Lifting representations to the mapping class group

\( B \) closed oriented surface of genus \( g(g(B) > 2) \).

\[ \rho : \pi_1(B) \rightarrow \text{Sp}(2h, \mathbb{R}) \]

\( \Sigma(r) = \Sigma(Ep) \), \( Ep \) flat \( \text{Sp}(2h, \mathbb{R}) \) bundle
  with a choice of (non flat, in general) \( \alpha \) struct.

Toledo invariant of \( \rho = \langle \Sigma(Ep), B \rangle \).

Consider \( H^1(B; Ep) \), this carries a symmetric bilinear form

\[ \begin{array}{c}
  \alpha, \beta \in H^1(B; Ep) \\
  \mapsto \langle \alpha \cup \beta, [B] \rangle
\end{array} \]

**Fact** \[ |\text{Sign}(I)| = 4 |\langle c_1(Ep), [B] \rangle| \]

U. Meyer (1972)

\[ |\alpha(I)| \]

\( \alpha \) group with generators \( x, y \)

If \( a, b \in \text{Sp}(2h, \mathbb{R}) \) consider \( \rho : \mathbb{Z}_2 \rightarrow \text{Sp}(2h, \mathbb{R}) \)

\[ \begin{array}{c}
  x \mapsto a \\
  y \mapsto b
\end{array} \]

\[ F_2 = \pi_1(\mathcal{O}) \]

The Meyer signature cocycle on \( \text{Sp}(2h, \mathbb{R}) \) can be defined as \( \sigma(a, b) = \sigma(I) \) where \( I \) is defined on \( Ep \rightarrow \mathcal{O} \)

Consider lifting problem:

mapping class group of \( \mathcal{O} \)

\[ \pi_1(\mathcal{O}) \rightarrow \text{Sp}(2h, \mathbb{C}) \]

\[ \text{Sp}(2h, \mathbb{C}) \rightarrow \text{Sp}(2h, \mathbb{R}) \]

Other possibilities:

[Diagram illustrating relationships between groups and mappings]

\[ \text{Symplectic}(F_2, \omega) \]

\[ \text{Diff}^+(F_2) \]
In this case, composing \( \tilde{\varphi} \) with the map \( \tilde{\varphi} \rightarrow \text{Sp}(12,\mathbb{R}) \) we obtain \( \varphi \) whose \( \varphi_0 \) is the fiberwise \( \mathcal{F}_1 \) of the fibration over \( \mathbb{R} \) the spectral sequence of \( X \) degenerates at \( E' \) and \[
abla^2(X) = H^2(B) \otimes H^2(F) \otimes H^2(B \otimes H^2(F))
\]
the hyperboloid space.

This implies \( \sigma(X) = \sigma(I) \), with \( I \) as above.

For \( \text{Sp}(2h,\mathbb{B}) \) we have \( |\langle G_{(p)}, [B_3] \rangle| \leq (g-1)h \). (proved by Thrall; Demir-Toldeo)

For \( X \rightarrow B \) as above, \( |\sigma(X)| \leq \beta_2(X) = X(X) - 2 + 2 \cdot \eta(X) \)

P.D. \( = 4(g-1)(h-1) - 2 + 9g + 4h \)
\( = 4g(h+2) \)

Using coverings, this implies (\( \sigma \) is multiplicative under unramified covering) \( |\sigma(X)| \leq 4(g-1)h \), which is the same as

Example \( h=1 \). A fibration \( T^2 \rightarrow X \) has \( \sigma(X) = 0 \), some \( X \) covers itself (fibers: \( t^2 \)).

Hence, if \( \varphi : \pi_1(B) \rightarrow \text{SL}(2;\mathbb{R}) \) can be lifted to \( \text{SL}(2;\mathbb{Z}) \)
then \( \varphi(\rho) = 0 \).

Furthermore, there is a section \( \rho \) if \( \varphi \) lifts \( \rho : \pi_1(B) \rightarrow \text{SL}(2;\mathbb{Z}) \)
then set fibration with \( T^2 \rightarrow X \) \( \gamma = \text{SL}(2;\mathbb{Z}) \)
with torsion fibration.
Theorem (Kwok-Chui-Lo 93) If $\phi: \pi_1(B) \to \pi_0(\mathcal{F}_h, \mathcal{P}_h)$, $h \geq 2$

Don't know sharp inequality.

Sketch of proof: By Thurston, $X$ admits a hyperbolic structure on both choices of orientations. Use Taniyama theorem to show canonical class is represented by embedded symplectic surface. (Maybe disconnected)

Apply the same argument to both orientations.

If $X$ admits a $\mathfrak{g}$ structure, then it is holomorphically fibered, and the inequality is

\[ |\sigma(X)| \leq \frac{1}{3} |\chi(X)| \quad \text{(Tan-Miyake)} \]

The example with the largest $|\sigma(X)|/|\chi(X)|$ are due to Bryant-Donagi (2002)

If we fix $g,h$, there are at most finitely many conjugacy classes of representations $\pi_1(B) \to \Gamma_h$ which correspond to $\mathfrak{g}$ surfaces. But there are always infinitely many conjugacy classes, and they all correspond to $C^0$ fibered manifolds.
Example: $n = 2$, $\langle g(p), [B] \rangle \leq 2(g-1)$ (Milnor word, etc.)

- If $p$ lifts to $\pi_1(B)$, then $g(p) = 0$.
  (For example, because $H^2(\mathbb{F}_2) = 0$)
- If $p$ lifts to $Sp(\mathbb{Q}, \mathbb{Z})$ then $g(p)$ may be nonzero.
  (W. Meyer thesis) (Result of Hecke imply that it can be maximal)

Theorem (K'99) If $\overline{\rho}: \pi_1(B) \to \mathbb{F}_2$, then the following are equivalent:

1. $g(p) = 0$ and the corresponding $X$ admits a $c^0$ structure
2. $\overline{\rho}$ has finite image

$(2) \Rightarrow (1)$ (Use Nielsen realization)

If $\Theta$ lifts

$\pi_1(B) \xrightarrow{\overline{\rho}} \mathbb{F}_2 \xrightarrow{\text{Diff}^+(\mathbb{F}_2)} X \xrightarrow{\text{is a flat bundle of surfaces, this is denoted } X}$

$X = (\mathbb{B} \times \mathbb{F}_2) / \pi_1(B)$

In fact, $\overline{\rho}: \pi_1(B) \to \mathbb{F}_2$ lifts to $\text{Diff}^+(\mathbb{F}_2)$ if the corresponding $X$ admits a horizontal foliation.

What is the pullback of $g \in H^2(\mathbb{F}_2, \mathbb{R})$ to $H^2(\text{Diff}^+(\mathbb{F}_2), \mathbb{R})$?

Theorem (K-Moist, 05) Every surface bundle over a surface can be fibered immersed with signature 0 bundle to obtain a bundle with a horizontal foliation with area preserving total holonomy.

This implies that the pullback of $g$ to $\text{H}^2(\text{Symp}(\mathbb{F}_2), \mathbb{R})$ is nonzero.