Lifting representations to the mapping class group

1

KOTSCHICK
23/3/7

B closed oriented surface of genus $g = g(B) \geq 2$.

$\rho : \pi_1(B) \longrightarrow Sp(2h, \mathbb{R})$

$c_1(\rho) = c_1(E_\rho)$, E_ρ flat $Sp(2h,\mathbb{R})$ bundle
with a choice of (non flat, in general) cx. struct.

Toledo invariant of $\rho = \langle c_1(E_\rho), B \rangle$.

Consider $H^1(B; E_\rho)$. ⟵ this carries a symmetric bilinear form

$$I : H^1(B;E_\rho) \times H^1(B;E_\rho) \longrightarrow \mathbb{R}$$
$$(\alpha, \beta) \longmapsto \langle \alpha \cup \beta [B] \rangle$$

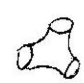
(combining cup product on B with sympl. form on E_ρ).

<u>Fact</u> $|\mathrm{Sign}(I)| = 4\,|\langle c_1(E_\rho), [B] \rangle|$. W. Meyer ('72)
$\quad\; = |\sigma(I)|$

free groups with generators X, Y

If $a, b \in Sp(2h,\mathbb{R})$ consider $\rho : F_2 \longrightarrow Sp(2h,\mathbb{R})$
$$X \longmapsto a$$
$$Y \longmapsto b$$

$F_2 = \pi_1($  $)$

The Meyer signature cocycle on $Sp(2h,\mathbb{R})$ can be defined as $\sigma(a,b) = \sigma(I)$ where I is defined on $E_\rho \to$ 

Consider <u>lifting problem</u>:

$$\begin{tikzcd}
 & \Gamma_h \arrow[d] \\
 & Sp(2h,\mathbb{Z}) \arrow[d] \\
\pi_1(B) \arrow[r] \arrow[ruu, dashed] \arrow[ru, dashed] & Sp(2h,\mathbb{R})
\end{tikzcd}$$

Γ_h ⟵ mapping class group of F_h closed oriented surface of genus h

$\Gamma_h = \mathrm{Diff}^+(F_h) / \mathrm{Diff}_0(F_h)$

other possibilities:

$$\begin{tikzcd}
 & \mathrm{Sympl}(F_h,\omega) \arrow[d] \\
 & \mathrm{Diff}^+(F_h) \arrow[d] \\
 & \Gamma_h \arrow[d] \\
 & {}
\end{tikzcd}$$

$$\{\pi_1(B) \xrightarrow{\tilde{\rho}} \Gamma_h\}/\sim \quad \longleftrightarrow \quad \left\{ \begin{array}{c} X \longleftarrow F \\ \downarrow \\ B \end{array} \right\} \Big/ \sim$$

smooth

In this case, composing $\tilde{\rho}$ with the map $\Gamma_h \to Sp(2h,\mathbb{R})$ we obtain a ρ whose E_ρ is the fiberwise H^1 of the fibration.

Over $\mathbb{R}$ the spectral sequence of X degenerates at E^1 and $H^2(X) = \underbrace{H^2(B) \oplus H^2(F)}_{\text{hyperbolic space}} \oplus H^2(B; H^1(F))$

This implies $\sigma(X) = \sigma(I)$, with I as above.

For $Sp(2h,\mathbb{R})$ we have $|\langle c_1(\rho), [B]\rangle| \leq (g-1)h$. (proved by Turaev, Domic – Toledo)

For $X \to B$ as above, $|\sigma(X)| \leq b_2(X) = \chi(X) - 2 + 2b_1(X)$

P.D.

$$= 4(g-1)(h-1) - 2 + 4g + 4h$$
$$= 4gh + 2$$

Using coverings, this implies (σ is multiplicative under unramified coverings) $|\sigma(X)| \leq 4(g-1)h$, which is the same as

Example $h=1$. A fibration $T^2 \to X \to B$ has $\sigma(X) = 0$, since X covers itself (fiberwise).

Hence, if $\rho: \pi_1(B) \to SL(2;\mathbb{R})$ can be lifted to $SL(2;\mathbb{Z})$ then $c_1(\rho) = 0$.

Furthermore, there is a section

$$\mathrm{Diff}^+(T^2) \to \Gamma_1 = SL(2;\mathbb{Z})$$

so if $\exists$ lift $\rho: \pi_1(B) \to SL(2,\mathbb{Z})$, then get fibration ~~with~~ $T^2 \to X \to B$

with transverse foliation.

KOTSCHICK
23/3/7

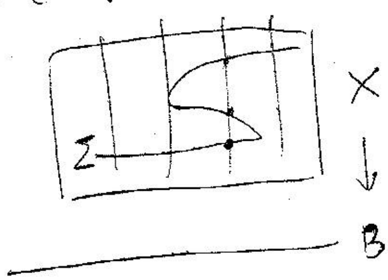
<u>Theorem</u> (Kotschick '98) If $\rho : \pi_1(B) \to Sp(2h, \mathbb{R})$, $h \geq 2$ lifts to Γ_h then:

$$|\langle c_1(\rho), [B] \rangle| \leq \frac{1}{2}(g-1)(h-1) \iff |\sigma(X)| \leq \frac{1}{2}|\chi(X)|$$

Don't know sharp inequality.

<u>Sketch of proof</u> By Thurston, X admits symp. form, for both choice of orientations. Use Taubes' theorem that on minimal symp. mfld canonical class is represented by embedded symplectic surface. (maybe disconnected)

$PD[\Sigma] = c_1(K)$



$$\begin{tikzcd} X \supset \Sigma \arrow[d, "\pi"'] \arrow[dr, "\pi|\Sigma"] & \\ B & \end{tikzcd}$$

then the genus of Σ cannot be too small compared to the genus of B.
then apply adjunction formula.

#

Apply the same argument to both orientations.

If X admits a cx. structure, then it is holomorphically fibered and the inequality is $|\sigma(X)| \leq \frac{1}{3}|\chi(X)|$ (Yau - Miyaoka)

$$\Updownarrow$$

$$|\langle c_1(\rho), [B] \rangle| < \frac{1}{3}(g-1)(h-1)$$

The examply with the largest $\frac{|\sigma(X)|}{|\chi(X)|}$ are due to Bryan - Donagi (~2002)

If we fix g, h then there are at most finitely many conjugacy classes of representation $\pi_1(B) \to \Gamma_h$ wich correspond to cx. Surfaces. But there are always infinitely many conjugacy classes, and they all correspond to C^∞ fibered manifolds

Example $h=2$ $\quad |\langle c(\rho), [B]\rangle| \le 2(g-1)$ (Milnor Wood, etc.)

- if ρ lifts to the Γ_2 then $c(\rho)=0$.
  (for example, because $H^2(\Gamma_2)=0$)
- If ρ lifts to $Sp(4,\mathbb{Z})$ then $c(\rho)$ may be nonzero.
  (W. Meyer thesis). (Results of Hecke imply that it can be maximal.)

Theorem (K'99) if $\tilde{\rho}: \pi_1(B) \to \Gamma_h$ $(h\geq 2)$ then the following are equivalent:

(1) $c(\rho)=0$ and the corresponding X admits a cx. structure

(2) $\tilde{\rho}$ has finite image

((2) $\Rightarrow$ (1) uses Nielsen realization)

If $\exists$ lift

$$\begin{tikzcd} & \mathrm{Diff}^+(F_h) \arrow[d] \\ \pi_1(B) \arrow[r] \arrow[ur, dashed] & Sp(2h,\mathbb{R}) \end{tikzcd}$$

then $\begin{matrix} X \\ \downarrow \\ B \end{matrix}$ is a flat bundle of surfaces: this is denoted $\begin{matrix} X \\ \not\Downarrow \\ B \end{matrix}$ $\leftarrow$ "foliation"

$$X = \left(\tilde{B} \times F_h\right) / \pi_1(B)$$

In fact, $\tilde{\rho}: \pi_1(B) \to \Gamma_h$ lifts to $\mathrm{Diff}^+(F_h)$ iff the corresponding X admits a horizontal foliation.

What is the pullback of $c \in H^2(Sp(2h,\mathbb{R}))$ to $H^2((\mathrm{Diff}^+ F_h)^\delta)$? discrete topology

Theorem (K.-Morita '05) Every surface bundle over a surface can be fibered summed with signature 0 bundle to obtain a bundle with a horizontal foliation with area preserving total holonomy.

This implies that the pullback of c to $H^2(\mathrm{Sympl}(F_h)^\delta)$ is nonzero.