h(s,s) = O(r^{2\gamma})$.

Theorem (Simpson) $\left\{\begin{array}{l} \text{~~stable~~ Irreducible representations} \\ \pi_1(\Sigma \setminus x_i) \to GL(n,\mathbb{C}) \\ \text{semisimple monodromy} \\ \text{around punctures} \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} \text{stable parabolic} \\ \text{Higgs bundles par-deg} = 0 \\ \mathrm{Re}\,\lambda = 0 \end{array}\right\}$

Parabolic Higgs bundle: parabolic bundle E, $D = \{x_1 \dots x_n\}$.

$$\varphi \in H^0(\underbrace{\Omega^1 \otimes \mathcal{O}(D)}_{\Omega^1_{\log D}} \otimes \mathfrak{g}_E) \quad \text{s.t.} \quad \mathrm{Res}_{x_i}\varphi \in \mathfrak{p}_i \subset (\mathfrak{g}_E)_{x_i}$$

parabolic subalg corresponding to P_i

For filtered local systems have a stability condition, and φ is arbitrary.

The nilpotent part Y of $\mathrm{res}_{x_i}\varphi$ corresponds to the unipotent part e^Y of the monodromy at x_i.

Moduli spaces

hyperkähler: fixing the conjugacy class of monodromy at punctures.
(+ filtered structure). (~~Higgs side~~ local system)

On the loc. system side, fix parabolic structure at x_i

at x_i, $\mathfrak{p}_i = \mathfrak{l}_i \oplus \mathfrak{n}_i$

When projecting $\pi_{\mathfrak{l}_i}(\mathrm{Res}_{x_i}\varphi_i) \in \mathfrak{l}_i$ is ~~fixed~~ in a fixed orbit in the Levi.

Example: Unitary monodromy at puncture $\Longleftrightarrow$ $\mathrm{Res}_{x_i}\varphi \in \mathfrak{n}_i$ — then $\mathfrak{n}_i = 0$, so $\mathfrak{p}_i = \mathfrak{l}_i$

unipotent " " " $\Longleftrightarrow$ trivial parabolic structure, fix nilpotent adjoint orbit to which $\mathrm{Res}_{x_i}\varphi$ belongs

Real representations

G real noncpct, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ Cartan decomposition

Higgs bdl side: $H^{\mathbb{C}}$-principal bdl E — parabolic $P < H^{\mathbb{C}}$

$\varphi \in H^0(K \otimes E(\mathfrak{m}^{\mathbb{C}}))$

extend to $G^{\mathbb{C}}$

$\varphi \in H^0(K_{\log D} \otimes E(\mathfrak{m}^{\mathbb{C}}))$

$\mathrm{Res}_{x_i} \varphi \in \mathfrak{p}_i$

Repr $\{ \pi_1(\Sigma \setminus \{x_i\}) \to G \} \longleftrightarrow \{ G\text{-parabolic Higgs bundle} \}$

moduli space? fix adjoint orbit in $\mathfrak{g}$

nilpotent case Kostant–Sekiguchi–Vergne correspondence.

$$\left\{ \begin{array}{c} \text{nilpotent } G \text{ orbits} \\ \text{in } \mathfrak{g} \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{c} \text{nilpotent } H^{\mathbb{C}}\text{-orbits} \\ \text{in } \mathfrak{m}^{\mathbb{C}} \end{array} \right\}$$

Vergne proved that ~~f~~ each orbit on the left is diffeomorphic to the orbit on the right.

Can generalize to other orbits.

Toledo invariant G/H Herm. symm. $\Rightarrow Z(H) = U_1$

then $E \rightsquigarrow$ line bundle

Toledo = parabolic degree of this line bundle

Hitchin component $SL(2n, \mathbb{R})$ choose L s.t. $L^2 = K \otimes \mathcal{O}(D)$ — assuming $\deg D$ even

bundle: $L^{-2n+1} + L^{-2n+3} + \dots + L^{-1} + L + \dots + L^{n-1}$ $SO(2n, \mathbb{C})$ bundle

$$\varphi = \begin{pmatrix} 0 & 1 & & 0 \\ a_2 & \ddots & \ddots & \\ \vdots & 0 & \ddots & 1 \\ a_n & \cdots & a_2 & 0 \end{pmatrix}$$

$a_d \in H^0(K^d \otimes \mathcal{O}(dD))$

if $a_d \in H^0(K^d \otimes \mathcal{O}((d-1)D)) \to$ unipotent monodromy

if $a_d \in H^0(K^d \otimes \mathcal{O}(dD)) \to$ split

adding a parabolic structure $\to$ unitary monodromy at x_i