Diploma-Thesis

Basics on Hermitian Symmetric Spaces

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Contents

0 Introduction

Consider a space on which we have a notion of length and curvature. When should one call it symmetric? Heuristically a straight line is more symmetric than the graph of $x^5 - x^4 - 9x^3 - 3x^2 - 2x + 7$, a circle is more symmetric than an ellipse and a ball is more symmetric than an egg. Why? Assume we have a ball and an egg, both with blue surface without pattern. If you close your eyes and I rotate the ball, you can't reconstruct the rotation. But if I rotate the egg, you can reconstruct at least the rotations along two axes. The reason is the curvature. The curvature of the ball is the same in each point (i.e. constant), the curvature of the egg is not. Should we call a space symmetric if its curvature is constant? No, because there are to less. Every manifold of constant curvature can be obtained by one of the following: from a n-dimensional sphere if the curvature is positive, from a n-dimensional euclidian space if the curvature is zero and from a n-dimensional hyperbolic space if the curvature is negative. So we go a step back and consider spaces on which a "derivative" of the curvature vanishes. This gives an interesting class of spaces, called *locally symmetric spaces*. They are defined as Riemannian manifold, where the covariant derivative of the curvature tensor vanishes. We'll show that this happens if and only if there exists for every point x a local isometry which fixes x and acts by multiplication by -1 on the tangent spaces at x. A Riemannian manifold is called *symmetric* if this local symmetry extends to a global isometry. Surprisingly its group of automorphisms acts transitively on the symmetric space and it is homogeneous. The study and classification can be reduced to the study of Lie algebras equipped with an involutive automorphism. We treat this relation explicitly in the first three sections. Élie Cartan used this fact in the early 20th century to classify them. In the 60th Koecher studied them via a relation between symmetric cones and Jordan algebras. We will sketch this relation in Section 2.7.2.

This text should be an introduction to symmetric spaces readable for a third year student. We presuppose only the knowledge of an introduction to differential geometry and basic knowledge in the theory of Lie groups and Lie algebras.

The first chapter gives an introduction to the notion and the basic theory of symmetric spaces. The first section is a short introduction to Riemannian geometry. We present there in short the theory needed later in the text. In the second section we define symmetric spaces as Riemannian manifold whose curvature tensor is invariant under parallel transport. Further we give some examples and deduce from the definition that a symmetric space has the form G/K for a Lie group G (the automorphism group) and a compact subgroup K. The Lie algebra $\mathfrak{g}$ of G decomposes in a natural way into $\mathfrak{k} + \mathfrak{p}$. In the third section we discuss the decomposition of Lie algebras, which can be uses to decompose symmetric spaces. The fourth and last section treats Hermitian symmetric spaces. They are symmetric spaces which have a complex structure. There can be characterized as the spaces G/K where the center of K is non-trivial.

In the second chapter we treat some topics on symmetric spaces. The first section explains the notion of a Cartan decomposition of a Lie algebra, which is generalization of decomposition of $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$. The second and the third section introduce the rank. The next two chapters show that Hermitian symmetric spaces of the non-compact type are exactly the bounded domains in $\mathbb{C}^n$. In the sixth section gives an application of Hermitian symmetric spaces as the moduli space of variations of Hodge structures on a vector space V. The last chapter explains a one-to-one correspondence between algebraic objects (Jordan algebras) and geometric ones (symmetric cones).

A very good introduction to Riemannian geometry is Lees Book [Le97]. THE standard book for symmetric spaces is Helgason [He78]. It is complete and can be used as an introduction to Riemannian geometry too, but it is technical and its structure is not very good. A short and precise treatment of symmetric spaces can be found in [Bo98]. Further good introductions are the book of Wolf [Wo67] and the text of Korányi [Ko00]. The latter is a very nice introduction for the short reading. The texts [Ma06] and [Pa06] should be read together. The first gives an introduction from the differential geometers point of view, the latter from the algebraists point of view. Delignes course notes [De73] are very interesting, since his way to symmetric spaces is different from the usual ways. But the text is not so easy to read, since the proof are discontinuous and he omitted big parts without telling what he omits.

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1 Basics

1.1 Basics in Riemannian Geometry

In this section we will introduce notions and some general facts from Riemannian geometry. We will need them for the definition and the study of symmetric spaces. A very well readable and detailed introduction is [Le97].

All manifolds and vector fields are assumed to be C^∞.

Definition 1.1.1. Let M be a manifold and $\mathfrak{X}(M)$ the vector space of vector fields on M. An *affine connection* ∇ assigns to each $X \in \mathfrak{X}(M)$ a linear mapping ∇_X of $\mathfrak{X}(M)$ into itself, satisfying the following two conditions:

(i) $\nabla_{fX+gY} = f\nabla_X + g\nabla_Y$;

(ii) $\nabla_X(fY) = f\nabla_X Y + (Xf)Y$.

for $f, g \in C^\infty(M)$ and $X, Y \in \mathfrak{X}(M)$. The second condition is sometimes called *Leibniz rule*. The operator ∇_X is called *covariant derivative along* X.

Let ∇ be a affine connection on M and φ a diffeomorphism of M. Put

$$\nabla'_X Y := d\varphi^{-1}\nabla_{d\varphi X} d\varphi Y.$$

One can easily check that this defines a connection. The affine connection ∇ is called *invariant* under φ, if $\nabla' = \nabla$. In this case φ is called an *affine transformation* of M.
A *tensor* of type (r, s) over a vector space V is an element of $(V^*)^{\otimes^r} \otimes V^{\otimes^s}$. A tensor of type (r, s) over a manifold is a section in the bundle $(TM^*)^{\otimes^r} \otimes (TM)^{\otimes^s}$. By abuse of notation we call them tensor, too. The tensors T of type $(2, 1)$ and R of type $(3, 1)$ are defined by

$$T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]; \qquad R(X, Y) := \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}.$$

T is called *torsion*, R is the *curvature*.

Let (U, x) be a local chart on M. We write ∂_i for $\frac{\partial}{\partial x_i}$ and with this notation we can write

$$\nabla_{\partial_i}\partial_j = \sum_k \Gamma^k_{ij}\partial_k,$$

with $\Gamma^k_{ij} \in C^\infty(U)$. The Γ^k_{ij} are called *Christoffel symbols.*

Remark 1.1.2. The vector $(\nabla_X Y)_p$ depends only on the values of X and Y on a neighborhood of p. We show this for Y, for X the proof works similarly. Let Y and $\tilde{Y}$ be vector fields which coincide on a neighborhood U of p. We want to prove that $\nabla_X Y_p = \nabla_x \tilde{Y}_p$. By linearity this is true if and only if $Y - \tilde{Y} = 0$ on U implies $\nabla_X(Y - \tilde{Y})_p = 0$. Now let Y be a vector field which vanishes on a neighborhood of p. We show that $\nabla_X Y_p = 0$. To do this, choose a bump function φ with support in U and with $\varphi(p) = 1$. Note that $\varphi Y \equiv 0$. With the linearity we have $\nabla_X(\varphi Y) = 0$. And by Definition 1.1.1 (ii) we get

$$0 = \nabla_X(\varphi Y) = (Y \cdot \varphi)X + \varphi\nabla_X Y.$$

The first term of the right hand side is zero, since Y vanishes on the support of φ. Therefore $(\nabla_X Y)_p = 0$.

Now we use this result to show, that $(\nabla_X Y)_p$ depends only on the values of Y in a neighborhood of p and on $X(p)$. We choose local coordinates and we can write $X = \sum X^i \partial_i$. Again by linearity we can assume that the $X^i(p) = 0$. Since we proved that $(\nabla_X Y)_p$ depends only on the values of X and Y in a neighborhood of p we get with Definition 1.1.1 (ii)

$$(\nabla_X Y)_p = (\nabla_{\sum X^i \partial_i} Y)_p = \sum X^i(p)(\nabla_{\partial_i} Y)_p = 0.$$

Definition 1.1.3. Let γ be a curve in M and X a vector field on M with $X(\gamma(t)) = \dot\gamma(t)$. Such a vector field exists always, because on can construct X with the help of local charts in a neighborhood of γ and a partition of unity. A vector field Y is called *parallel (along γ)* if $(\nabla_X Y)_{\gamma(t)} = 0$ for all t.
A *geodesic* is a curve for which every vector field X with $X(\gamma(t)) = \dot\gamma(t)$ is parallel along γ, i.e.

$$\nabla_{\dot\gamma(t)} \dot\gamma(t) = 0 \ \forall t.$$

Let γ be a curve. A *vector field along* γ is a smooth map $Y : I \subset \mathbb{R} \to TM$ with $Y(t) \in T_{\gamma(t)}M$ for all t.

Remark 1.1.4. The definition of parallelity makes sense, since by Remark 1.1.2 the vector $(\nabla_X Y)_p$ depends only on Y and the value of X in p. Therefore the definition is independent of the extension of X on M.

Let γ be a curve and X and Y a vector fields, where $\dot\gamma(t)_i = X^i(\gamma(t))$. In local coordinates X and Y can be written as $\sum X^i \partial_i$ respectively $\sum Y^i \partial_i$ and we have

$$\nabla_X Y = \sum_k \left(\sum_i X^i (\partial_i Y^k) \partial_k + \sum_{i,j} X^i Y^j \Gamma_{ij}^k \partial_k \right).$$

Since on γ the $\dot\gamma_i(t) = X^i(\gamma(t))$, for the vector field Y being parallel is equivalent to

$$\frac{dY^k}{dt} + \sum_{i,j} \Gamma_{i,j}^k \frac{dx_i}{dt} Y^j = 0. \qquad (1)$$

for all k.

Proposition 1.1.5. *Let $p \in M$ and $X \neq 0$ in T_pM. There exists a unique maximal geodesic γ in M such that*

$$\gamma(0) = 0 \text{ and } \dot\gamma(0) = X.$$

We will denote this geodesic by γ_X.

One can find straight from the definitions of Γ_{ij}^k and the geodesic a system of differential equation for a geodesic (the *geodesic equation*):

$$\frac{d^2 x_k}{dt^2} + \sum_{i,j} \Gamma_{ij}^k \frac{dx_i}{dt} \frac{dx_j}{dt} = 0.$$

By the existence and uniqueness theorem for linear differential equations it has a unique solution. See [He78, Ch. I.6] for details.

Remark 1.1.6. Let $X \in T_pM$ and γ be a geodesic with $\gamma(0) = p$. Then there exists a vector field along γ with $X(\gamma(0))$, which is parallel along γ. Indeed let $(x_1, \ldots, x_m)$ be a local chart with $(x_1(p), \ldots, x_m(p)) = 0$. By the existence and uniqueness theorem for ordinary differential equations and Equation (1) we know that there exists, for every vector $X = (X^1, \ldots, X^m)$ in T_pM, functions

$$Y^i(t) = \varphi_i(t, X^1, \ldots, X^m)$$

with $Y^i(0) = \varphi_i(0, X^1, \ldots, X^m)) = X_i$. The vector field $(Y^1, \ldots, Y^m)$ is therefore parallel with respect to γ. The mapping $Y(0) \mapsto Y(t)$ is linear, since $\varphi_i(0, X^1, \ldots, X^m)$ is linear in $X^1, \ldots, X^m$ and φ_i is unique. From the uniqueness also follows that $Y(0) \mapsto Y(t)$ is in isomorphism from $T_{\gamma(0)}M$ to $T_{\gamma(t)}M$. We call this map *parallel transport* (see [He78, Ch. I.5]).
Let X be a vector field and denote by $\tau_{s,p}$ the parallel transport from T_pM to $T_{\gamma(s)}M$ along the geodesic $\gamma_{X(p)}$. Let A be a tensor of type (l, k). Since $\tau_{s,p}$ is an isomorphism between T_pM and $T_{\gamma(s)}M$ we can transport the tensor A_p on T_pM to $T_{\gamma(0)}M$ via

$$(\tau_{s,p}A_p)(a_1, \ldots, a_l, b^1, \ldots, b^k) := A_p(\tau_{s,p}^{-1}a_1, \ldots, \tau_{s,p}^{-1}a_l, \tau_{s,p}^{*}b_1, \ldots, \tau_{s,p}^{*}b^k),$$

for $a_1, \ldots, a_l \in T_{\gamma(s)}M$ and $b^1, \ldots, b^k \in T^*_{\gamma(s)}M$. Define

$$(\nabla_X A)_p := \lim_{s \to 0} \frac{1}{s}((\tau_{s,p}^{-1}A_{\gamma(s)}) - A_p).$$

This is the *covariant derivative* for tensor fields. A tensor is called *invariant* under parallel transport, if $\tau_s A = A$ for all s. This is the case if and only if $\nabla_X A = 0$ for all vector fields X.

Definition 1.1.7. Let p be a point in M. For every $X \in T_pM$ there exists a geodesic γ_X. This defines a function from T_pM to M

$$\exp_p : X \mapsto \gamma_X(1)$$

if $\gamma_X(1)$ is defined. It is called the *exponential map.* Its differential exists and it can be seen as a map $T_pM \to T_pM$ and a direct calculation shows that it is the identity. By the local inversion theorem $\exp_p$ is a local diffeomorphism between non-empty open sets $U \subset T_pM$ and $V \subset M$. Its inverse is noted $\log_p$.
A neighborhood N_0 of $0 \in T_pM$ is called *normal* if $\exp_p$ is a diffeomorphism of N_0 onto an open neighborhood N_p of p and for all $X \in N_0$ and $0 \leq t \leq 1$ then $tX \in N_0$. N_p is also called normal.

For a more precise treatment of the exponential map see [He78, Ch. I.6] or [Le97, Ch. 5].
Let X be a vector field. Define the 1-forms

$$\omega^i(X_j) := \delta^i_j \qquad\qquad \omega^i_j = \sum_k \Gamma^i_{kj}\omega^k.$$

The following theorem shows that these two forms are determined by the torsion and curvature tensor. Since they define the connection ∇, the torsion and curvature tensor determine the connection.

Theorem 1.1.8. *Let T be the torsion tensor and R the curvature tensor. Then*

$$d\omega^i = -\sum_p \omega^i_p \wedge \omega^p + \frac{1}{2}\sum_{j,k} T^i_{jk}\omega^j \wedge \omega^k$$

$$d\omega^i_l = -\sum_p \omega^i_p \wedge \omega^p_l + \frac{1}{2}\sum_{j,k} R^i_{ljk}\omega^j \wedge \omega^k.$$

These equations are called the Cartan structure equations.
Let $Y_1, \ldots, Y_m$ be a basis for the tangent space T_pM and let N_0 be a normal neighborhood of the origin in T_pM and $N_p := \exp N_0$. Let $Y_1^*, \ldots, Y_m^*$ be the vector fields on N_p adapted to $Y_1, \ldots, Y_m$. Let $\Phi : \mathbb{R} \times \mathbb{R}^n \to M$ be given by

$$\Phi : (t, a_1, \ldots, a_m) \to \exp(ta_1Y_1 + \ldots + ta_mY_m).$$

Then the dual forms $\Phi^*\omega_i$ and $\Phi^*\omega^j_i$ are given by

$$\Phi^*\omega^i = a_i dt + \bar{\omega}^i, \qquad \Phi^*\omega^i_l = \bar{\omega}^i_l,$$

where $\bar{\omega}^i$ and $\bar{\omega}^i_j$ are 1-forms in $da_1, \ldots, da_m$. They are given by

$$\frac{\partial \bar{\omega}^i}{\partial t} = da_i + \sum_k a_k\bar{\omega}^i_k + \sum_{j,k} T^i_{jk}a_j\bar{\omega}^k. \qquad \bar{\omega}^i_{t=0}(t : aj; da_k) = 0 : \tag{2}$$

$$\frac{\partial \bar{\omega}^i_l}{\partial t} = \sum_{j,k} R^i_{ljk}a_j\bar{\omega}^k \qquad \bar{\omega}^i_l(t; a_j; da_k)_{t=0} = 0 \tag{3}$$

For a proof see [He78, I.8].

Theorem 1.1.9. *Let M and M' be differentiable manifolds with affine connections ∇ and ∇'. Assume that the torsion tensors T and T' and the curvature tensors R and R' are invariant under parallel transport, i.e.:*

$$\nabla T = 0 = \nabla' T', \quad \nabla R = 0 = \nabla' R'.$$

Let $x \in M$ and $x' \in M'$ and $A : T_xM \to T_{x'}M'$ be an isomorphism of the tangent spaces at x and x' sending T and R onto T' and R'. Then there exists a local isomorphism[1] $a : (M, \nabla, x) \to (M', \nabla', x')$ whose differential in x is A.

Proof. Define

$$a(y) := \exp_{x'}(A \log_x y).$$

It seems to be a good candidate for our local isomorphism. It is in fact a local diffeomorphism. Now we have to check, that it is an affine transformation. Before we show that, we take a closer look on what it means to be an affine transformation. On M respectively on M' we have

$$\nabla_{X_i} X_j = \sum \Gamma^k_{ij} X_k \qquad \nabla'_{X'_i} X'_j = \sum \Gamma'^k_{ij} X'_k,$$

[1] An isomorphism is a diffeomorphism which is an affine transformation in both directions.

if X_i and X'_i are local basis' for vector fields in M respectively M'. A map $a : M \to M'$ is an affine transformation if and only if

$$\nabla'_X Y = da \nabla_{da^{-1}X} da^{-1} Y.$$

If $X'_i = da X_i$ and a is affine, we have

$$\begin{aligned}\sum_k \Gamma^k_{ij} X_k &= \nabla_{X_i} X_j = \nabla_{da^{-1}X'_i} da^{-1} X'_j = da^{-1}\left(\sum \Gamma'^k_{ij} X'_k\right)\\ &= da^{-1}\left(\nabla'_{X'_i} X_j\right) = \sum (\Gamma'^k_{ij} \circ a) X_k.\end{aligned}$$

hence $\Gamma'^k_{ij} \circ a = \Gamma^k_{ij}$ is a sufficient and necessary condition for a to be an affine transformation. Lets check this for our a.
Choose normal neighborhoods N_0 and N'_0 of the origins in T_xM and $T_{x'}M'$ and a basis $Y_1, \dots, Y_m$ of T_xM. Put $N_x := \exp(N_0)$ and $N'_{x'} := \exp N'_0$. Then $AY_1, \dots, AY_n$ is a basis of $T_{x'}M'$ and the coefficients of T and T' with respect to these basis' are constant, since all derivatives vanishes by $\nabla T = 0$. Furthermore they are the same for T and T' since A maps T to T'. This also holds for the coefficients of R and R'. Put

$$\Phi(t, a_1, \dots, a_m) = \exp_x(ta_1Y_1 + \cdots ta_mY_m) \tag{4}$$
$$\Phi'(t, a_1, \dots, a_m) = \exp_{x'}(ta_1Y_1 + \cdots ta_mY_m). \tag{5}$$

We have by construction $\Phi' = a \circ \Phi$. The Cartan structure equations say

$$\Phi^*\omega^i = a_i dt + \bar{\omega}^i, \qquad (\Phi')^*\omega'^i = a_i dt + \bar{\omega}'^i$$
$$\Phi^*\omega^i_j = \bar{\omega}^i_j, \qquad (\Phi')^*\omega'^i_j = \bar{\omega}'^i_j$$

By equations (2) and (3) in Theorem 1.1.8 $\bar{\omega}^i = \bar{\omega}'^i$ and $\bar{\omega}'^i_j = \bar{\omega}^i_j$, since both sides are solutions of the same differential equation because the coefficients of T and T' respectively R and R' are the same. With (4), (5) and $\Phi'^* = \Phi^* \circ a^*$ we get

$$\Phi^*\omega^i = a_i dt + \bar{\omega}^i = a_i dt + \bar{\omega}'^i = (\Phi')^*\omega'^i = \Phi^* \circ a^*(\omega'^i)$$

and with a similar calculation

$$\Phi^*\omega^i_j = \Phi^* \circ a^* \omega'^i_j.$$

Putting $t = 1$ we obtain

$$\omega^i = a^*(\omega'^i), \qquad \omega^i_j = a^*(\omega'^i_j).$$

Finally we have

$$\sum_k \Gamma^i_{kj} \omega^k = \omega^i_j = a^*(\omega'^i_j) = \sum_k (\Gamma'^i_{kj} \circ a) a^*(\omega'^k) = \sum_k (\Gamma'^i_{kj} \circ a) \omega^k,$$

so $\Gamma^i_{kj} = \Gamma'^i_{kj} \circ a$. □

The theorem above is Lemma IV.1.2 in [He78].

Proposition 1.1.10. *Under the hypotheses of Theorem* 1.1.9 *assume that M and M' are simply connected and geodesically complete. Then a can be continued to a unique global isomorphism.*

Proof. Let $a := \exp'_p(A \log_p)$ and x be a point in M. We are searching for a point y in M', such that $y = a(x)$. Since M is geodesically complete, we can join p and x by a geodesic γ (see Theorem A.1.1), such that $p = \gamma(0)$ and $x = \gamma(1)$. A transports the initial vector in T_pM to a vector in $T_{p'}M$ which generates a geodesic $\tilde{\gamma}$ in M'. Put $a(x) := \tilde{\gamma}(1)$. Since M and M' are simply connected, this is welldefined and unique. See [He78, Thm. IV.5.6] or [Wo67, Ch. 1.8] for details. □

Corollary 1.1.11. *Let (M, ∇) be such that $\nabla T = 0 = \nabla R$ and M simply connected and geodesically complete. Let $\gamma(t)$ be a geodesic (i.e. $\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) = 0$ for every t). There exists a unique one-parameter group of automorphisms $\tau(\gamma, u)$ of M that induces translation $\gamma(t) \mapsto \gamma(t + u)$ on γ and displaces the tangent bundle on γ by parallel transport.*

Proof. Fix $t \in \mathbb{R}$ and denote by ϑ the parallel transport from $T_{\gamma(t)}M$ to $T_{\gamma(t+u)}$. Since ϑ is an isomorphism we have by Theorem 1.1.9 and Proposition 1.1.10 a unique automorphism $\tau(\gamma, u)$ which maps $\gamma(t)$ to $\gamma(t+u)$ and $d\tau(\gamma, u)_{\gamma(t)} = \vartheta$. Being an automorphism $\tau(\gamma, u)$ maps geodesics on geodesics. Since it acts as parallel transport from $T_{\gamma(t)}M$ to $T_{\gamma(t)}M$ if maps $\dot{\gamma}(t)$ to $\dot{\gamma}(t + u)$, hence γ on itself. Using the geodesic equation one can show that $\tau(\gamma, u)$ maps $\gamma(s)$ to $\gamma(s + u)$ for all s. By the uniqueness of the solution of the Equation (1) $\tau(\gamma, u)$ acts by parallel transport on the tangent bundle of γ. □

Remark 1.1.12. Conversely if such a one-parameter group of automorphisms exists, M is geodesically complete. If a geodesic γ exists on $[-1, 1]$, then we have $\gamma(n + a) = \tau(\gamma, n)\gamma(a)$, if $n \in \mathbb{Z}$ and $a \in [0, 1)$.

Corollary 1.1.13. *Let M a manifold with an affine connection ∇. The following conditions are equivalent:*

(i) $T = 0 = \nabla R$;

(ii) for all $x \in M$ there exist s_x, a local automorphism of (M, ∇) inducing $t \mapsto -t$ on T_xM.

Proof.(i)⇒ii) Use Theorem 1.1.9 with $A = -1$.

(ii)⇒i) Fix $x \in M$. $s := s_x$ is a local automorphism of (M, ∇), i.e. we have

$$(\nabla_X Y)_y = (ds^{-1}(\nabla_{dsX} dsY))_{s(y)}$$

for all vectorfields X and Y and y in a neighborhood of x. Using this fact and the definition of R and T we see that $\nabla_Z R(X, Y) = ds^{-1}\nabla_{ds^{-1}Z} R(dsX, dsY)ds$ and $T(X, Y) = ds^{-1}T(dsX, dsY)$. $ds_x = -1$, hence

$$T(X, Y)_x = -T(X, Y)_x \qquad \nabla_Z R(X, Y)_x = -\nabla_Z R(X, Y)_x,$$

since ∇ is invariant under s. But x was arbitrary, so $\nabla R \equiv 0 \equiv T$.

□

Definition 1.1.14. Let M be a manifold and g a $(2,0)$ tensor field on M such that g_p is positive definite for all $p \in M$, i.e. g_p is a scalar product on T_pM. Such a g is a *Riemannian metric.* A connection ∇ on a manifold equipped with a Riemannian metric is called *Levi-Civita-Connection*[2] if

(i) $\nabla_X Y - \nabla_Y X = [X, Y]$ (torsion free)

(ii) $\nabla_X g \equiv 0$ for all X (invariant under parallel transport).

By a *Riemannian manifold* we mean a triple (M, g, ∇) where M is a connected manifold, g a riemannian metric and ∇ a Levi-Civita-Connection. Usually we simply write M.

Remark 1.1.15. The condition (ii) for the Levi-Civita Connection is equivalent to $X \cdot g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$.
The Levi-Civita Connection exists and it is unique. From the definition one can get the formula

$$2g(X, \nabla_Z Y) = Z \cdot g(X, Y) + g(Z, [X, Y]) + Y \cdot g(X, Y) + g(Y, [X, Y]) - X \cdot g(Y, Z) - g(X, [Y, Z]).$$

This formula and the fact that g is non-degenerate can be used to prove uniqueness and existence (see [Le97, Ch.5]).
The tensor g is called metric, because one can use it to measure the length of a path γ by

$$L(\gamma) := \int_\gamma \sqrt{g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))}dt.$$

For $p, q \in M$ we define

$$d(p, q) := \inf\{L(\gamma) |\ \gamma \text{ a path joining } p \text{ and } q\}.$$

This defines a distance function on M.

Definition 1.1.16. Let M be a Riemannian manifold and $p \in M$ a point. Let S be a two-dimensional subspace of T_pM. The *sectional curvature* is defined by

$$K(S) = -\frac{g_p(R_p(Y, Z)Y, Z)}{|Y \wedge Z|^2},$$

(c.f. [He78, Thm. I.12.2]) where Y and Z are linearly independent vectors in S and $|Y \wedge Z|$ denotes the area of the parallelogram spanned by Y and Z.

For details see [He78, I.12]).

For the proof of the next proposition we need a technical lemma, it is Lemma I.12.4 from [He78].

Lemma 1.1.17. *Let E be a vector space over $\mathbb{F}$ (field of characteristic 0). Suppose $B : E \times E \times E \times E \to \mathbb{F}$ is quadrilinear and satisfies the identities*

(i) $B(X, Y, Z, T) = -B(Y, X, Z, T)$*;*

(ii) $B(X, Y, Z, T) = -B(X, Y, T, Z)$*;*

[2] Sometimes it is called Riemannian connection.

(iii) $B(X,Y,Z,T) + B(Y,Z,X,T) + B(Z,X,Y,T) = 0$*;*

(iv) $B(X,Y,X,Y) = 0$*;*

then $B \equiv 0$*.*

Proof. The proof is a pure calculation without tricks. □

Proposition 1.1.18. *Let* M *be a Riemannian manifold. The following conditions are equivalent:*

(i) $\nabla R = 0$*;*

(ii) the sectional curvature is invariant under parallel transport;

(iii) for all $x \in M$*,* $y \mapsto \exp_x(-\log_x(y))$ *is a local isometry.*

Proof. (i) ⇒ (ii) Let $p \in M$ and S a two dimensional subvectorspace of T_pM. The sectional curvature K in p is defined as

$$K(S) = -\frac{g_p(R_p(Y,Z)Y,Z)}{|Y \wedge Z|^2}.$$

The metric g and the curvature R are invariant under parallel transport (since $\nabla R \equiv 0$) and the same holds for the length of the two vectors and the angle between them, since length and angle are measured with the help of the invariant g. Hence K is invariant under parallel transport.
ii) ⇒ iii) Denote by τ the parallel transport from p to q along a geodesic. By the invariance of K we have

$$g_p(R_p(X,Y)X,Y) = g_q(R_q(\tau X,\tau Y)\tau X,\tau Y) \tag{6}$$

and by the invariance of g

$$g_p(R_p(X,Y)X,Y) = g_q(\tau R_p(X,Y)X,\tau Y) \tag{7}$$

holds for every $X, Y \in T_pM$. Let

$$B(X,Y,Z,T) := g_q(R_q(\tau X,\tau Y)\tau Z,\tau T) - g_q(\tau R_p(X,Y)Z,\tau T)$$

If we show that $B \equiv 0$, then $\tau R_p = R_q$, hence $\nabla R = 0$, since g is non-degenerate. To show $B \equiv 0$ we use Lemma 1.1.17. Lets check the assumptions. (i) is true, since R is skew-symmetric. Checking (ii) needs a straightforward calculation which uses the definition of R and the fact that g is invariant under parallel transport and hence $Z \cdot g(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$. (iii) is exactly the Bianchi-identity. (iv) is clear since the difference of the equations (6) and (7) is zero. By Corollary 1.1.13 $\varphi(y) := \exp_q(-\log_q(y))$ is a local automorphism of (M,∇). It remains to show, that it is an isometry. Multiplication with $-1 (= d\varphi_q)$ is an isometry on T_qM. Then

$$g_p(X,Y) = g_q(\tau X,\tau Y) = g_{\varphi(q)}(d\varphi_q \tau X, d\varphi_q \tau Y).$$

This last equals $g_{\varphi(p)}(d\varphi_p X, d\varphi_q Y)$ because φ is an automorphism of (U,∇), i.e. the parallel transport τ commutes with $d\varphi$.
(iii) ⇒ (i) follows from Corollary 1.1.13. □

1.2 Riemannian Symmetric Spaces

Definition 1.2.1. If a Riemannian manifold M satisfies the equivalent conditions of Proposition 1.1.18, it is called *locally symmetric*. We call the map $y \mapsto \exp_x(-\log_x y)$ the geodesic symmetry in x. If M is connected and for all $x \in M$ there exists an isometric involution s_x with x as an isolated fixed point, M is called *(Riemannian) symmetric*.

Remark 1.2.2. Since x is an isolated fixed point, $ds_x|_x$ acts as multiplication with -1 on the tangent space in x.
A symmetric space is complete. Indeed note first that for every point x on a geodesic γ the symmetry s_x maps the geodesic on itself because s_x is an isometry. Therefore every geodesic is either closed or can be continued to infinity. By the Hopf-Rinow-Theorem (Theorem A.1.1) two arbitrary points can be joined by a geodesic arc and M is complete. Conversely if M is locally symmetric, complete and simply connected it is symmetric, because by Hopf-Rinow-Theorem the completeness implies that M is geodesically closed and therefore by Proposition 1.1.10 such a local symmetry extends to a global one.
Let M be a complete locally symmetric space. Its universal cover is locally symmetric too. Since it is complete an simply connected, it is globally symmetric. Therefore complete, locally connected spaces are quotients of global ones.

Example 1.2.3. The euclidian spaces $\mathbb{E}^n$, the spheres $\mathbb{S}^n$ and the hyperbolic spaces $\mathbb{H}^n$ seen as Riemannian manifolds are symmetric spaces. There curvatures are constant, hence they are locally symmetric by Proposition 1.1.18. They are simply connected and complete, therefore symmetric. For $n = 2$ it is easy to see how the geodesic symmetry acts on the space. Since it acts as multiplication with -1 on the tangent space, s_p is the rotation by π around p in $\mathbb{E}^2$. In $\mathbb{S}^2$ the symmetry is rotation by π around the axis through p and $-p$. If one looks at the upper half plane model for $\mathbb{H}^2$, it is the inversion at the halfcircle through p perpendicular to the x-axis with radius $\Im p$, composed with inversion at the geodesic line parallel to the y-axis through p.

Example 1.2.4. Let $P := P(n, \mathbb{R})$, the set of the symmetric positive definite matrices in $SL(n, \mathbb{R})$. It is clearly an open subset of the vector space of the symmetric matrices. Its dimension is $n(n + 1)/2$. First we equip it with a Riemannian metric. On T_pP (which is the vector space of symmetric matrices) we define a scalar product by

$$< X, Y >_p := \operatorname{tr}(p^{-1}Xp^{-1}Y).$$

This is a Riemannian metric.
$SL(n, \mathbb{R})$ acts on P via

$$g \cdot p := gpg^\top,$$

where $g \in SL(n, \mathbb{R})$ and $p \in P$. The differential is $X \mapsto gXg^\top$. The metric given above is invariant under this action since

$$\begin{aligned} < gXg^\top, gYg^\top >_{g \cdot p} &= \operatorname{tr}((gpg^\top)^{-1}gXg^\top(gpg^\top)^{-1}gYg^\top) = \operatorname{tr}(g^\top p^{-1}Xp^{-1}Y(g^\top)^{-1}) \\ &= \operatorname{tr}(p^{-1}Xp^{-1}Y) =< X, Y >_p . \end{aligned}$$

This explains why we defined the metric in this way.
Now we want for each point $p \in P$ a symmetry s_p. Put $s_p(q) := p(q^\top)^{-1}p$. It is clearly an involution which fixes p. We need to show that s_p is an automorphism. It is a composition

of the maps $\sigma : q \mapsto q^{-1}$ and $q \mapsto p \cdot q$. We showed above, that the latter is an isometry. To show that σ is an isometry we calculate its derivative. Let $X \in T_pP$ be a tangent vector. It is the initial vector of a curve $p + tX$. Let $p^{-1} + t\bar{X}$ be it inverse curve. Then

$$1 = (p + tX)(p^{-1} + t\bar{X}) = 1 + t(Xp^{-1} + p\bar{X}) + \mathcal{O}(t^2),$$

hence $d\sigma_p(X) = \bar{X} = -p^{-1}Xp^{-1}$. Therefore

$$< \bar{X}, \bar{Y} >_{p^{-1}} = \mathrm{tr}(p\bar{X}p\bar{Y}) = \mathrm{tr}(Xp^{-1}Yp^{-1}) = \mathrm{tr}(p^{-1}Xp^{-1}Y) =< X, Y >_p,$$

where we have the general fact $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. The differential of the map $q \mapsto p \cdot q$ is the map itself and the composition of the two differentials is multiplication with -1. Hence P is a symmetric space.

Proposition 1.2.5. *Let M be a Riemannian symmetric space γ be a geodesic. Denote by s_x the symmetry in $x \in M$. We write s_t for $s_{\gamma(t)}$. Then:*

i) $s_t(\gamma(t + u)) = \gamma(t - u)$*;*

ii)
$$\tau(\gamma, u) = s_{u/2} \circ s_0$$
and
$$s_t \circ \tau(\gamma, u) \circ s_t^{-1} = \tau(\gamma, -u).$$

Proof. i) The geodesics $\gamma(t+u)$ and $\gamma(t-u)$ are generated by $\dot{\gamma}(t)$ respectively by $-\dot{\gamma}(t) = ds_t\dot{\gamma}(t)$. By uniqueness of this geodesics we have $s_t(\gamma(t + u)) = \gamma(t - u)$.

ii) With i): $s_u(\gamma(t)) = s_u(\gamma(u+(t-u))) = \gamma(u-(t-u)) = \gamma(2u-t)$. Hence $s_{u/2} \circ s_0(\gamma(t)) = s_{u/2}(\gamma(-t)) = \gamma(u+t) = \tau(\gamma, u)(t)$. And $s_t \circ \tau(\gamma, u), \circ s_t(\gamma(r)) = s_t \circ \tau(\gamma, u)(\gamma(2t-r)) = s_t(\gamma(2t - r + u) = \gamma(r - u) = \tau(\gamma, -u)\gamma(r)$.

□

Let M be a symmetric Riemannian space and G its group of isometries. We endow G with the compact-open topology It is generated by sets $W(C, U) := \{f \in G | f(C) \subset U\}$, where $C \subset G$ is compact and $U \subset G$ is open.It is the coarsest topology such that the evaluation map is constant. It turns G into a topological group which acts continuously on M. It is a finite dimensional Lie group (see Theorem A.1.3). For details on this topology see [Ha02, p. 529f] and [He78, Ch. IV.2].

It acts transitively on M. Indeed take two points p and q and join them by a geodesic γ so that $p = \gamma(0)$ and $q = \gamma(1)$. The symmetry in $\gamma(1/2)$ maps one to the other, since it is an isometry. Further $\tau(\gamma, 1)$ maps p to q. Since τ is a one parameter subgroup of G, (write τ_u for $\tau(\gamma, u)$) for $u, v \in \mathbb{R}$ $\tau_{v+u} = \tau_v \circ \tau_u$.

Now we want to show that the identity component of G, denoted by G_0, acts transitively on M. To do that let $V = \bigcap_{i=1}^{l} W(C_i, U_i)$ be an open subset of the identity component which contains the identity. Since $e \in V$ we have $C_i \subset U_i$. The metric $d(x, \tau_u(x))$ measures how far a point x is moved by τ_u. It depends continuously of x, because d and τ_u are continuous in x .It has for a fixed u a maximum on C_i, since C_i is compact. This maximum is continuous as a function of u. We call this function m and we have $m(0) = 0$. Furthermore we can cover C_i by a finite number of open balls which are contained in U_i and such that the open balls with half radius still cover C_i. Denote by r the minimal radius of these balls. If one chooses u_i such that $m(u_i) < r/2$, one has $\tau_{u_i}(C_i) \subset U_i$. For all $0 < u < u_i$ for all i the map τ_u is contained in V. Since τ is a one-parameter group, we have $\tau_{u+v} = \tau_u \circ \tau_v$. Therefore G_0 acts transitively on M.

Let K be the stabilizer of a point $x \in M$. It is compact (Theorem A.1.2). Hence $M = G/K$ (see Theorem A.1.4). On G there exists a natural involution $\sigma : g \mapsto s_x g s_x$. For $k \in K$ we have $d(s_x k s_x) = (-1)dk(-1) = dk$ and since K injects into $O(T_xM)$ (Lemma A.1.5) we have $s_x k s_x = k$ and all elements of K are left fixed by σ. Therefore $d\sigma$ is the identity on $\mathfrak{k} := \text{Lie}(K)$. Let $\mathfrak{p} \subset \text{Lie}(G)$ be the set of infinitesimal generators of the one parameter subgroups $\tau(\gamma, u)$, where γ is a geodesic through x. The involution σ acts on $\mathfrak{p}$ by multiplication with -1 because of Proposition 1.2.5 (ii). The vector space $\mathfrak{p}$ can be identified with T_xM, because for two geodesics γ and $\tilde{\gamma}$ with $\gamma(0) = \tilde{\gamma}(0)$ and $\dot{\gamma}(0) = s\dot{\tilde{\gamma}}(0)$ with $s > 0$ we have $\gamma(t) = \tilde{\gamma}(st)$ and therefore $\tau(\sigma, u) = \tau(\tilde{\gamma}, u/s)$. Hence $\tau(\gamma, u)$ is uniquely determined by a vector in T_xM, its direction gives γ and its length u and vice versa. By dimension reasons we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{p}$ is the eigenspace of σ to the eigenvalue -1 and $\mathfrak{k}$ is the eigenspace to the eigenvalue 1. If not otherwise stated, in the following G denotes the isometry group of the Riemannian symmetric space M and K the stabilizer of a point $p \in M$. The Lie algebra $\mathfrak{g}$ of G admits always the decomposition $\mathfrak{k} \oplus \mathfrak{p}$ with respect to the involution σ.

This shows

Theorem 1.2.6. *Let M be a symmetric space and G its group of isometries. Then $M = G/K$ where K is the stabilizer of a point $x \in M$.*

Example 1.2.7. Lets look how our examples are as homogeneous spaces. The isometry group of $\mathbb{R}^n$ is $\mathbb{R}^n \ltimes O(n)$ and we have $O(n+1)$ for S^n. For $\mathbb{H}^n$ we have $O(n,1)_+$. It is the subgroup of $O(n,1)$ of index 2, which preserves the upper sheet $\mathbb{H}^n$. The group of isometries of $P(n,\mathbb{R})$ is $SL(n,\mathbb{R})$. The point-stabilizer in the first three cases is $O(n)$, in the last case it is $SO(n)$. Therefore we have:

$$\begin{aligned} \mathbb{R}^n &= \mathbb{R}^n \ltimes O(n)/O(n) \\ S^n &= O(n+1)/O(n) \\ \mathbb{H}^n &= O(n,1)_+/O(n) \\ P(n,\mathbb{R}) &= SL(n,\mathbb{R})/SO(n) \end{aligned}$$

For details for the calculation of the isometry groups see [Br99, Ch. I].

Remark 1.2.8. Now we want to see how we can build a symmetric space from a given Lie group G and a compact subgroup K. We assume that

(i) $M = G/K$ is connected and

(ii) there exists an involution σ of G such that $\text{Lie}(G)^{d\sigma_e} = \text{Lie}(K)$ and $K^\sigma = K$.

We want to put a structure of a Riemannian symmetric space on M. First we choose Q a G-invariant Riemannian structure. Such a structure exists always. We can define a K-invariant inner product on T_eM via

$$< a, b >= \int_K (dk \cdot a, dk \cdot b) d\mu(k),$$

where $(\cdot,\cdot)$ is an arbitrary positive definite bilinear form on T_eM and $d\mu(k)$ is the Haar-measure[3] on K. The integral exists since K is compact and acts continuously on M. Now

$$h_{gK}(X,Y) := < dg^{-1}X, dg^{-1}Y >$$

defines an invariant riemannian structure on M. It is well-defined since $< \cdot, \cdot >$ is K-invariant, hence it does not depend on the representant of gK.
Now we are searching for a suitable point-symmetry for our M. Our σ is an involution, therefore $\mathfrak{g}$ decomposes into the direct sum $\mathfrak{k} + \mathfrak{p}$, where $\mathfrak{k}$ is the eigenspace to the eigenvalue 1 (and the Lie algebra of K) and $\mathfrak{p}$ is the eigenspace to the eigenvalue -1. Therefore it has eK as an isolated fixed point and it is a symmetry in eK. Since M is homogeneous this gives a symmetry for every $gK \in M$, putting $s_{pK}(gK) := p\sigma(p^{-1}q)K$. It is clearly an involution and it is an automorphism of M, since σ and multiplication with an element of G are automorphisms.

In Proposition 1.3.10 we will see that if G is semi-simple, connected and acts faithfully on M, it is the identity component of the group of isometries of M.

Example 1.2.9. Let $G = SO(2,n)$ (with $n > 3$) the group of matrices in $GL(2+n,\mathbb{R})$ which leave the bilinear form of type $(2,n)$ invariant, which is given by

$$(x|y) := x_1y_1 + x_2y_2 - x_3y_3 - \ldots - x_{n+2}y_{n+2}$$

for $x = (x_1, \ldots, x_{n+2})$ and $y = (y_1, \ldots, y_{n+2})$. This form is defined with respect to a basis $(e_1, \ldots, e_{2+n})$. We can use this basis to define the standard scalar product and therefore a norm on $\mathbb{R}^2$ and $\mathbb{R}^n$. We denote the scalar product by $\langle\cdot,\cdot\rangle$ and the norm by $\|\cdot\|$.
The Lie algebra of G consists of matrices of the form

$$\begin{pmatrix} X_1 & X_2 \\ X_2^\top & X_3 \end{pmatrix}$$

where $X_1 \in M(2,2,\mathbb{R})$ skew-symmetric, $X_2 \in M_(2,n,\mathbb{R})$ arbitrary and $X_3 \in M_(n,n,\mathbb{R})$ skew-symmetric. One sees directly, that the subalgebra $\mathfrak{k}$ consisting of matrices

$$\begin{pmatrix} X_1 & 0 \\ 0 & X_3 \end{pmatrix}$$

is compact. The corresponding Lie group K is $S(O(2) \times O(n))$. Let $\sigma : G \to G$ the map which maps g to $(g^\top)^{-1}$. It is clearly an involution. The fixed points of σ are exactely the matrices in K. Its differential acts as $X \mapsto -X^\top$ on the Lie algebra $\mathfrak{g}$ of G. The fixed points are exactly the matrices in $\mathfrak{k}$. Therefore G/K is a symmetric space.

A model of this symmetric space is the space $\mathcal{G}$ of the two dimensional subspaces of $\mathbb{R}^2 \oplus \mathbb{R}^n$ on which $(\cdot|\cdot)$ is positive definite. The group G acts on it linearly. We show now that the action is transitive. We have clearly $V_0 := \mathbb{R}^2 \oplus \{0\} \in \mathcal{G}$. If we can show, that every point in $\mathcal{G}$ can be mapped to V_0, we are done. Let $V \in \mathcal{G}$. By $V^\perp$ we denote the orthogonal complement of V with respect to $(\cdot|\cdot)$. Since this bilinear form is non-degenerated on V we can decompose $\mathbb{R}^2 \oplus \mathbb{R}^n = V \oplus V^\perp$. If we show, that our bilinear form is negative definite on $V^\perp$, we can find a orthogonal basis $\{w_1, \ldots, w_n\}$ of $V^\top$ and a orthogonal basis $\{v_1, v_2\}$ of V. The matrix $(v_1, v_2, w_1, \ldots, w_n)$ is by orthogonality in $SO(2,n)$ and it maps V_0 to V. Hence the action is

[3] The Haar-measure exists since K is compact.

transitive. It remains to show that, $(\cdot|\cdot)$ is negative definite on $V^\top$. Lets do this: let $\{v_1, v_2\}$ be a orthogonal basis of V. Lets decompose them as $v_1 = \xi_1 + \eta_1$ and $v_2 = \xi_2 + \eta_2$ with $\xi_i \in \mathbb{R}^2$ and $\eta_i \in \mathbb{R}^n$ and we have $\|\xi_i\| > \|\eta_i\|$. A vector $\xi + \eta \in V^\top$ with $\xi \in \mathbb{R}^2$ and $\eta \in \mathbb{R}^n$ satisfies $(u_1 + u_2|\alpha v_1 + \beta v_2) = 0$, hence $\langle \xi, \alpha\xi_1 + \beta\eta_2\rangle = \langle \eta, \alpha\eta_1 + \beta\eta_2\rangle$. Since $\{\xi_1, \xi_2\}$ form a basis of $\mathbb{R}^2$ we have

$$\begin{aligned}\|\xi\| &= \sup_{\|\alpha\xi_1+\beta\xi_2\|} |\langle \xi, \alpha\xi_1 + \beta\xi_2\rangle| = \sup_{\|\alpha\xi_1+\beta\xi_2\|} |\langle \eta, \alpha\eta_1 + \beta\eta_2| \\ &\leq \sup_{\|\alpha\xi_1+\beta\xi_2\|} \|\eta\| \cdot \|\alpha\eta_1 + \beta\eta_2\| < \sup_{\|\alpha\xi_1+\beta\xi_2\|} \|\eta\| \cdot \|\alpha\xi_1 + \beta\xi_2\| = \|\eta\|.\end{aligned}$$

We used in the first step a characterization of the norm, the third step is the Cauchy-Schwarz inequality and the fourth step uses $\alpha v_1 + \beta v_2 \in V$. It is important that we have there "$<$" and not "$\leq$"!

The space $\mathcal{G}$ can be realized as a subset of $M(2, n, \mathbb{R})$. Let $v = v_1 + v_2 \in V$ with $v_1 \in \mathbb{R}^2$ and $v_2 \in \mathbb{R}^n$. The projection on $\mathbb{R}^2$ is injective since $v_1 = 0$ if and only if $v_2 = 0$. By dimension reasons it is also surjective and we can choose a basis $\{\xi_1, \xi_2\}$ of V with $\xi_1 = \{1, 0, Z_{11}, \ldots, Z_{1n}\}$ and $\xi_2 = \{0, 1, Z_{21}, \ldots, Z_{2n}\}$. By definition of $\mathcal{G}$ we have $\sum_j Z_{ij}^2 < 1$ for $j = 1, 2$. Conversely every $2 \times n$ matrix defines an element of $\mathcal{G}$ via

$$V := \{(v, Zv) | v \in \mathbb{R}^2\}.$$

Therefore we have a one-to-one correspondence between $\mathcal{G}$ and the set of matrices $M := \{Z = (Z_1, Z_2) \in M_{2,n}(\mathbb{R}) | Z_1 \cdot Z_1 < 1,\ Z_2 \cdot Z_2 < 1\}$.

Example 1.2.10. Let $G := SL(n, \mathbb{R})$, $K := SO(n, \mathbb{R})$ and $\sigma : G \to G$ with $\sigma : g \mapsto (g^\top)^{-1}$. $M := G/K$ is connected, since G and K are. σ is an involution and its fix-point set is by definition exactly K. If the fixed point set of $d\sigma_e$ is $\mathrm{Lie}(K)$ we know, that M is a symmetric space. Let $X \in \mathrm{Lie}(G)$. It generates a path $e + tX$ in G. The path $\sigma(e + tX) = (e + tX^\top)^{-1}$ can be written in the form $e + td\sigma_e X$ and is satisfies therefore

$$e = (e + tX^\top)(e + td\sigma_e X) = e + t(X^\top + d\sigma_e X) + \mathcal{O}(t^2).$$

This means $d\sigma_e X = -X^\top$. The fixed point set of this involution is the vector space of skew-symmetric matrices, which is equal to the Lie algebra of $SO(n, \mathbb{R})$. Therefore $M = G/K$ is a symmetric space.

In Example 1.2.4 we constructed a symmetric space P as the set of symmetric positive definite matrices in $SL(n, \mathbb{R})$. The identity matrix I is clearly in P. The involution on the isometry group of P given by $\sigma : g \to s_I g s_I$ can be calculated explicitly. Let $q \in P$:

$$s_I(g \cdot (s_I(q))) = s_I(g \cdot (q^\top)^{-1}) = s_I(gq^\top)^{-1}g^\top) = (g^{-1})^\top q g^{-1} = (g^\top)^{-1} \cdot q.$$

Hence σ maps g to $(g^\top)^{-1}$. This is the same action as above. Hence $G/K = P$.

Proposition 1.2.11. *Let R denote the curvature tensor of the space Riemannian symmetric space G/K with G and K as usual and let Q be the Riemannian structure on G/K. Then at the point $o \in G/K$*

$$R_o(X, Y)Z = -[[X, Y], Z], \quad \textit{for } X, Y, Z \in \mathfrak{p}.$$

This proposition is Theorem IV.4.2 in [He78]. In the following section we will use the fact that a symmetry s_x in a symmetric space M gives in a natural way an involution on its group of automorphisms. We will use its differential, which is an involutive Lie algebra automorphism, to study the structure of symmetric spaces.

Proposition 1.2.12. *Let G and K be as above. The following conditions are equivalent*

(i) G acts faithfully on $M = G/K$;

(ii) K acts faithfully on $\mathfrak{p} = Lie(G)/Lie(K)$=tangent space to M at the origin.

Proof. (i)⇒ (ii). $K < G$ acts faithfully on M and K is the stabilizer of the point $eK \in M$. If K acts not faithfully on $\mathfrak{p}$ there exists $k_1, k_2 \in K$ with $k_1 \neq k_2$ and $dk_1 = dk_2$ in $T_{eK}M$. But this is by Lemma A.1.5 a contradiction.
(ii) ⇒ (i). Take $h \in G$ with $h(gK) = gK$ for all $g \in G$. This holds in particular for $g = e$ and have $h \in K$. From $h = \mathrm{id}$ on M follows that $dh = \mathrm{id}$ on $\mathfrak{p}$. K acts faithfully on $\mathfrak{p}$ and therefore $h = e$. Hence G acts transitively on M. □

If the two conditions hold, σ is the identity on K, because it acts as identity on Lie(K).

Now we need some notions and theory of Lie algebras.

Definition 1.2.13. Let $\mathfrak{g}$ be a Lie algebra. By ad we denote the *adjoint representation* of $\mathfrak{g}$ in itself by

$$\begin{aligned} \mathrm{ad}(X) : & \mathfrak{g} \to \mathrm{End}(\mathfrak{g}), \\ & X \mapsto \{\mathrm{ad}(X) : Y \mapsto [X, Y]\} \end{aligned}$$

with $X, Y \in \mathfrak{g}$.
The *Killing form* of $\mathfrak{g}$ is defined as

$$B(X, Y) := \mathrm{tr}(\mathrm{ad}(X)\mathrm{ad}(Y)).$$

We denote the Killing form of a Lie algebra always by B or by $B_{\mathfrak{g}}$ if there is danger of confusion.

Proposition 1.2.14. *The Killing form has the following properties:*

(i) It is a symmetric bilinear form, i.e. $B(X, Y) = B(Y, X)$ for all $X, Y \in \mathfrak{g}$.

(ii) It is infinitesimally invariant under ad, i.e. for all $X, Y, Z \in \mathfrak{g}$ we have

$$B(ad(Z)X, Y) = -B(X, ad(Z)Y).$$

This is equivalent to $B([X, Y], Z) = B(X, [Y, Z])$.

(iii) Let $\mathfrak{i} \subset \mathfrak{g}$ be an ideal. Then the Killing form on $\mathfrak{i}$ equals the Killing form on $\mathfrak{g}$ restricted to $\mathfrak{i} \times \mathfrak{i}$.

Proof. (i) The bilinearity comes from the bilinearity of $\mathrm{tr}(XY)$ and the linearity of ad. The symmetry is a consequence of the general fact that for matrices A and B we have $\mathrm{tr}(AB) = \sum A_{ij}B_{ji} = \sum B_{ij}A_{ji} = \mathrm{tr}(BA)$.

(ii) With the Jacobi-Identity we get:

$$\begin{aligned}\mathrm{ad}([Z,Y])(A) =& [A,[Y,Z]] = -[Y,[Z,A]] - [Z,[A,Y]] \\ =& -\mathrm{ad}(Y)\mathrm{ad}(Z)(A) + \mathrm{ad}(Z)\mathrm{ad}(Y)(A)\end{aligned}$$

Therefore

$$\begin{aligned}B(\mathrm{ad}(Z)X,Y) &= \mathrm{tr}(\mathrm{ad}([Z,X])\mathrm{ad}(Y)) = \mathrm{tr}(\mathrm{ad}(Z)\mathrm{ad}(X)\mathrm{ad}(Y)) - \mathrm{tr}(\mathrm{ad}(X)\mathrm{ad}(Z)\mathrm{ad}(Y)) \\ &= \mathrm{tr}(\mathrm{ad}(X)\mathrm{ad}(Y)\mathrm{ad}(Z)) - \mathrm{tr}(\mathrm{ad}(X)\mathrm{ad}(Z)\mathrm{ad}(Y)) = \mathrm{tr}(\mathrm{ad}(X)\mathrm{ad}([Y,Z])) \\ &= -B(X,\mathrm{ad}(Z)Y)\end{aligned}$$

(iii) Since $\mathfrak{i}$ is a subvectorspace of $\mathfrak{g}$, one can complete a basis of $\mathfrak{i}$ to a basis of $\mathfrak{g}$. Since it is an ideal in $\mathfrak{g}$ (i.e. $\mathrm{ad}(x)\mathfrak{g} \subset \mathfrak{i}$) for $x \in \mathfrak{i}$ the matrix of $\mathrm{ad}(x)$ in $\mathfrak{g}$ has the form

$$\begin{pmatrix} A & * \\ & 0 \end{pmatrix}$$

where A is the matrix of $\mathrm{ad}(x)$ in $\mathfrak{i}$.

□

Definition 1.2.15. The *adjoint representation* Ad of G in $\mathfrak{g}$ is given by $\mathrm{Ad}(g) := dc_g|_e$ where $c_g : h \to ghg^{-1}$ is the conjugation by g.

The differential of Ad in e is ad. The Killing form is therefore Ad-invariant, since by Proposition 1.2.14 (ii) $B(\mathrm{ad}(Z)X,Y) = B(X,\mathrm{ad}(Z)Y) = 0$.

We have for $k \in K$ and a geodesic through e

$$k\tau(\gamma,u)k^{-1} = \tau(k\gamma,u).$$

Since γ is generated by a vector $X = \dot{\gamma}(0) \in \mathfrak{p}$ and K acts faithfully on $\mathfrak{p}$ via $k \mapsto dk$, we see that $k\gamma$ is generated by dkX. Hence the adjoint representation of K on $\mathfrak{p}$ is faithful.

Further we have that $\mathrm{ad}(X)$ is a skew-symmetric matrix (Appendix Lemma A.1.6), if $X \in \mathfrak{k}$. Therefore it is diagonalisable with eigenvalues in $i\mathbb{R}$ (Appendix Lemma A.1.7) and hence $B(X,X) = \mathrm{tr}(\mathrm{ad}(X)^2) = -\sum \lambda_j^2 \leq 0$, where $\lambda_j \in \mathbb{R}$ and $i\lambda_j$ are the eigenvalues of $\mathrm{ad}(x)$. If $\mathrm{tr}(\mathrm{ad}(X)^2) = 0$ then the only eigenvalue of $\mathrm{ad}(X)$ is zero and therefore $\mathrm{ad}(X) = 0$. But then

$$\mathrm{Ad} \circ \exp(X) = \exp \circ \mathrm{ad}(X) = 0,$$

and since the action of K on $\mathfrak{p}$ is faithful, X is zero. This shows that $B|_{\mathfrak{k}\times\mathfrak{k}}$ is negative definite.

Definition 1.2.16. A Lie algebra is *semi-simple* if the Killing form is non-degenerate.

1.3 Involutive Lie Algebras

We have seen, that one can introduce on the group of automorphims G of a symmetric space M an involution in a natural way. We fixed a point $x \in M$ and set $\sigma : g \mapsto s_x g s_x$. This is clearly an involution, since s_x is. This involution gives an involution on $\mathfrak{g}$ (the Lie algebra of G) by taking the differential at the identity $d\sigma|_e$. We have also seen, that we can write $\mathfrak{g}$ as $\mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k}$ and $\mathfrak{p}$ are the eigenspaces to the eigenvalues 1 respectively -1 of $d\sigma_e$. We will always write $\mathfrak{p}$ and $\mathfrak{k}$ for these eigenspaces. By abuse of notation we write σ instead of $d\sigma_e$. First we define two properties and show that they are fulfilled for Lie algebras that come from symmetric spaces.

Definition 1.3.1. An *involutive Lie algebra* $(\mathfrak{g}, \sigma)$ is a Lie algebra together with an involutive Lie algebra automorphism[4] σ.

Let $(\mathfrak{g}, \sigma)$ be an involutive Lie algebra. By Remark A.1.8 σ is diagonalisable with eigenvalues 1 and -1. Therefore we can decompose $\mathfrak{g}$ into $\mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k}$ and $\mathfrak{p}$ are the eigenspaces to the eigenvalues 1 respectively -1 of σ. We have

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \ [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \ [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}. \tag{8}$$

because $\mathfrak{k}$ and $\mathfrak{p}$ are defined as eigenspaces of σ. $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$. Since σ is an Lie algebra automorphism it commutes with the bracket and we have $\mathrm{ad}(\sigma X) = \sigma \mathrm{ad}(X)\sigma^{-1}$. As a general fact we know that the trace is invariant under conjugation. Therefore we have for $k \in \mathfrak{k}$ and $p \in \mathfrak{p}$

$$\begin{aligned} B(k, p) =& \mathrm{tr}(\mathrm{ad}(k)\mathrm{ad}(p)) = \mathrm{tr}(\sigma \mathrm{ad}(k)\sigma\sigma^{-1}\mathrm{ad}(p)\sigma^{-1}) = \mathrm{tr}(\mathrm{ad}(\sigma k)\mathrm{ad}(\sigma p)) = -\mathrm{tr}(\mathrm{ad}(k)\mathrm{ad}(p)) \\ =& - B(k, p), \end{aligned}$$

hence $B(\mathfrak{p}, \mathfrak{k}) = 0$. Lets start our discussion of $(\mathfrak{g}, \sigma)$.
Our main reference here is [Bo98].

Definition 1.3.2. An involutive Lie algebra $(\mathfrak{g}, \sigma)$ is said to be *reduced*, if $\mathfrak{k}$ contains no non-zero ideal of $\mathfrak{g}$.

Let $\mathfrak{i} \subset \mathfrak{k}$ be an ideal of $\mathfrak{g}$. Then by definition of an ideal we have $[\mathfrak{p}, \mathfrak{i}] \subset \mathfrak{i} \subset \mathfrak{k}$ and by from (8) we deduce $[\mathfrak{p}, \mathfrak{i}] \subset \mathfrak{p}$. Since $\mathfrak{p} \cap \mathfrak{k} = \{0\}$ the ideal $\mathfrak{i}$ is contained in the centralizer of $\mathfrak{p}$ in $\mathfrak{k}$.

Proposition 1.3.3. *An involutive Lie algebra $(\mathfrak{g}, \sigma)$ is reduced if and only if the representation of $\mathfrak{k}$ in $\mathfrak{p}$ given by $k \mapsto ad_{\mathfrak{g}}(k)|_{\mathfrak{p}}$ is faithful.*

Proof. The kernel of this representation is an ideal contained in $\mathfrak{k}$ and it is by definition of the representation equal to the centralizer of $\mathfrak{p}$ in $\mathfrak{k}$, denoted by $\mathfrak{z}(\mathfrak{p})_{\mathfrak{k}}$. To show that the kernel is in fact an ideal, we need to show, that for r in the kernel and $g = k + p \in \mathfrak{k} \oplus \mathfrak{p}$ we have that $[g, r]$ is in the kernel, i.e. $[[g, r], \tilde{p}] = 0$ for each $\tilde{p} \in \mathfrak{p}$. We use the Jacobi-Identity:

$$[[k + p, r], \tilde{p}] = [[k, r], \tilde{p}] = [k, [r, \tilde{p}]] + [r, [\tilde{p}, k]].$$

The right hand side is zero because $\tilde{p}$ and $[\tilde{p}, k]$ are in $\mathfrak{p}$.
If $(\mathfrak{g}, \sigma)$ is reduced the kernel of the adjoint representation is zero, since it is an ideal in $\mathfrak{k}$ and vice versa. □

From this proposition we see that an involutive Lie algebra which comes from a symmetric space is reduced, since by Proposition 1.2.12 the representation $k \mapsto \mathrm{ad}_{\mathfrak{g}}(k)|_{\mathfrak{p}}$ is faithful.
Since $\mathfrak{z}(\mathfrak{p})_{\mathfrak{k}}$ is the greatest ideal of $\mathfrak{g}$ contained in $\mathfrak{k}$, it is invariant under σ and σ induces an involution on $\mathfrak{g}/\mathfrak{z}(\mathfrak{p})_{\mathfrak{k}}$, which becomes a reduced involutive Lie algebra, denoted the reduced involutive Lie algebra associated to $(\mathfrak{g}, \sigma)$.

Definition 1.3.4. An *orthogonal involutive Lie algebra* is an involutive Lie algebra $(\mathfrak{g}, \sigma)$ on which there exists a positive non-degenerate quadratic form which is invariant under σ and infinitesimally invariant[5] under $\mathrm{ad}_{\mathfrak{g}}\mathfrak{k}$.

[4] A vector space isomorphism with $\sigma([X, Y]) = [\sigma(X), \sigma(Y)]$ for all $X, Y \in \mathfrak{g}$.
[5] i.e. $Q(\mathrm{ad}(Z)X, Y) = -Q(X, \mathrm{ad}(Z)Y)$.

Proposition 1.3.5. *An involutive Lie algebra is orthogonal if and only if $\mathfrak{k}$ is compact and $ad(\mathfrak{k})|_{\mathfrak{p}}$ leaves a positive non-degenerate quadratic form infinitesimally invariant.*

Proof. If $(\mathfrak{g}, \sigma)$ is orthogonal, there exists Q a positive non-degenerate quadratic form invariant under σ. By definition of $\mathfrak{k}$ and $\mathfrak{p}$ as eigenspaces of σ we get $Q(\mathfrak{k}, \mathfrak{p}) = 0$ and $Q = Q_1 + Q_2$ where $Q_1 = Q|_{\mathfrak{k}\times\mathfrak{k}}$ and $Q_2 = Q|_{\mathfrak{p}\times\mathfrak{p}}$. Since ad$(\mathfrak{k})$ leaves the positive non-degenerate quadratic form Q_1 invariant, $\mathfrak{k}$ is compact. Q_2 is invariant under ad$(\mathfrak{k})$, too.
Conversely if $\mathfrak{k}$ is compact, it leaves a positive non-degenerate quadratic form Q_1 infinitesimally invariant. Denote by Q_2 the positive non-degenerate form on $\mathfrak{p}$ invariant under ad$(\mathfrak{k})$. $Q := Q_1 + Q_2$ is by construction invariant under σ and infinitesimally invariant under ad$(\mathfrak{k})$, hence $(\mathfrak{g}, \sigma)$ is orthogonal. □

Let $(\mathfrak{g}, \sigma)$ be the involutive Lie algebra that comes from a symmetric space. We know from our discussion above, that $\mathfrak{k}$ is compact, hence ad$(\mathfrak{k})$ contains only skew-symmetric matrices. Therefore we can construct on $\mathfrak{p}$ a positive non-degenerate quadratic form Q invariant under $\mathrm{ad}_{\mathfrak{g}}\mathfrak{k}$ by fixing a basis of $\mathfrak{p}$ and take Q as the usual quadratic form $(x, y) := x^\top \cdot y$. We have shown, that ad(k) is skew-symmetric for each $k \in \mathfrak{k}$, therefore this bilinear form has all desired properties and $(\mathfrak{g}, \sigma)$ is orthogonal.

Definition 1.3.6. A Lie algebra is said to be *flat* if its Lie bracket is constantly zero.

Theorem 1.3.7. *Let $(\mathfrak{g}, \sigma)$ be a reduced orthogonal involutive Lie algebra. Then $(\mathfrak{g}, \sigma)$ is the direct product of a reduced orthogonal involutive flat Lie algebra and of reduced semi-simple irreducible orthogonal involutive Lie algebras $(\mathfrak{g}_i, \sigma_i)$,$(i = 1, \ldots, a)$. This decomposition is unique up to the order of the factors and $\mathfrak{z}(\mathfrak{p}) = \mathfrak{p}_0$.*

Proof. Let Q be a positive non-degenerate quadratic form on $\mathfrak{p}$, infinitesimally invariant under $\mathrm{ad}_{\mathfrak{g}}\mathfrak{k}$. Q exists since $(\mathfrak{g}, \sigma)$ is orthogonal. Let A be the linear map of $\mathfrak{p}$ into itself, defined by $Q(Ax, y) = B(x, y)$ for every $x, y \in \mathfrak{p}$. If Q and B are the matrices of the corresponding bilinear forms $A = Q^{-1}B$, where Q is invertible since it is non-degenerate. Since Q and B are both symmetric forms, we have $Q(Ax, y) = B(x, y) = B(y, x) = Q(Ay, x) = Q(x, Ay)$. Therefore A is symmetric, hence diagonalisable. We have a decomposition of $\mathfrak{p}$ into eigenspaces

$$\mathfrak{p} = \bigoplus_{i=0}^{r} \mathfrak{q}_i,$$

such that

$$A|_{\mathfrak{q}_i} = c_i \cdot \mathrm{id} \ (c_i \in \mathbb{R},\ c_0 = 0,\ c_i \neq c_j \text{ for } j \neq i).$$

$\mathfrak{q}_0$ is the kernel of the Killing form, i.e. it contains every $x \in \mathfrak{g}$ with $B(x, y) = 0$ for all $y \in \mathfrak{q}$. It is contained in $\mathfrak{p}$ since B is non-degenerate on $\mathfrak{k}$. The $\mathfrak{q}_i$ are subvectorspaces of $\mathfrak{p}$. For $x \in \mathfrak{q}_i$ and $y \in \mathfrak{q}_j$, we have $B(x, y) = c_iQ(x, y) = c_jQ(x, y) = 0$ if $i \neq j$, hence

$$B(\mathfrak{q}_i, \mathfrak{q}_j) = Q(\mathfrak{q}_i, \mathfrak{q}_j) = 0,\ (i \neq j), B(\mathfrak{q}_0, \mathfrak{p}) = 0$$

and

$$B|_{\mathfrak{q}_i} = c_iQ|_{\mathfrak{q}_i}.$$

Since B and Q are non-degenerate and infinitesimally invariant under $\mathrm{ad}_{\mathfrak{g}}\mathfrak{k}$, A commutes elementwise with $\mathrm{ad}_{\mathfrak{g}}\mathfrak{k}$. Therefore ad$(\mathfrak{k})$ leaves the eigenspaces of A fixed and hence we have

$[\mathfrak{k}, \mathfrak{q}_i] \subset \mathfrak{q}_i$. Let $U \subset \mathfrak{p}$ be a subvectorspace invariant under $\mathrm{ad}(\mathfrak{k})$ and $U^\perp$ its orthogonal complement in $\mathfrak{p}$ with respect to Q. Then we have for all $u \in U, v \in U^\perp, k \in \mathfrak{k}$:

$$Q(u, \mathrm{ad}(k)v) = -Q(\mathrm{ad}(k)u, v) = 0.$$

Hence $\mathrm{ad}(k)v$ is contained in $U^\perp$, because it is orthogonal to u. Therefore we can decompose $\mathfrak{q}_i$ in subvectorspaces invariant minimal under $\mathrm{ad}\mathfrak{k}$ and there exists a direct sum decomposition of $\bigoplus_{i=1}^r \mathfrak{q}_i$ in $\bigoplus_{i=1}^a \mathfrak{p}_i$, where each $\mathfrak{p}_i$ is invariant minimal under $\mathrm{ad}\mathfrak{k}$. Each $\mathfrak{p}_i$ is contained in a $\mathfrak{q}_j$ and $\mathfrak{p}_i$ and $\mathfrak{p}_l$ are orthogonal with respect to Q if $l \neq i$.
Furthermore we have $[\mathfrak{p}_i, \mathfrak{p}_j] = 0$ for $i \neq j$, since for $k \in \mathfrak{k}$:

$$B(k, [\mathfrak{p}_i, \mathfrak{p}_j]) = B([k, \mathfrak{p}_i], \mathfrak{p}_j) \subset B(\mathfrak{p}_i, \mathfrak{p}_j) = 0,$$

since $\mathfrak{p}_i$ and $\mathfrak{p}_j$ are orthogonal with respect to Q. B is non-degenerate on $\mathfrak{k}$, therefore $[\mathfrak{p}_i, \mathfrak{p}_j] = 0$. The same argumentation shows that $[\mathfrak{q}_i, \mathfrak{q}_j] = 0$, if $i \neq j$. We will need this fact later.
Let now

$$\mathfrak{g}_i := [\mathfrak{p}_i, \mathfrak{p}_i] + \mathfrak{p}_i, \ (i = 1, \ldots, a).$$

$[\mathfrak{p}_i, \mathfrak{p}_i]$ is contained in $\mathfrak{k}$ because σ acts as the identity on it, thus the sum is direct. Furthermore we have

$$[\mathfrak{g}_i, \mathfrak{g}_j] = 0, \ (i \neq j), \ [\mathfrak{k}, \mathfrak{g}_i] \subset \mathfrak{g}_i \ (1 \leq i, j \leq a)$$

and

$$[\mathfrak{p}_0, \mathfrak{g}_i] = 0, \ [\mathfrak{p}, \mathfrak{g}_i] = [\mathfrak{p}_i, \mathfrak{g}_i] \subset \mathfrak{g}_i.$$

$\mathfrak{g}_i$ is an ideal of $\mathfrak{g}$ which follows from the Jacobi identity and the equations above. Therefore the Killing form on $\mathfrak{g}_i$ is the Killing form on $\mathfrak{g}$ restricted to $\mathfrak{g}_i \times \mathfrak{g}_i$. It is non-degenerate on the $\mathfrak{p}_i$, since it is a multiple of Q and it is non-degenerate on $[\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{k}$. Hence it is non-degenerate on $\mathfrak{g}_i$ and therefore $\mathfrak{g}_i$ is semi-simple. We have, again by the above formulae and the Jacobi-identity for $i \neq j$

$$B(\mathfrak{g}_i, \mathfrak{g}_j) = B(\mathfrak{p}_i, \mathfrak{p}_j) + B(\mathfrak{p}_i, [\mathfrak{p}_j, \mathfrak{p}_j]) + B(\mathfrak{p}_j, [\mathfrak{p}_i, \mathfrak{p}_i]) + B([\mathfrak{p}_i, \mathfrak{p}_i], [\mathfrak{p}_j, \mathfrak{p}_j]) = 0.$$

Therefore the $\mathfrak{g}_i$ are linearly independent. Let $\mathfrak{m}$ be their sum. It is a semi-simple ideal of $\mathfrak{g}$ invariant under σ, therefore $\mathfrak{g} = \mathfrak{g}_0 \times \mathfrak{m}$, with $\mathfrak{g}_0 = \mathfrak{z}(\mathfrak{g})$. Denoting by σ_i the restriction of σ to $\mathfrak{g}_i$ we have:

$$(\mathfrak{g}, \sigma) = (\mathfrak{g}_0, \sigma_0) \times (\mathfrak{g}_1, \sigma_1) \times \ldots (\mathfrak{g}_a, \sigma_a),$$

with $(\mathfrak{g}_i, \sigma_i)$ semi-simple, irreducible for $i \geq 1$ and $(\mathfrak{g}_i, \sigma_i)$ reduced for all i's. $(\mathfrak{g}_0, \sigma_0)$ is flat. □

This proof shows that for $i \neq 0$, $B|_{\mathfrak{p}_i}$ is either negative definite or positive definite, since it is a multiple of the positive definite bilinear form Q. Therefore either $\mathfrak{g}_i$ is compact (if $B|_{\mathfrak{p}_i}$ is negative definite) or $\mathfrak{g}_i$ is non-compact and $[\mathfrak{p}_i, \mathfrak{p}_i]$ is a maximal compact subalgebra. In the first case we say that it is of *compact type*, in the second it is of *non-compact type*. This property is directly related to the sectional curvature.

Corollary 1.3.8. *Let $(\mathfrak{g}, \sigma)$ an orthogonal involutive Lie algebra, G a Lie group with Lie algebra $\mathfrak{g}$ and K a subgroup corresponding to $\mathfrak{k}$. Assume that K is connected and closed. Let Q be an arbitrary K-invariant Riemannian structure on G/K. Then*

i) If G/K is of Euclidean type, then G/K has sectional curvature everywhere $= 0$.

ii) If G/K is of compact type, then G/K has sectional curvature everywhere ≥ 0.

iii) If G/K is of non-compact type, then G/K has sectional curvature everywhere ≤ 0.

Proof. The tangent space at o can be identified with $\mathfrak{p}$. Let S be two-dimensional subspace of T_pM and X and Y two orthonormal vectors in S. Proposition 1.2.11 tells us that $R(X,Y)X = -[[X,Y],X]$ and by the definition of the sectional curvature (Definition 1.1.16, we get

$$K(S) = -Q_0(R(X,Y)X,Y) = Q_0([[X,Y],X],Y)$$

since the latter is zero in the Euclidean case, we have proved i).
The other two cases are not difficult. We assume that $\mathfrak{g}$ is semi-simple and use the results of Theorem 1.3.7. We decomposed $\mathfrak{p}$ into subspaces $\mathfrak{q}_i$ as eigenspaces to the eigenvalues c_i of the matrix A defined by $Q_0(AX,Y) = B(X,Y)$ and for $X_i, Y_i \in \mathfrak{q}_i$ we have

$$B(X_i, Y_i) = c_i Q_0(X_i, Y_i).$$

And $[\mathfrak{q}_i, \mathfrak{q}_j] = 0$ holds if $i \neq j$. With $X = \sum X_i$ and $Y = \sum Y_i$ ($X_i, Y_i \in \mathfrak{q}_i$ we get $[X,Y] = \sum [X_i, Y_i]$ and $[[X_i,Y_i],X] = [[X_i,Y_i],X_i]$. Hence

$$K(S) = Q_0([[X,Y],X],Y) = \sum_i Q_0([[X_i,Y_i],X_i],Y_i) = \sum \frac{1}{c_i} B([X_i,Y_i],[X_i,Y_i]). \tag{9}$$

If G/K is of compact type, by definition the c_i are all negative and the sectional curvature is negative. If G/K is of non-compact type, the c_i are all positive and the sectional curvature is positive. □

Proposition 1.3.9. *For an involutive semi-simple reduced Lie algebra* $(\mathfrak{g}, \sigma)$

$$\mathfrak{g} = \mathfrak{p} \oplus [\mathfrak{p}, \mathfrak{p}]$$

holds.

Proof. First note, that the Killing-form B on $\mathfrak{g}$ is non-degenerate, since $\mathfrak{g}$ is semi-simple. Therefore we can decompose $\mathfrak{g}$ in a direct sum of $\mathfrak{p} + [\mathfrak{p},\mathfrak{p}]$ and $\mathfrak{c} := (\mathfrak{p} + [\mathfrak{p},\mathfrak{p}])^\top$. Since $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$, the first sum is also direct. $\mathfrak{c}$, being orthogonal to $\mathfrak{p}$, is contained in $\mathfrak{k}$. If we show, that $\mathfrak{c}$ is zero, we are done, because B is non-degenerate. To do that we will show that it is an ideal of $\mathfrak{g}$ contained in $\mathfrak{k}$ and must therefore be zero.
Fix $c \in \mathfrak{c}$ and let $p, \tilde{p} \in \mathfrak{p}$ and $g_0 = p_0 + k_0 \in \mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$.

$$B(g_0, [p,c]) = B([g_0,p],c) = B(\underbrace{[p_0,p]}_{\in[\mathfrak{p},\mathfrak{p}]},c) + B(\underbrace{[k_0,p]}_{\mathfrak{p}},c) = 0,$$

hence $[p,c] = 0$, since g_0 was arbitrary and B is non-degenerate. With the Jacobi-identity we get immediately that $[[p,\tilde{p}],c] = 0$. We are not far from our result, since

$$\begin{aligned} B([g_0,c],p) &= B(g,[c,p]) = 0 \\ B([g_0,c],[p,\tilde{p}]) &= B(g,[c,[p,\tilde{p}]]) = 0. \end{aligned}$$

This shows that $\mathfrak{c}$ is an ideal of $\mathfrak{g}$ contained in $\mathfrak{k}$ which must be zero, since $\mathfrak{g}$ is reduced. Therefore

$$\mathfrak{g} = \mathfrak{p} \oplus [\mathfrak{p}, \mathfrak{p}]$$

holds. □

This property characterizes the isometry group of a symmetric space. In Remark 1.2.8 we announced the following proposition:

Proposition 1.3.10. *Let G be a connected semi-simple Lie group, K a compact subgroup and σ an involution on G. Assume that*

i) $M = G/K$ is connected,

ii) there exists an involution σ of G such that $Lie(G)^{d\sigma_e} = Lie(K)$ and $K^\sigma = K$ and

iii) G acts faithfully on M.

Then G is the identity component of the group of isometries on M.

Proof. Let G' denote the group of isometries of M. M can be written in the form G'/K' where K' is a compact subgroup of G'. G acts by definition of M isometrically on M and since it is connected, it can be identified with a subgroup of the identity component of G'. $\mathfrak{g}'$, the Lie algebra of G', is reduced since, by Proposition 1.2.12 K' and therefore by Lemma A.1.5 $\mathfrak{k}$ act faithfully on $\mathfrak{p} \simeq T_pM$. Furthermore $\mathfrak{g}'$ is semi-simple, since G' has no abelian factor. Hence $\mathfrak{g}' = \mathfrak{p} \oplus [\mathfrak{p}, \mathfrak{p}]$.
The Lie algebra $\mathfrak{g}$ of G is a subalgebra of $\mathfrak{g}'$ and it decomposes therefore into eigenspaces $\tilde{p} \oplus \tilde{k}$ with respect to σ. Since $\tilde{\mathfrak{p}} \simeq T_pM \simeq \mathfrak{p}$ and $\mathfrak{g}$ is by assumption reduced and semi-simple, we have

$$\mathfrak{g} = \mathfrak{p} \oplus [\mathfrak{p}, \mathfrak{p}] = \mathfrak{g}'$$

and therefore G is the identity component of G'. □

Remark 1.3.11. Helgason classifies in [He78] *irreducible* orthogonal involutive Lie algebras, where a Lie algebra is irreducible, if $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$ acts irreducibly and faithful on $\mathfrak{p}$. By assumption in our case here, $(\mathfrak{g}, \sigma)$ is reduced, hence $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$ acts faithful on $\mathfrak{p}$. Let $\mathfrak{i} \subset \mathfrak{k}_i$ be an ideal of $\mathfrak{g}_i$, i.e. $[\mathfrak{i}, \mathfrak{g}_i] \subset \mathfrak{i}$. Then $\mathfrak{i}$ is an ideal of $\mathfrak{g} = \bigoplus_{i=0}^{a} \mathfrak{g}_i$ since $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$ if $(i \neq j)$. We have assumed $(\mathfrak{g}, \sigma)$ to be reduced, therefore $\mathfrak{i}$ is trivial and hence $\mathfrak{g}_i$ is reduced. By construction of $\mathfrak{p}_i$ the action of $\mathrm{ad}_{\mathfrak{g}}\mathfrak{k}$ is irreducible, hence the $(\mathfrak{g}_i, \sigma_i)$ are irreducible in the sense of Helgason.

By Theorem 1.3.7 we can assume that $(\mathfrak{g}, \sigma)$ is either flat or a semi-simple reduced orthogonal involutive Lie algebra. If $(\mathfrak{g}, \sigma)$ is flat, it corresponds to an euclidean space, since we have a vector space with vanishing Lie bracket.
A compact semi-simple Lie algebra $\mathfrak{g}$ (ignoring the involution) decomposes in a product of simple ideals

$$\mathfrak{g} = \bigoplus_{i=1}^{n} \mathfrak{h}_i.$$

$\sigma(\mathfrak{h}_i)$ is simple, hence $\sigma(\mathfrak{h}_i) = \mathfrak{h}_j$. We have two cases:

i) $i = j$. In this case $(\mathfrak{h}_i, \sigma|_{\mathfrak{h}_i})$ is a simple reduced orthogonal involutive Lie algebra. $\mathfrak{h}_i = \mathfrak{k}_i \oplus \mathfrak{p}_i$ where $\mathfrak{k}_i$ and $\mathfrak{p}_i$ are the eigenspaces with respect to the eigenvalues 1 and -1.

ii) $i \neq j$. Then $\mathfrak{h}_j$ and $\mathfrak{h}_i$ are isomorphic via σ and $(\mathfrak{h}_i \oplus \sigma(\mathfrak{h}_i), \sigma|_{\mathfrak{h}_i \oplus \sigma(\mathfrak{h}_i)})$ is a reduced orthogonal involutive Lie algebra. σ acts on it via $(x, y) \mapsto (y, x)$.

They are both invariant under σ. This gives us a new decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \bigoplus_i \mathfrak{l}_i,$$

where $\mathfrak{l}_i$ is an ideal in $\mathfrak{g}$, invariant under σ. It can therefore be decomposed into eigenspaces $\mathfrak{l}_i = \mathfrak{k}_i + \mathfrak{p}_i$. The $\mathfrak{l}_i$ are ideals, hence $[\mathfrak{l}_i, \mathfrak{l}_j] \subset \mathfrak{l}_i$ for all i and j. Therefore $[\mathfrak{l}_i, \mathfrak{l}_j] \subset \mathfrak{l}_i \cap \mathfrak{l}_j$. The last is trivial if $i \neq j$, therefore $[\mathfrak{k}_i, \mathfrak{p}_j] = [\mathfrak{k}_i, \mathfrak{k}_j] = 0$ if $i \neq j$. Since $\mathfrak{k}$ acts by assumption irreducibly on $\mathfrak{p}$ and $\operatorname{ad}(\mathfrak{k}_i)(\mathfrak{p}_j) = 0$ for $i \neq j$ we have $\mathfrak{p} = \mathfrak{p}_i$ for some i. We can assume $i = 1$ without loss of generality. For $i \neq 1$ we have $[\mathfrak{k}_j, \mathfrak{p}_1] = 0$, hence $\mathfrak{k}_j$ is an ideal of $\mathfrak{g}$ in $\mathfrak{k}$. But $\operatorname{ad}(\mathfrak{k})$ acts faithfully on $\mathfrak{p}$, hence $\mathfrak{k} = \mathfrak{k}_1$. This shows that $\mathfrak{g} = \mathfrak{l}_1$.
An irreducible Lie algebra $\mathfrak{g}$ of compact type has the form

I. $(\mathfrak{g}, \sigma)$ is a compact simple Lie algebra and σ is any involution on it.

II. $\mathfrak{g}$ is the direct sum $\mathfrak{l}_1 \oplus \mathfrak{l}_2$ and σ acts on it via $(x, y) \mapsto (y, x)$.

If $(\mathfrak{g}, \sigma)$ is a non-compact irreducible orthogonal involutive Lie algebra, we can construct a dual compact Lie algebra by putting

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \mapsto \tilde{\mathfrak{g}} := \mathfrak{k} \oplus i\mathfrak{p} \subset \mathfrak{g}^{\mathbb{C}}.$$

$\tilde{\mathfrak{g}}$ is also irreducible, since taking this dual does not change the action of $\mathfrak{k}$ on $\mathfrak{p}$ respectively $i\mathfrak{p}$.
Let $\tilde{\mathfrak{g}}$ be in case I. Assume $\mathfrak{g}$ to be not simple. Then there exists a decomposition in non-zero ideals $\mathfrak{n}_1$ and $\mathfrak{n}_2$. They can be decomposed into $\mathfrak{k}_1 + \mathfrak{p}_1$ and $\mathfrak{k}_2 + \mathfrak{p}_2$, hence $(\mathfrak{k}_1 + i\mathfrak{p}_1) + (\mathfrak{k}_2 + i\mathfrak{p}_2)$ is a decomposition of $\tilde{\mathfrak{g}}$ into non-zero ideals. Since this is impossible, $\mathfrak{g}$ is simple. A similar argumentation shows that the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ is simple, too.
If $\tilde{\mathfrak{g}}$ is in case II., its dual admits a complex structure (see [He78, Thm. V.2.4]) and it is simple. Therefore we have the following two possibilities for the non-compact case.

III. $\mathfrak{g}$ is a simple, noncompact Lie algebra over $\mathbb{R}$, the complexification $\mathfrak{g}_{\mathbb{C}}$ is a simple Lie algebra over $\mathbb{R}$ and σ is an involutive automorphism of $\mathfrak{g}$ such that the fixed points form a compactly imbedded subalgebra.

IV. $\mathfrak{g}$ is a simple Lie algebra over $\mathbb{C}$. Here σ is the conjugation of $\mathfrak{g}$ with respect to a maximal compactly imbedded subalgebra.

Theorem 1.3.12. *Let M be a simply connected symmetric space of compact or non-compact type. Then M is product*

$$M = M_1 \times \ldots \times M_n,$$

where the factors M_i are irreducible.

Proof. We proofed that the Lie algebra $\mathfrak{g}$ of its group of isometries decomposes into semi-simple factors (Theorem 1.3.7)

$$\mathfrak{g} = \mathfrak{g}_1 \times \ldots \times \mathfrak{g}_n$$

with $\mathfrak{g}_i = \mathfrak{p}_i \oplus \mathfrak{k}_i$. Hence $G = G_1 \times \ldots \times G_n$ where $\mathfrak{g}_i$ is the Lie algebra of G_i. Further $K = K_1 \times \ldots \times K_n$ and $\mathfrak{k}_i$ is the Lie algebra of K_i. Therefore $M = M_1 \times \ldots \times M_n$. □

1.4 Hermitian Symmetric Spaces

Definition 1.4.1. A *Hermitian manifold* is a differentiable Riemannian manifold with the Riemannian connection ∇ whose tangent bundle is endowed with a Hermitian structure, i.e. a $(1,1)$-tensor J with $J^2 = -1$, which leaves the metric invariant ($g(JX, JY) = g(X, Y)$).

Like in the study of riemannian symmetric spaces we start with a local property.

Proposition 1.4.2. *Let M be a Hermitian manifold. The following conditions are equivalent:*

i) $\nabla R = 0 = \nabla J$*;*

ii) for all $x \in M$, $s_x : y \mapsto \exp_x(-\log_x y)$ is a local automorphism at x of M, i.e. s_x leaves ∇ and J invariant.

Proof. i)$\Rightarrow$ii). From Corollary 1.1.13 we know that s_x is a local diffeomorphism which leaves ∇ invariant, because $\nabla R = 0$. J is invariant under parallel transport, since $\nabla J = 0$. Since ds_x acts by multiplication with -1 on T_xM, it commutes with J_x and hence J and s_x are compatible.
ii)$\Rightarrow$i). If ii) is satisfied every canonical tensor of odd degree is zero, because it is invariant by s_x. Therefore ∇J and ∇R are both zero. (Same argumentation as in Corollary 1.1.13.) $\square$

Definition 1.4.3. A Hermitian manifold M is *locally symmetric* if the equivalent conditions of Proposition 1.4.2 are satisfied. It is called *Hermitian symmetric* if it is connected and for all $x \in M$ there exits an involutive automorphism s_x of M with x as isolated fixed point.

Everything we said about Riemannian symmetric spaces applies to Hermitian symmetric spaces, too.

Corollary 1.4.4. *A locally symmetric Hermitian manifold is Kähler. If g is the metric on M, then*

$$\omega(X, Y) := g(X, JY)$$

is a Kähler form, i.e. ω is a symplectic, closed 2-form.

The following fact is a central fact in our study of hermitian symmetric spaces.

Corollary 1.4.5. *Let M be a simply connected symmetric Hermitian space and let G be the identity component of its group of automorphisms. Let x be a point of M and let K its stabilizer. There exists a homomorphism $u_x : U^1 \to K$ such that $u_x(z)$ induces multiplication*[6] *by z on T_xM.*

Proof. We use Theorem 1.1.9 and Proposition 1.1.10 to construct u_x. For a given $z \in U^1$, we have in the notation of Theorem 1.1.9 $M = M'$, $x = x'$, $\nabla = \nabla'$ and A is multiplication with z. By Proposition 1.4.2 and the definition of the Riemannian connection we have $\nabla T = 0 = \nabla R$. If we show that R is invariant under the automorphism $v \mapsto zv$ on T_pM (i.e. $\bar{z}R(zX, zY)zZ = R(X, Y)Z$), then we are done. Note that

$$g(R(X, Y)Z, T) = g(R(Z, T)X, Y) \tag{10}$$

[6] $U^1 := \{z \in \mathbb{C} : |z| = 1\}$ and for $v \in T_xM$ and $z = a + ib \in U^1$we define $z \cdot v$ by $av + bJv$.

holds. A proof of this fact can be found[7] in [He78, Lem. I.12.5].
By assumption J is invariant under parallel transport. Hence it commutes with the contravariant derivative ∇_X for all X. Therefore

$$R(X,Y)(JZ) = \nabla_X\nabla_Y(JZ) - \nabla_Y\nabla_X(JZ) - \nabla_{[X,Y]}(JZ) = JR(X,Y)Z.$$

Now we can show what we wanted:

$$\begin{aligned} g(\bar{z}R(zX,zY)zZ,T) &= g(R(zX,zY)\bar{z}zZ,T) = g(R(Z,T)zX,zY) = g(\bar{z}R(Z,T)zX,Y) \\ &= g(R(Z,T)\bar{z}zX,Y) = g(R(X,Y)Z,T). \end{aligned}$$

□

Remember the maps $\tau(\gamma,u)$ introduced in Corollary 1.1.11. They are composites of symmetries and are therefore automorphisms of M. As before, we have

$$M = G/K \text{ and } \mathrm{Lie}(G) = \mathfrak{k} \oplus \mathfrak{p}$$

where G is the identity component of the Lie group of automorphisms of M and K the stabilizer of a $x \in M$. σ denotes the automorphism of G induced by s_x.

Before we continue, we have a look at an example.

Example 1.4.6. Let V be a n-dimensional complex vector space endowed with a non-degenerate Hermitian form h of signature (p,q), i.e. p is the maximal dimension of a subspace $L \subset V$ such that $h|_L$ is positive definite. We can choose a basis $(e_1,\dots,e_p,e_{p+1},\dots,e_{p+q})$ of V such that h is of the form

$$h(z,w) = \sum_{i=1}^{p} z_i\bar{w}_i - \sum_{j=1}^{q} z_{j+p}\bar{w}_{j+p}.$$

Denote W^+ the space spanned by $e_1,\dots,e_p$. Its complement with respect to h is spanned by $e_{p+1},\dots,e_{p+q}$. Denote by X the space of p-dimensional subspaces $W \subset V$ with $h|_W$ is positive definite. X is an open subset of the Grassmannian $Gr_p(V)$, hence a manifold. Fix $W \in X$. Since $h|_W$ is non-degenerated, we can decompose V into the direct sum of W and $W^\perp$, the orthogonal complement with respect to h. An element x of V can be decomposed into $w + w^\perp$. Denote s_W the map which maps $w + w^\perp$ to $w - w^\perp$. It is the reflection with respect to W. For two orthogonal vectors $x, y \in V$(i.e. $h(x,y) = 0$) we have

$$\begin{aligned} h(x+y,x+y) =& h(x,x) + h(y,y) + h(x,y) + h(y,x) = h(x,x) + h(y,y) - h(y,x) - h(x,y) \\ =& h(x-y,x-y). \end{aligned}$$

Therefore the reflection s_W lies in $SU(p,q)$, the subgroup of $SL(n,\mathbb{C})$ which contains all matrices which leaves h invariant and s_W induces a map from X into X, also denoted by s_W. It is involutive with W as an isolated fixed point. This makes X into a symmetric space.

Now we want to see how X can be realized as a homogeneous space G/K. Let $G := SU(p,q)$ and fix $W^+ \in X$ as a base point. G acts transitively on X, since one can choose

[7] Or one can proof it oneself by using the definition of R and the invariance of g under paralleltransportation in the form $X \cdot g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ to show that $g(R(X,Y)Z,T) = -g(R(X,Y)T,Z)$ and use this, the Bianchi identity and Lemma 1.1.17 to prove Equation 10.

a basis of W and one of $W^\perp$ and write down a matrix for h with respect to this basis. This matrix is hermitian, hence diagonalizable and by scaling it can be brought to the form $diag(1,\ldots,1,-1,\ldots,-1)$ with p-times 1 and q-times -1. Therefore W can be mapped onto W^+ by an element of G and G acts transitively on X.
Let g be in G. Write g in the form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A \in M(p,p,\mathbb{C})$, $B \in M(p,q,\mathbb{C})$, $C \in M(q,q,\mathbb{C})$ and $D \in M(q,p,\mathbb{C})$. As a direct calculation shows the following conditions must hold:

$$\begin{aligned} &A^*A - C^*C = \mathrm{id}_p \\ &B^*B - D^*D = -\mathrm{id}_q \\ &B^*A - D^*C = 0 \\ &\det(g) = 1. \end{aligned}$$

Such a matrix g maps W^+ into itself if and only if $C = 0$. From the third equation follows that in this case $B = 0$ because by the first equation A is invertible. The first two equations tell us that A and D are hermitian, since they leave a positive-definite Hermitian form invariant. Therefore the stabilizer of W is equal to $S(U(p) \times U(q))$ the matrices of $U(p) \times U(q)$ with $\det(u_p)\det(u_q) = 1$ for $u_p \in U(p)$ and $u_q \in U(q)$. We can write X as the quotient $G/K = SU(p,q)/S(U(p) \times U(q))$. A discussion of $SU(1,1)$ and the associated symmetric space, the hyperbolic plane, can be found in Example 2.5.3 and Proposition A.4.7.

Remark 1.4.7. The complex structure on M induces on $\mathfrak{p}(= T_xM)$ a complex structure J, which is invariant under K, i.e. J commutes with $dk|_e$ for $k \in K$. Conversely, if $M = G/K$ is Riemannian and symmetric, every complex K-invariant structure on $\mathfrak{p}$ endows M with a hermitian symmetric structure, since the structure can transported along the geodesics to every point and this is well-defined since the hermitian structure on $\mathfrak{p}$ is K-invariant.

Remark 1.4.8. If J_e is a $\mathfrak{k}$-invariant complex structure on $\mathfrak{p}$, i.e. $J([k,p]) = [k, Jp]$ then the $\mathfrak{p}_i$ from theorem 1.3.7 are complex subspaces, because they are disjoint representations of $\mathfrak{k}$ and hence $[\mathfrak{k}, J\mathfrak{p}_i] \subset \mathfrak{p}_i$. By Theorem 1.3.7 the Lie algebra of the group of automorphisms decomposes into a direct sum of a flat Lie algebra and semi-simple involutive Lie algebras $\mathfrak{g}_i$ which can be decomposed as $\mathfrak{k}_i + \mathfrak{p}_i$. There exists subgroups K_i corresponding to $\mathfrak{k}_i$. Hence M decomposes into Hermitian factors

$$M = M_0 \times \prod M_i,$$

where M_0 is flat.

Remark 1.4.9. Let G be the identity component of the group of isometries of M. Assume that it is semi-simple. Then it has the same Lie algebra $[\mathfrak{p},\mathfrak{p}] \oplus \mathfrak{p}$ as the analogous group for the universal covering $\tilde{M}$ of M (Proposition 1.3.9). Therefore we have by Corollary 1.4.5 a map $u_x : U^1 \to \tilde{K}$ where $\tilde{K}$ is the stabilizer of a point $x \in \tilde{M}$. Composing u_x with the canonical projection $\tilde{M} \to M$ gives a map from U^1 to K the stabilizer of a point in M. Since u_x acts as multiplication with z on $\tilde{\mathfrak{p}}$ and since the canonical projection is a local diffeomorphism, the composition acts as multiplication with z on $\mathfrak{p}$.

Proposition 1.4.10. *i) The subgroup K is the centralizer of $u_x(U^1)$ in G and K is connected.*

ii) The center of G *is trivial.*

iii) The Hermitian symmetric space M *is simply connected.*

iv) If $(\mathfrak{g}, \sigma)$ *is indecomposable, then* $u_x(U^1)$ *is the center of* K*.*

Proof. i) Let K' be the centralizer of $u_x(U^1)$. First we show $K' \subset K$ using $Lie(K') \subset \mathrm{Lie}(K)$. Remark that the Lie algebra of K is the fixed point set of the action of $d\sigma_e$ in $\mathfrak{g}$. As σ is multiplication with -1 on $\mathfrak{p}$, we have $\sigma = \mathrm{Ad}u_x(-1)$. By definition $\mathrm{Ad}u_x(-1)(= \sigma)$ is the differential of $g \mapsto u_x(-1)gu_x(-1)^{-1}$, which is the identity on K'. Hence $d\sigma_e$ leaves $\mathrm{Lie}(K')$ fixed and therefore $\mathrm{Lie}(K') \subset \mathrm{Lie}(K)$. K' is the centralizer of a torus, thus it is connected (see Appendix, Proposition A.4.1). Therefore K' is a subgroup of K.

Conversely, note that the representation of $\mathfrak{k}$ on $\mathfrak{p}$ and the complex structure J on $\mathfrak{p}$ commute and and that the representation of $\mathfrak{k}$ on $\mathfrak{p}$ is faithful. Therefore we have $K \subset K'$, hence $K = K'$ and K is connected.

ii) We have seen above that $k\tau(\gamma, u)k^{-1} = \tau(k\gamma, u)$ for all $k \in K$, where γ is a geodesic starting in x. By i) the center Z of G must be contained in K. If k is in Z it must commute with $\tau(\gamma, u)$ and $u \in \mathbb{R}$ and hence $\tau(\gamma, u) = \tau(k\gamma, u)$ for all $k \in Z$. But since K acts faithful on $\mathfrak{p}$ this is only possible for $k = e$, i.e. Z is trivial.

iii) Let $\pi : \tilde{G} \to G$ be the universal covering of G. The Lie algebras of $\tilde{G}$ and G are both equal to $\mathfrak{p} \oplus [\mathfrak{p}, \mathfrak{p}]$. On $\tilde{G}$ exists an involution $\tilde{\sigma}$ with $d\tilde{\sigma} = d\sigma$. Denote by $\tilde{K}$ the fixed point group of $\tilde{G}$, it is connected. We have $\pi(\tilde{K}) = K$, since the Lie algebras of $\tilde{K}$ and K are equal and they are both connected. From Remark 1.2.8 we know that $\tilde{G}/\tilde{K}$ is a symmetric space. We can apply i) and $\tilde{K}$ contains the center of $\tilde{G}$. Therefore $\pi^{-1}(K) = \tilde{K}$. Hence

$$\tilde{G}/\tilde{K} = G/K = M,$$

is simply connected, since $\tilde{G}$ is and $\tilde{K}$ is connected.

iv) Since $\mathfrak{g}$ is indecomposable, the representation of K in $\mathfrak{p}$ is irreducible. The commutator of K in $\mathrm{End}(\mathfrak{p})$ the same as the commutator of K in $\mathrm{End}_{\mathbb{C}}(\mathfrak{p})$, since the complex structure on J commutes with the K-action. By Schur's lemma the center of K consists only of multiples of the identity matrix. Since K is compact, the center is, as a closed subgroup, compact to. Hence the center is $u_x(U^1)$.

□

Proposition 1.4.11. *Let* M *be a symmetric Riemannian space and let* G *be the identity component of its group of automorphisms,* $x \in M$*,* K *its stabilizer in* G *and* σ *the involution on* G *generated by conjugation with* s_x*. Assume that* G *is semi-simple,* K *is connected and* $(\mathfrak{g}, \sigma)$ *is indecomposable. The following conditions are equivalent:*

i) the representation $\mathfrak{p}$ *of* $\mathfrak{k}$ *is not absolutely simple (i.e.* $\mathfrak{p}_{\mathbb{C}}$ *is not simple);*

ii) the center of $\mathfrak{k}$ *is non-zero;*

iii) M *admits the structure of a symmetric Hermitian space (compatible with its Riemannian structure).*

Proof. i)$\Rightarrow$ iii). By assumption $\mathfrak{p}_{\mathbb{C}}$ decomposes into a direct sum of two non-trivial complex subrepresentations $\mathfrak{p}_-$ and $\mathfrak{p}_+$. We can embed $\mathfrak{p}$ into $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p} \oplus i\mathfrak{p}$ and project $\mathfrak{p}_{\mathbb{C}}$ onto $\mathfrak{p}_{\pm}$. Composing these maps we get two maps

$$\mathfrak{p} \to \mathfrak{p}_{\mathbb{C}} \to \mathfrak{p}_{\pm}.$$

Note that $p \in \mathfrak{p} \subset \mathfrak{p}_{\mathbb{C}}$ can be written uniquely as $p_+ + p_-$ with $p_- \in \mathfrak{p}_-$ and $p_+ \in \mathfrak{p}_+$. Since $\mathfrak{p} \cap \mathfrak{p}_- = \{0\} = \mathfrak{p} \cap \mathfrak{p}_+$ this maps are injective. Therefore $\mathfrak{p}_+$ and $\mathfrak{p}_-$ must have the same dimension, hence the maps are isomorphisms. Therefore we can transport the complex structure of $\mathfrak{p}_+$ to $\mathfrak{p}$. Since $\mathfrak{p}$ can be identified with the tangent space of M in some point this tangent space can be endowed with a complex structure too. By Remark 1.4.7 this induces a complex structure on M.
iii) $\Rightarrow$ i). Let J denote the complex structure on $\mathfrak{p}$. Since J^2 is symmetric it can be diagonalized over $\mathbb{C}$ with eigenvalues i and $-i$. The eigenspace to i is an $\mathrm{ad}(\mathfrak{k})$ invariant subspace of the complexification of $\mathfrak{p}$, therefore the representation is not absolutely simple.
iii) $\Rightarrow$ ii) follows from Proposition 1.4.10.
ii) $\Rightarrow$ iii) By Schur's Lemma (Lemma A.3.11) the center of K acts on $\mathfrak{p}_{\mathbb{C}}$ as multiplication with complex diagonal matrices. Since K is compact, it contains only multiples of the identity matrix with U^1. Since the representation is faithful, we get a complex structure on $\mathfrak{p}$ and therefore one on M. □

Example 1.4.12. This shows that $P(n,\mathbb{R})$ is a Hermitian symmetric space if and only if $n = 2$. We saw that $P(n,\mathbb{R}) = SL(n,\mathbb{R})/SO(n,\mathbb{R})$ and the stabilizer of a point equals $SO(n,\mathbb{R})$, which has non-trivial center if and only if $n = 2$ (see [Hi91, p. 328]).

Example 1.4.13. Remember Example 1.2.9. We saw there that the symmetric space

$$SO(2,n)/S(O(2) \times O(n))$$

can be realized as the space $\mathcal{G}$ of two dimensional subspaces of $\mathbb{R}^2 \oplus \mathbb{R}^n$ on which the bilinear form of signature $(2,n)$ is positive definite. Since we are dealing with two-dimensional real subspaces of $\mathbb{R}^{n+2}$ it seems possible, that we can put a complex structure on $\mathcal{G}$. And this is in fact true: since the center of $S(O(2) \times O(n))$ contains at least $O(2)$, it is non-trivial and by Proposition 1.4.11, there exists a complex structure on $\mathcal{G}$. There exists a holomorphic diffeomorphism of

$$D := \{z \in \mathbb{C}^n | 1 - 2\langle z, z\rangle + |zz^\top| > 0,\ |z| < 1\}$$

to M.

Remark 1.4.14. Let $M = G/K$ be a compact simple symmetric hermitian space. For $x \in M$ let $u_x : U^1 \to G$ be as before. In the representation $\mathrm{Ad}u_x$ on $Lie(G)_{\mathbb{C}}$ it acts only through the characters 1 (on $\mathfrak{k}_{\mathbb{C}}$) and z and $\bar{z}$ (on $\mathfrak{p}_{\mathbb{C}}$).

Conversely, if G is a compact adjoint simple group and $u : U^1 \to G$ acts like above on $Lie(G)_{\mathbb{C}}$. Its centralizer K is connected since it is the centralizer of a torus. It has for Lie algebra the subspace fixed by the involution $\mathrm{Ad}u(-1)$ of $Lie(G)$ and G/K is Hermitian symmetric.

The classification problem of irreducible Hermitian symmetric spaces is therefore reduced to the classification of compact adjoint simple groups with a morphism $u : U^1 \to G$ such that its representation on $\mathrm{Lie}(G)_{\mathbb{C}}$ acts only through the characters 1 (on $\mathfrak{k}_{\mathbb{C}}$) and z and $\bar{z}$ (on $\mathfrak{p}_{\mathbb{C}}$).

2 Some Topics on Symmetric Spaces

2.1 Complexification and Cartan Decomposition

Let V be a finite dimensional vector space over $\mathbb{R}$. A *complex structure* of V is a $\mathbb{R}$-linear endomorphism J with $J^2 = -1$. This endomorphism is something like the multiplication with i in a complex vector space. A vector space V with a complex structure can be turned into a vector space $\tilde{V}$ over $\mathbb{C}$ by putting

$$(a + ib)X := aX + bJX,$$

for $a, b \in \mathbb{R}$ and $X \in V$. The dimension of $\tilde{V}$ over $\mathbb{C}$ is $\frac{1}{2}\dim_{\mathbb{R}} V$.

A Lie algebra $\mathfrak{v}$ over $\mathbb{R}$ is said to have a complex structure, if it has a complex structure J as a vector space and

$$[X, JY] = J[X, Y] \tag{11}$$

holds for $X, Y \in \mathfrak{v}$. Remark that the Lie bracket on a complex Lie algebra is $\mathbb{C}$-linear, hence it commutes with i. If (11) would not hold, it would not be possible to turn it into a complex Lie algebra like we did it with the vector space V. Bur if (11) hold, it is possible as one can easily check.

Now let W be a finite dimensional vector space over $\mathbb{R}$ (without further structure). If one wants to extend W to a complex vector space $W_{\mathbb{C}}$ of the same dimension[8], i.e. make a scalar extension, one has several possibilities (which all yield the same result). The most obvious way may be to take a basis $\{e_1, \ldots, e_n\}$ of W and define

$$W_{\mathbb{C}} := \sum_{i=1}^{n} \mathbb{C}e_i.$$

This works but one has to check that $W_{\mathbb{C}}$ does not depend on the chosen basis. Another possibility uses the observation that for $a, b \in \mathbb{R}$ we have $i(a+ib) = -b+ia$, i.e. multiplication with i maps the vector (a, b) to $(-b, a)$. Defining $J : W \times W \to W \times W$ by $(X, Y) \mapsto (-Y, X)$ for $X, Y \in W$ gives us a complex structure on $W \times W$ which we also denote by $W_{\mathbb{C}}$. Furthermore one can write $W_{\mathbb{C}} := W + iW$ as a formal sum and define multiplication with i in the obvious way. Last but not least one can use some theory and put $W_{\mathbb{C}} := W \otimes_{\mathbb{R}} \mathbb{C}$. All four possibilities are isomorphic. The vector space $W_{\mathbb{C}}$ is called *complexification* of W.

The complexification works for Lie algebra too. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g}_{\mathbb{C}}$ its complexification (as a vector space). It consists of all symbols $X + iY$ with $X, Y \in \mathfrak{g}$. For $X + iY, Z + iT \in \mathfrak{g}_{\mathbb{C}}$ we define

$$[X + iY, Z + iT] := [X, Z] - [Y, T] + i([Y, Z] + [Z, T]).$$

This makes $\mathfrak{g}_{\mathbb{C}}$ to a complex Lie algebra.

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$. A *real form* of $\mathfrak{g}$ is a subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$ considered as a real Lie algebra, s.t.

$$\mathfrak{g} = \mathfrak{g}_0 \oplus J\mathfrak{g}_0$$

as a $\mathbb{R}$ Lie algebra.

In this case each $Z \in \mathfrak{g}$ can be written as $X + iY$ with unique $X, Y \in \mathfrak{g}_0$. The map $\sigma : X + iY \mapsto X - iY$ is called *conjugation* with respect to $\mathfrak{g}_0$.

An important theorem is the following:

[8] $\dim_{\mathbb{R}} W = \dim_{\mathbb{C}} W_{\mathbb{C}}$

Theorem 2.1.1. *Every semi-simple Lie algebra over $\mathbb{C}$ has a real form which is compact.*

See [He78, Thm. III.6.3] for a proof. It uses a root decomposition to write down explicitly the compact real form.

Definition 2.1.2. Let $\mathfrak{g}_0$ be a semi-simple Lie algebra over $\mathbb{R}$. Denote by $\mathfrak{g}$ its complexification an by σ the conjugation in $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. A decomposition of $\mathfrak{g}_0$

$$\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$$

where $\mathfrak{k}_0$ is a subalgebra of $\mathfrak{g}_0$ and $\mathfrak{p}_0$ is a subvectorspace is a *Cartan decomposition* if there exists a compact real form $\mathfrak{g}_k$ of $\mathfrak{g}$ such that

$$\sigma(\mathfrak{g}_k) \subset \mathfrak{g}_k, \quad \mathfrak{k}_0 = \mathfrak{g}_0 \cap \mathfrak{g}_k, \quad \mathfrak{p}_0 = \mathfrak{g}_0 \cap (i\mathfrak{g}_k).$$

The following theorem and Theorem 2.1.3 show that each semi-simple Lie algebra admits a Cartan decomposition.

Theorem 2.1.3. *Let $\mathfrak{g}_0$ be a semi-simple Lie algebra over $\mathbb{R}$, $\mathfrak{g}$ ist complexification and $\mathfrak{u}$ any compact real from of $\mathfrak{g}$. Let σ denote the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. Then there exists an automorphism φ of $\mathfrak{g}$ such that the compact real form $\varphi(\mathfrak{u})$ is invariant under σ.*

One can show that a decomposition of a semi-simple Lie algebra $\mathfrak{g}_0$ into a subalgebra $\mathfrak{k}_0$ and a vector space $\mathfrak{p}_0$ is a Cartan decomposition if an only if the bilinear form

$$B_\tau(X,Y) := -B(X,\tau Y),$$

where $\tau : k + p \mapsto k - p$, is positive definite. An involution τ on a semi-simple Lie algebra is called *Cartan involution*, if $B_\tau(X,Y) := -B(X,\tau Y)$ is positive definite. There exists a one-to-one correspondence between Cartan involutions and Cartan decompositions. Therefore the decomposition of $\mathfrak{g}$ in the direct sum of $\mathfrak{k}$ and $\mathfrak{p}$ in the context of the non-compact symmetric spaces is a Cartan decomposition of $\mathfrak{g}$.

2.2 Totally Geodesic Submanifolds

Definition 2.2.1. Let M be a Riemannian manifold. A submanifold S is called *geodesic* in a point $p \in S$, if each geodesic in M, which is tangent to S in p, is a curve in S (i.e. a differentiable map from $I \subset \mathbb{R}$ to S). The submanifold S is called *totally geodesic* if it is geodesic in each point $p \in S$.

One can show (see [He78, Lem I.14.2]) that geodesics in M which are contained in S are geodesics in S and vice versa.

Note that a totally geodesic submanifold of a locally symmetric space is locally symmetric, since the restriction of the geodesic symmetry is again a geodesic symmetry.
For symmetric spaces there exists a nice description of totally geodesic submanifold in terms of the Lie algebra of the group of automorphisms. For the central theorem we need some facts, which we don't want to prove it here, since it would be just copying from Helgason. We will only sketch the proofs. All details can be found in [He78], Chapters I.14 and V.6.

Theorem 2.2.2. *Let M be a Riemannian manifold an S a connected, complete submanifold of M. Then S is totally geodesic if and only if parallel transport in M along curves in S transports tangents to S into tangents to S.*

The proof uses the differential equations for parallelity which we introduced for Remark 1.1.6 (Equation (1)).

Theorem 2.2.3. *Let $M = G/K$ be a symmetric space. For $X \in \mathfrak{p}$ we put $T_X := (ad(X))^2$. We fix a point $p \in M$ and, as usual, we identify T_pM with $\mathfrak{p}$. The latter can be considered as a manifold whose tangent space at each point is identified with itself. The exponential map* $\exp : \mathfrak{p} \mapsto G/K$ *has in $X \in \mathfrak{p}$ the differential $d\exp_X : \mathfrak{p} \mapsto \mathfrak{p}$, which is given by*

$$d\exp_X = dL_{\exp X}|_p \circ \sum_{n=0}^{\infty} \frac{(T_X)^n}{(2n+1)!},$$

where $L_g : M \to M$ denotes the left-action of G on M.

This is [He78, Thm. IV.4.1]. The next theorem is the main theorem on totally geodesic submanifolds in Riemannian symmetric spaces. It relates totally geodesic submanifolds to subspaces of $\mathfrak{p}$, called *Lie triple systems.* A subspaces $\mathfrak{s}$ of $\mathfrak{p}$ is called Lie triple system, if it satisfies $[X, [Y, Z]] \in \mathfrak{s}$ for all $X, Y, Z \in \mathfrak{s}$.

Theorem 2.2.4. *Let M be a Riemannian symmetric space and $\mathfrak{s}$ be a Lie triple system contained in $\mathfrak{p}$. Then $S = \exp(\mathfrak{s})$ has a natural differentiable structure in which it is a totally geodesic submanifold of M with $T_pS = \mathfrak{s}$.*
Conversely, if S is a totally geodesic submanifold of M with $p \in S$, then $T_pM = \mathfrak{s}$.

Proof. Let $\mathfrak{s}$ be a Lie triple system. Since $\mathfrak{s} \subset \mathfrak{p}$, we have $[\mathfrak{s}, \mathfrak{s}] \subset [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. From the Jacobi identity we get

$$[[X, Y], [U, V]] + [U, [V, [X, Y]]] + [V, [[X, Y], U] = 0.$$

Combined with the fact, that $\mathfrak{s}$ is a Lie triple system we get that $[\mathfrak{s}, \mathfrak{s}]$ is a subalgebra of $\mathfrak{k}$. Therefore the $\mathfrak{g}' := \mathfrak{s} + [\mathfrak{s}, \mathfrak{s}]$ is a subalgebra of $\mathfrak{g}$. Denote by G' the corresponding subgroup of G and by K' the stabilizer of p in G'. Define $S := G' \cdot p$. By definition of K' we have a bijection between G'/K' and S via $g \cdot p \to gK'$. Therefore we can pull the topology and the differential structure from G'/K' to S. The tangent space of S in p is $\mathfrak{s}$. The geodesics in M through p have the form $\exp(tX) \cdot p$ with $X \in \mathfrak{p}$. Such a geodesic is tangent to S if and only if $X \in \mathfrak{s}$. Hence S is geodesic in p. Since G acts transitively on M, it is everywhere geodesic, hence totally geodesic and $S = \exp(\mathfrak{s})$.

Conversely: let S be a totally geodesic submanifold of M. Denote by $\mathfrak{s}$ the tangent space at a point p. For $t \in \mathbb{R}$ the vector $A := d\exp_{tY}(X)$ is tangent to S at the point $\exp(tY)$, since $\exp_{tX}$ maps $\mathfrak{s}$ to S because the exponential function on S is the restriction of the one on M. As above we have $dL_{\exp(-tY)}A$ is parallel along $\exp(tY)$, hence $dL_{\exp(-tY)}A \in \mathfrak{s}$. From Theorem 2.2.3 we know that this is equal to

$$\sum_{n=0}^{\infty} \frac{T_Y}{(2n+1)!}(X),$$

therefore this is in $\mathfrak{s}$ to. This implies that $T_X(Y) \in \mathfrak{s}$ for all $X, Y \in \mathfrak{s}$. We have by definition of T_X

$$T_{Y+Z} - T_Y - T_Z = \mathrm{ad}Y\mathrm{ad}Z + \mathrm{ad}Z\mathrm{ad}Y.$$

The left hand side is in $\mathfrak{s}$, hence the same holds for the right hand side. Applying it to $X \in \mathfrak{s}$ we get with the Jacobi identity

$$2[Y, [Z, X]] + [X, [Y, Z]] \in \mathfrak{s}.$$

Interchanging X and Y gives

$$4[X,[Z,Y]]+2[Y,[X,Z]]\in\mathfrak{s}.$$

Added to the equation above this shows that $[X,[Y,Z]]\in\mathfrak{s}$. □

This is [He78, Thm. IV.7.2].

Remark 2.2.5. A totally geodesic submanifold S of a symmetric space M is a symmetric space. We saw in the proof of Theorem 2.2.4, that $\mathfrak{g}' := \mathfrak{s}+[\mathfrak{s},\mathfrak{s}]$ is a subalgebra of $\mathfrak{g}$. The automorphism σ of $\mathfrak{g}$ which maps $g\in G$ to $s_p g s_p$ can be restricted to a map $\mathfrak{g}'\to\mathfrak{g}'$, therefore S is a symmetric space and can be written as G'/K'.

2.3 Rank

Definition 2.3.1. A Riemannian manifold is *flat* if the curvature vanishes identically. The *rank* of a symmetric space M is the maximal dimension of a flat, totally geodesic submanifold of M.

The rank is always bigger or equal 1, since a one-dimensional subspace is always flat. One can describe maximal flat totally geodesic submanifolds in terms of the Lie algebra, using Theorem 2.2.4.

Theorem 2.3.2. *Let M be a symmetric space of the compact or the non-compact type. Let p be any point in M and we identify the tangent space in p with $\mathfrak{p}$. Let $\mathfrak{s}$ be a Lie triple system contained in $\mathfrak{p}$. Then the totally geodesic submanifold $S := \exp(\mathfrak{s})$ is flat if and only if $\mathfrak{s}$ is abelian.*

Proof. Remember Corollary 1.3.8 where we studied the connection between (non-)compactness and the scalar curvature of symmetric spaces. Recall Equation (9)

$$K(S)=Q_0([[X,Y],X],Y)=\sum_i Q_0([[X_i,Y_i],X_i],Y_i)=\sum\frac{1}{c_i}B([X_i,Y_i],[X_i,Y_i]),$$

where in our case the c_i are all positive or all negative since our M, since we assumed M to be of compact or non-compact type. If we want to apply this formula, we have to proof that S is symmetric and can be written as G'/K', since we proved Corollary 1.3.8 for spaces of this form. But this is true by Remark 2.2.5. □

Theorem 2.3.3. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ a Cartan decomposition and $(\mathfrak{g},\sigma)$ and $(\mathfrak{g}_U,\sigma)$ the corresponding (dual) orthogonal Lie algebras and K be the subgroup with Lie algebra $\mathfrak{k}$. Let $\mathfrak{a}\subset\mathfrak{p}$ be any maximal abelian subspace and $A\subset G$ respectively $A_U\subset G_U$ the corresponding subgroups to $\mathfrak{a}$ respectively $\mathfrak{a}_U$. Then*

$$G=KAK,\qquad G_U=KA_UK.$$

For the proof we need a Lemma

Lemma 2.3.4. *Let $\mathfrak{a}$ and $\mathfrak{a}'$ be two maximal abelian subgroups of $\mathfrak{p}$. Then*

i) There exists an element $H\in\mathfrak{a}$ whose centralizer in $\mathfrak{p}$ is $\mathfrak{a}$.

ii) There exists an element $k \in K$, such that $Ad(k)\mathfrak{a} = \mathfrak{a}'$.

iii) $\mathfrak{p} = \bigcup_{k \in K} Ad(k)\mathfrak{a}$.

This is [He78, Lem. V.6.3]. A short and nice proof can be found there.
Now we want to prove Theorem 2.3.3. If K_0 is the identity component of K, we have by completeness (Hopf-Rinow) that $G \cdot K_0 = \exp(\mathfrak{p})$. With the Lemma above, we can find for $g \in G$ elements $H \in \mathfrak{p}$ and $k \in K$ such that $gK_0 = \exp(\mathrm{Ad}(k)H)$. Since $\exp(\mathrm{Ad}(k)H) = k\exp(H)k^{-1}$, therefore $G = k\exp(H)k^{-1}k_0$ for some $k_0 \in K_0$.

2.4 Bounded Symmetric Domains

A *domain* is a connected open set in $\mathbb{C}^n$. A domain D is called *bounded symmetric domain*, if D is bounded and for every $z \in D$ there exists a biholomorphic map $s_z : D \to D$ which is involutive and has z as an isolated fixed point.

Example 2.4.1. Let D be the unit disc in $\mathbb{C}$. It is clearly a bounded domain. Lets make it symmetric. The map $s_0 : z \mapsto -z$ is a biholomorphic map from D to D. and it has 0 as an isolated fixed point. If we know, that the group $SU(1,1)$ acts transitive holomorphically on D, we can choose for $z \in D$ a $g \in SU(1,1)$, which maps g to 0 and put $s_z := g^{-1}s_0g$. This is for every $z \in D$ a point symmetric.
Lets discuss the action of $SU(1,1)$ on D. The definition and of $SU(1,1)$ can be found in Example 1.4.6. It acts via Möbius transformations

$$\begin{pmatrix} \bar{a} & b \\ \bar{b} & a \end{pmatrix} \cdot Z = \frac{az+b}{\bar{b}z+\bar{a}}$$

on D. The point 0 is mapped to b/a. For an arbitrary $z \in D$ we have $1-|z|^2 > 0$. Put $a := 1/(1-|z|^2)$ and $b := z/(1-|z|^2)$. The matrix $\begin{pmatrix} \bar{a} & b \\ \bar{b} & a \end{pmatrix}$ is in $SU(1,1)$ and it maps 0 to z. Therefore the action is transitive.

The natural question at this point is: Given a bounded symmetric domain. Is there a (natural?) metric on D, such that D is a Riemannian/Hermitian symmetric space? The answer is yes. The metric is called Bergman metric. It is discussed shortly in [Ko00] and more explicitly in [He78].

Let D be a domain. By $\mathcal{H}^2(D)$ we denote the space of holomorphic functions that are quare integrable with respect to the Lebesgue measure. If D is bounded the polynomials are in $\mathcal{H}^2(D)$, hence it is infinite dimensional. Furthermore it is a complete Hilbert space. We denote the scalar product by $(\cdot|\cdot)$. Consider for $w \in D$ the linear functional on $\mathcal{H}^2(D)$ given by $f \mapsto f(w)$. By the Riesz representation theorem there exists $K_w \in \mathcal{H}^2(D)$ with $f(w) = (f|K_w)$ for all $f \in \mathcal{H}^2(D)$. Sometimes we write $K(z,w)$ for $K_w(z)$. The function K is defined on $D \times D$ and it has the property

$$K(z,w) = K_w(z) = (K_w|K_z) = \overline{(K_z|K_w)} = \overline{K_z(w)} = \overline{K(w,z)}.$$

Hence $K(z,z) = |K_z\|^2 \geq 0$. The function K is called *Bergman kernel.*
For any complete orthogonal system $\{\varphi_k\}$ we have

$$K(z,w) = \sum_k \varphi_k(z)\overline{\varphi_k(w)}.$$

Since $K_w \in \mathcal{H}^2(D)$, the Bergman kernel is holomorphic in the first and antiholomorphic in the second variable.

If $F : D \to D'$ is an isomorphism[9] between two domains with Bergman kernels K respectively K', we have

$$K(z,w) = K'(F(z),F(w))j_F(z)\overline{j_F(w)},$$

where j_F is the (complex) Jacobi determinant of F. A proof of this fact can be found in the Appendix (Proposition A.4.5).

Let D be a bounded domain and K its Bergman kernel. Let for $z \in D$

$$g_{jk}(z) := \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log K(z,z).$$

This defines an invariant Hermitian metric on D. Hermitian means, that for $z \in D$ the matrix $g_{jk}(z)$ defines a Hermitian form on the tangent space T_pD. To see that, note that $g_{jk}(z) = \overline{g_{kj}(z)}$. It remains to show that it is positive definite on T_zD. This can be done by a direct calculation using a orthonormal system $\{\varphi_n\}$ of $\mathcal{H}^2(D)$. A direct calculation shows also that F is an isometry between D and D' equipped with their Bergman metrics.

2.5 Embedding

Our reference in this section is [Ko00] and [Wo72].

2.5.1 The Borel Embedding

In this section we will introduce the Borel embedding. It embeds a Hermitian symmetric space of non-compact type into its compact dual. In the text one will find numbers (n). They refer to the corresponding number in Example 2.5.3.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a non-compact, orthogonal, involutive Lie algebra. Since we are looking at Hermitian symmetric spaces we assume that the center of $\mathfrak{k}$ is non-trivial. An equivalent assumption is, that $\mathfrak{k}$ is not semi-simple. In this case we have a $n \in \mathfrak{k}$, such that the $B_{\mathfrak{k}}(n,\mathfrak{k}) = 0$. Since n is in a compact Lie algebra the matrix of $\mathrm{ad}(n)$ is skew-symmetric, hence diagonalisable with eigenvalues of the form it with $t \in \mathbb{R}$. Therefore $B(n,n)$ being the sum of squares is zero if and only if every summand is zero. Hence $ad(n) \equiv 0$ and n is in the center of $\mathfrak{k}$. This shows that a non-semi simple Lie algebra has a non-trivial center. However, in the following the center of $\mathfrak{k}$ is not trivial. (1)

Further we assume that $\mathfrak{k}$ acts faithful on $\mathfrak{p}$. $\exp \mathfrak{k} =: K$ acts via orthogonal linear transformations (via $k \mapsto dk$) faithful on $\mathfrak{p}$ and therefore on the complexification $\mathfrak{p}_{\mathbb{C}}$ too. The center of K (which is by the above discussion at least one-dimensional) consists of automorphisms of the representation of K in $\mathfrak{p}$. By Schur's lemma (Lemma A.3.11) it can only contain diagonal matrices. Since K is compact and the center is closed, is must be isomorphic to U^1. Since the representation is faithful there exists a complex structure $J = \mathrm{ad}(z)$ on $\mathfrak{p}$, with $z \in U^1$. J commutes with the $\mathrm{ad}(u)$ and it is diagonalisable. Therefore $\mathfrak{p}_{\mathbb{C}}$ decomposes into two eigenspaces $\mathfrak{p}^+$ and $\mathfrak{p}^-$ with respect to the eigenvalues i and $-i$. Let $p_1, p_2 \in \mathfrak{p}^+$ and $J = \mathrm{ad}(z)$ be the complex structure of $\mathfrak{p}$ (2). With the Jacobi-Identity we get

$$J[p_1,p_2] = \mathrm{ad}(z)[p_1,p_2] = [z,[p_1,p_2]] = [[z,p_1],p_2] + [p_1,[z,p_2]] = 2i[p_1,p_2],$$

[9] Biholomorphic map.

hence $[p_1, p_2] = 0$, since J has only eigenvalues i and $-i$. This shows that $\mathfrak{p}^+$ is abelian. Since J is in the center of $\mathfrak{k}$ the spaces $\mathfrak{p}^\pm$ are invariant under $\mathrm{ad}(\mathfrak{k})$.

Let now $G_\mathbb{C}$ be the simply connected group with Lie algebra $\mathfrak{g}_\mathbb{C}$. Denote by $K_\mathbb{C}$, K, P^+, P^-, G and G_U the subgroups corresponding to the Lie algebras $\mathfrak{k}_\mathbb{C}, \mathfrak{k}, \mathfrak{p}^+, \mathfrak{p}^-, \mathfrak{g}$ and$\mathfrak{g}_U$ (where $\mathfrak{g}_U := \mathfrak{k} + i\mathfrak{p}$) (4). On $G_\mathbb{C}$ there exists an involution σ which is induced by the map $g_1 + ig_2 \mapsto g_1 - ig_2$ on $\mathfrak{g}_\mathbb{C}$ since this map induces a local involution on $G_\mathbb{C}$, which can be continued by Proposition 1.1.10. In the same way we can find involutions τ and θ, which leave $\mathfrak{g}_U$ respectively $\mathfrak{k}_\mathbb{C}$ fixed. G, $K_\mathbb{C}$ and G_U are closed since they are the connected components of fixed point subgroups of the involutions σ, τ and θ. K is closed, since it is the connected component of $G \cap K_\mathbb{C}$. Lets consider the case $P^\pm$. The subalgebra $\mathrm{ad}(\mathfrak{p}^+)$ of $\mathrm{ad}(\mathfrak{g}_\mathbb{C})$ consists of nilpotent elements. This can be seen by applying $\mathrm{ad}(p)$ a few times to a $\tilde{p} \in \mathfrak{p}_\mathbb{C}$ and a $k \in \mathfrak{k}$. For example

$$\mathrm{ad}(p)\tilde{p} \in \mathfrak{k}_\mathbb{C} \;\Rightarrow\; \mathrm{ad}^2(p)\tilde{p} \in \mathfrak{p}^+ \;\Rightarrow\; \mathrm{ad}^3(p)\tilde{p} = 0,$$

since $\mathfrak{p}^+$ is abelian. From [He78, Cor VI.4.4] we know, that $\exp : \mathrm{ad}(\mathfrak{p}^+) \to \mathrm{Ad}(P^+)$ is regular[10] and surjectiv. P^+ is the pre-image under this covering homomorphism of this simply connected group. This shows that $\exp : \mathfrak{p}^+ \to P^+$ is a homeomorphism and P^+ is closed.

Lets have a look at $K_\mathbb{C}P^+$. It is a subgroup of $G_\mathbb{C}$ with Lie algebra $\mathfrak{k}_\mathbb{C} + \mathfrak{p}^-$. It is closed because $\mathfrak{k}_\mathbb{C}$ is the normalizer of $\mathfrak{p}^-$ in $\mathfrak{g}_\mathbb{C}$ and hence $K_\mathbb{C}P^-$ must be the identity component of the normalizer of $\mathfrak{p}^-$ (5).
Let $M^* = G_\mathbb{C}/K_\mathbb{C}P^-$. Since it is the quotient of a complex group by a complex subgroup, M^* is a complex manifold.

With these notations we have:

Theorem 2.5.1. *The space G_U/K is a compact Hermitian symmetric. The map $i : G_U/K \to M^*$, defined by $gK \mapsto gK_\mathbb{C}P^-$ is a holomorphic diffeomorphism (see Example* 2.5.3 *(6)).*

Proof. The tangent space of M^* at the point $o := eK_\mathbb{C}P^-$ is $\mathfrak{p}^+$. The map i is induced by the map from G_U into $G_\mathbb{C}$. It is well-defined since $K \subset K_\mathbb{C}P^-$. Its differential embeds $\mathfrak{g}_U$ into $\mathfrak{g}_\mathbb{C}$. Since $\mathfrak{g}_U \cap (\mathfrak{k}_\mathbb{C} + \mathfrak{p}^-) = \mathfrak{k}$, the differential of i in eK is the identity. Hence i is a local injection and it is holomorphic. The image of i is the orbit of o of the action of G_U on M^* via multiplication. It is open, since i is a local injection. But it is also compact, since it is the image of a compact set (G_U/K) under the continuous function i. Hence it is closed. This shows that the image of i is M^* and i is surjective and G_U/K is the universal covering of M^*. But M^* is simply connected since $G_\mathbb{C}$ is and $K_\mathbb{C}P^-$ is connected. Therefore i is a diffeomorphism. $\square$

Now we show that one can embed M into M^*.

Theorem 2.5.2. *$M = G/K$ is a Hermitian symmetric space. The map $j : M \to M^*$ defined by $j(gK) := gK_\mathbb{C}P^-$ is a G-equivariant holomorphic diffeomorphism onto an open subset of M^* (see Example* 2.5.3 *(7)).*

Proof. We have to show first, that the map is injective. To do that, remark that two points go and ho are equal in M^* if and only if $h^{-1}g \in K_\mathbb{C}P^-$. Since M equals G/K, two points gK and hK are equal if and only if $h^{-1}g \in K$. Therefore it suffices to show $G \cap K_\mathbb{C}P^- = K$. Since

[10] A map is regular if its differential is bijectiv.

$G = KP$ (with $P = \exp(\mathfrak{p})$) it suffices to show that $P \cap K_{\mathbb{C}}P^- = \{e\}$. Suppose $p = kp^- \in P$ with $k \in K_{\mathbb{C}}$ and $p^- \in P^-$. We have

$$p^{-1} = \sigma\tau p = \sigma\tau(kp^-) = k(p^-)^{-1}.$$

since a direct calculation, using only the definitions of σ and τ gives us

$$d\sigma d\tau : k_1 + p_1 + ik_2 + ip_2 = k_1 - p_1 - ip_2 + ik_2.$$

We get $p = kp^- = p^-k^{-1}$ and therefore $p^2 = (p^-)^2$. Applying τ at both sides gives $\tau(p^2) = p^2 = (p^-)^2 = \tau(p^-)^2$. By definition of τ this is in P^+. Since the $\mathrm{Ad}_{G_{\mathbb{C}}}$ is faithful we have $p^2 = e$, because P^+ and P^- are represented by upper respectively lower triangular matrices. Therefore $p^2 = e$ implies $p = e$.
The proof, that j is holomorphic diffeomorphism works like in the proof of the theorem above. □

Example 2.5.3. This example should make clear everything that happens in the section above. In the appendix (Details), one finds a list of the groups appearing here.

(1) Let G be $SU(1,1)$ the group of 2×2-matrices of determinant 1 which leave the bilinear form of signature $(1,1)$ invariant. Its elements are of the form

$$\begin{pmatrix} \bar{a} & b \\ \bar{b} & a \end{pmatrix}$$

where $a, b \in \mathbb{C}$ and $|a|^2 - |b|^2 = 1$ (see Proposition A.4.7. Its Lie algebra $\mathfrak{su}(1,1)$ consists of matrices of the form

$$\begin{pmatrix} ix & y + iz \\ y - iz & -ix \end{pmatrix}$$

with $x, y, z \in \mathbb{R}$. A proof of this fact can be found in [He78, p. 444]. $\mathfrak{su}(1,1)$ has therefore the matrices

$$e_1 := \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix},\ e_2 := \begin{pmatrix} 0 & i/2 \\ -i/2 & 0 \end{pmatrix},\ e_3 := \begin{pmatrix} i/2 & 0 \\ 0 & -i/2 \end{pmatrix}$$

as a basis and the Lie bracket relations are:

$$[e_1, e_2] = -e_3,\ [e_1, e_3] = -e_2,\ [e_2, e_3] = e_1.$$

This follows from a simple computation. Denote by $\mathfrak{k}$ the space spanned by e_3 and by $\mathfrak{p}$ the space spanned by e_1 and e_2. $\mathfrak{k}$ generates a compact subgroup K of G, namely the subgroup consisting of matrices of the form

$$\begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix},$$

with $z \in U^1$. The killing form of $\mathfrak{k}$ is constantly zero since $\mathfrak{k}$ is one-dimensional and hence abelian.

(2) The complexification of $\mathfrak{p}$ (denoted by $\mathfrak{p}_{\mathbb{C}}$) is given by $\mathfrak{p} \otimes \mathbb{C} = \mathfrak{p} + i\mathfrak{p}$. The Lie algebra $\mathfrak{k}$ acts via the adjoint representation on $\mathfrak{p}$ and the complex structure $J = \mathrm{ad}(e_3)$. It is diagonalisable over $\mathbb{C}$ with eigenvalues i and $-i$ and the eigenspaces $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are spanned by $\{e_1 - ie_2, e_2 + ie_1\}$ respectively $\{e_1 + ie_2, -e_2 + ie_1\}$. One sees immediately that they have complex structures and that they are conjugate.

(3) The vector space $\mathfrak{p}^+$ has the matrices $b_1 := e_1 - ie_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $b_2 := ib_1 = e_2 + ie_1 = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}$ as a basis. Calculating $\exp(zb_1) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ with $z \in \mathbb{C}$ one sees that P^+ is equal to $\mathbb{C}$, since it is $\mathbb{R}^2$ as a vector space and it is complex. The same holds for P^-.

(4) The Lie algebra $\mathfrak{g}_U$ is $\mathfrak{k} + i\mathfrak{p}$. It has the matrices

$$ie_1 = \begin{pmatrix} 0 & i/2 \\ i/2 & 0 \end{pmatrix},\ ie_2 = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix},\ e_3 = \begin{pmatrix} i/2 & 0 \\ 0, -i/2 & \end{pmatrix}$$

as a basis. Therefore $\mathfrak{g}_U$ consists of matrices of the form

$$\begin{pmatrix} ix & iy - z \\ iy + z & -ix \end{pmatrix}$$

with $x, y, z \in \mathbb{R}$. Consulting [He78, Ch. X.2] one finds that this is the Lie algebra of $SU^*(2)$, the group of matrices which commute with the map

$$\psi : \begin{cases} \mathbb{C}^2 \to \mathbb{C}^2 \\ (u, v) \mapsto (\bar{v}, -\bar{u}) \end{cases}$$

Direct calculation shows that

$$SU^*(2) = \left\{ \begin{pmatrix} \bar{a} & b \\ -\bar{b} & a \end{pmatrix} : \ a, b \in \mathbb{C},\ |a|^2 + |b|^2 = 1 \right\}.$$

The complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ has the matrices e_1, e_2 and e_3 as a complex basis. Another basis is

$$a_1 := e_1 - ie_2,\ a_2 := e_1 + ie_2,\ a_3 := -2ie_3.$$

Calculation its brackets

$$[a_1, a_2] = -a_3,\ [a_1, a_3] = -2a_2,\ [a_2, a_3] = 2a_2$$

and comparing with [He78, Ch. X.2], we see, that $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C})$. Therefore $G_{\mathbb{C}} = SL(2, \mathbb{C})$.
P^+ and P^- are isomorphic to $\mathbb{C}$ as we have seen in (3). $K_{\mathbb{C}}$ is isomorphic to $\mathbb{C}^*$ via $z \mapsto \begin{pmatrix} 1/z & 0 \\ 0 & z \end{pmatrix}$.

(5) What is G_U/K? Let $\begin{pmatrix} \bar{a} & b \\ -\bar{b} & a \end{pmatrix}$ in G_U. If $a = a_1 + ia_2$ and $b = b_1 + ib_2$, with $a_1, a_2, b_1, b_2 \in \mathbb{R}$, the condition $|a|^2 + |b|^2 = 1$ rewrites as

$$a_1^2 + a_2^2 + b_1^2 + b_2^2 = 1.$$

Therefore G_U can be identified with S^3, the 3-sphere in $\mathbb{C}^2 \simeq \mathbb{R}^4$. The circle-group $S^1 \simeq U^1$ acts on it via right-multiplication

$$\begin{pmatrix} \bar{a} & b \\ -\bar{b} & a \end{pmatrix} \begin{pmatrix} \bar{z} & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} \bar{z}\bar{a} & zb \\ -\bar{z}\bar{b} & za \end{pmatrix}.$$

The action is faithful, since a or b is non-zero. There is an important and well-known action of S^1 on S^3, namely the Hopf-Filtration. It has S^2 as the base space, S^3 as the total space and S^1 as the fiber. For details to this fibration see [Ha02]. The map

$$\begin{pmatrix} \bar{a} & b \\ -\bar{b} & a \end{pmatrix} \mapsto \frac{b}{a} \in \mathbb{C} \cup \{\infty\}$$

is a surjective map from G_U to S^2. If $b/a = \tilde{b}/\tilde{a}$, then $a/\tilde{a} = b/\tilde{b} =: z$ and we have

$$|a|^2 + |b|^2 = 1 = |\tilde{a}|^2 + |\tilde{b}|^2 = |z|^2(|a|^2 + |b|^2),$$

hence $z \in U^1 = K$. Therefore $G_U/K = S^2$.

Now we want to determine G/K. If one multiplies an arbitrary $\begin{pmatrix} \bar{a} & b \\ \bar{b} & a \end{pmatrix} \in G$ with an arbitrary $\begin{pmatrix} \bar{z} & 0 \\ 0 & z \end{pmatrix} \in K$ we get a matrix

$$\begin{pmatrix} \bar{z}\bar{a} & zb \\ \bar{z}\bar{b} & za \end{pmatrix}.$$

An element in M is uniquely determined by a and b. Since $z \in U^1$ we can assume that $a \in \mathbb{R}$. If $b = b_1 + ib_2$ with $a \in \mathbb{R}$ our condition $|a|^2 - |b|^2 = 1$ becomes $a^2 - b_1^2 - b_2^2 = 1$. But this is exactly the condition for a point $(a, b_1, b_2) \in \mathbb{R}$ to be in the upper sheet of the hyperboloid defined by the equation $x^2 - y^2 - z^2 = 1$.

In Example 1.4.6 we realized G/K as the space X of one-dimensional subspaces W of $\mathbb{C}^2$ on which the quadratic form $h(x, y) = x_1\bar{y}_1 - x_2\bar{y}_2$ is positive definite. Let (z_1, z_2) span a W. Since $|z_1|^2 - |z_2|^2 > 0$, $z_1 \neq 0$ and we can assume without loss of generality that $z_1 = 1$ (because $\frac{1}{z_1}(z_1, z_2) \in W$). Therefore W depends only of z_2 and since $1 - |z_2|^2 > 0$ we have $1 > |z_2|^2$ and X can be identified with the unit-disk.

What is $G_{\mathbb{C}}/K_{\mathbb{C}}P^-$? Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be in $G_{\mathbb{C}}$ and $\begin{pmatrix} x & 0 \\ y & z \end{pmatrix} \in K_{\mathbb{C}}P^-$. We have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x & 0 \\ y & z \end{pmatrix} = \begin{pmatrix} xa + yb & b/x \\ xc + yd & d/x \end{pmatrix}$$

Putting $x = d$ and $y = -c$ we get (with $ad - bc = 1$)

$$\begin{pmatrix} 1 & b/d \\ 0 & 1 \end{pmatrix},$$

with $b/d = \infty$ if $d = 0$. Therefore we have in every equivalence class an element of this form, hence we can identity the set of this equivalence classes with S^2.

(6) The map $i : G_U/K \to G_{\mathbb{C}}/K_{\mathbb{C}}P^-$ maps gK to $gK_{\mathbb{C}}P^-$. It is a map from S^2 to S^2. Let x be in $S^2 = \mathbb{C} \cup \{\infty\}$. If $x = \infty$ put $a := 0$ and $b := 1$. If $x \neq 0$ we can write x as b/a with $|a|^2 + |b|^2 = 1$. A representant of x is in both cases $\begin{pmatrix} \bar{a} & b \\ -\bar{b} & a \end{pmatrix}$. This matrix is contained in $G_{\mathbb{C}}$ and represent therefore an equivalence classes of $G_{\mathbb{C}}/K_{\mathbb{C}}P^-$. With the last part of (5), every class has one and only one element of the form $\begin{pmatrix} 1 & b/a \\ 0 & 1 \end{pmatrix}$. Therefore i is the identity on the Riemann sphere.

(7) The same calculation shows that j maps a point $(a, b_1, b_2) \in G/K$ to $b/a \in G_{\mathbb{C}}/K_{\mathbb{C}}P^-$, with $b := b_1 + ib_2$. Since $a^2 - b_1^2 - b_2^2 = 1$ we have $|a|^2 - |b|^2 = 1$ and therefore $|a|^2 = |b|^2 + 1 > |b|^2$. This shows that $j(G/K)$ is the lower hemisphere in $G_{\mathbb{C}}/K_{\mathbb{C}}P^-$.

2.5.2 The Harish-Chandra Embedding

In this section we will use the same notation as in the section on the Borel embedding.
We saw in the preceding section that one can embed a Hermitian symmetric space M of non-compact type into its compact dual M^*. In this section we will show that there exist a map $\xi : \mathfrak{p}^+ \to M^*$ which is an embedding. In our example 2.5.3 $\mathfrak{p}^+ \simeq \mathbb{C}$ and $M^* = S^2$. Later we will see that the image of M in M^* is contained in the image of $\mathfrak{p}^+$ under ξ. It is a diffeomorphism onto its image and we can use ξ^{-1} to send $i(M) \subset M^*$ onto a bounded domain in the vector space $\mathfrak{p}^+$.
Let now $\mathfrak{g}$ be a Lie algebra as in the preceding chapter. Let $\mathfrak{h} \subset \mathfrak{k}$ a maximal abelian subalgebra. This $\mathfrak{h}$ contains the center and its complexification $\mathfrak{h}_{\mathbb{C}}$ is maximal abelian in $\mathfrak{g}_{\mathbb{C}}$. Remember the map $\tau : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ defined by $p_1 + ip_2 + k_1 + ik_2 \mapsto -p_1 + ip_2 + k_1 - ik_2$. Define a bilinear form B_τ on $\mathfrak{g}$ by $B_\tau(x, y) := -B(x, \tau y)$. A straightforward calculation[11] shows that B_τ is positive definite Hermitian. The map $\mathrm{ad}(ih) : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ leaves the Killing form on $\mathfrak{g}$ infinitesimally invariant and one can easily check by direct calculation that $\mathrm{ad}(ih) \circ \tau = -\tau \circ \mathrm{ad}(ih)$ for all $h \in \mathfrak{h}$. These two facts combined tell us, that $\mathrm{ad}(ih)$ is Hermitian. Hence $\mathrm{ad}(ih)$ is semi-simple (i.e. every invariant subspace of $\mathfrak{g}_{\mathbb{C}}$ has an invariant complement). Therefore $\mathfrak{h}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. We can decompose $\mathfrak{g}_{\mathbb{C}}$ into root spaces $\mathfrak{g}_\alpha$ with respect to $\mathfrak{h}_{\mathbb{C}}$. Denote by Δ the set of roots. We define a positive root system by: $\alpha > 0$ if and only if $-i\alpha(z) > 0$, for the $z \in \mathfrak{h}_{\mathbb{C}}$ that induces the complex structure on $\mathfrak{p}^+$ via $J = \mathrm{ad}(z)$. The set of positive roots is denoted by Δ^+ and we put $\Delta^- := -\Delta^+$. Since for $x \in \mathfrak{g}_\alpha$ we have

$$-x = J^2 x = \mathrm{ad}(z)^2 x = \alpha(z)^2 x,$$

we have $\alpha(z) = \pm i$ and we defined a root to be positive if $\alpha(z) = +i$ and negative if $\alpha(z) = -i$. By Theorem A.2.4 there exists for every root α a unique $H_\alpha \in \mathfrak{h}_{\mathbb{C}}$ such that

$$\alpha(H) = 2\frac{B(H, H_\alpha)}{B(H_\alpha, H_\alpha)} \qquad \forall H \in \mathfrak{h}_{\mathbb{C}}.$$

Further we can find $e_\alpha \in \mathfrak{g}_\alpha$ with $\tau e_\alpha = -e_{-\alpha}$ and $[e_\alpha, e_{-\alpha}] = H_\alpha$. To see that, take for every $\alpha \in \Delta^+$ a nonzero vector $\tilde{e}_\alpha \in \mathfrak{g}^\alpha$ and put $\tilde{e}_{-\alpha} := -\tau \tilde{e}_\alpha$. Note that we have for $H \in \mathfrak{h}_{\mathbb{C}}$

$$B(H, [\tilde{e}_\alpha, \tilde{e}_{-\alpha}]) = B([H, \tilde{e}_\alpha], \tilde{e}_{-\alpha}) = \alpha(H) B(\tilde{e}_\alpha, \tilde{e}_{-\alpha}).$$

[11] This calculation needs only the definition of τ and the positivity respectively the negativity of B on $\mathfrak{k}$ and $\mathfrak{p}$.

Hence H_α is a multiple of $[\tilde{e}_\alpha, \tilde{e}_{-\alpha}]$ and we can find $x_\alpha \in \mathbb{C}$ such that for $e_\alpha := x_\alpha e_\alpha$ we have $[e_\alpha, e_{-\alpha}] = H_\alpha$.

The root space $\mathfrak{g}_\alpha$ is either in $\mathfrak{k}_\mathbb{C}$ of in $\mathfrak{p}_\mathbb{C}$, since the decomposition of $\mathfrak{g}_\mathbb{C}$ into $\mathfrak{k}_\mathbb{C} + \mathfrak{p}_+ + \mathfrak{p}_-$ is the decomposition with respect to eigenspaces of $\mathrm{ad}(z)$ with $z \in U^1$.

If $\mathfrak{g}_\alpha \subset \mathfrak{k}_\mathbb{C}$ we say that α is a compact root, if $\mathfrak{g}_\alpha \subset \mathfrak{p}_\mathbb{C}$, α is non-compact and we denote the set of positive non-compact roots by Φ. We denote the set of compact roots by Δ_K and the non-compact roots by Δ_P and we put $\Delta_K^+ := \Delta_K \cap \Delta^+$ and so on. The vector space $\mathfrak{p}_\mathbb{C}$ is the direct sum of the $\mathfrak{g}_\alpha$ with α non-compact. If α is a positive root, $-\alpha$ is a root too (Theorem A.2.4) and it is negative- Therefore the number of positive and negative non-compact roots are equal. For a positive root α we have

$$Je_\alpha = \mathrm{ad}(z)e_\alpha = \alpha(z)e_\alpha = ie_\alpha.$$

Therefore $\bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha \subset \mathfrak{p}^+$ and $\bigoplus_{-\alpha\in\Phi} \mathfrak{g}_\alpha \subset \mathfrak{p}^-$ and by dimension reasons we have

$$\mathfrak{p}^\pm = \bigoplus_{\alpha\in\pm\Phi} \mathfrak{g}_\alpha. \qquad (12)$$

Definition 2.5.4. Two roots $\alpha, \beta \in \Delta$ are *strongly orthogonal*, denoted $\alpha \Vdash \beta$, if neither $\alpha \pm \beta$ is a root.

If $\alpha \Vdash \beta$ then $[\mathfrak{g}_\alpha, \mathfrak{g}_{\pm\beta}] \subset \mathfrak{g}_{\alpha\pm\beta} = 0$ and by general theory of root they are orthogonal. One can construct inductively a maximal set of strongly orthogonal non-compact roots

$$\Psi := \{\psi_1, \dots, \psi_r\}$$

where ψ_{j+1} is a minimal root, strongly orthogonal to $\psi_1, \dots, \psi_j$.
Since $e_\alpha = -\tau e_{-\alpha}$ the vectors

$$x_{\alpha,0} := e_\alpha + e_{-\alpha} \text{ and } y_{\alpha,0} := i(e_\alpha - e_{-\alpha}), \qquad \alpha \in \Delta_P^+$$

form a basis of $\mathfrak{p}$, since they are in $\mathfrak{p}$ by definition of τ and they are linearly independent. Therefore

$$x_\alpha := ix_{\alpha,o} = i(e_\alpha + e_{-\alpha}) \text{ and } y_\alpha = iy_{\alpha,0} := -e_\alpha + e_{-\alpha}, \qquad \alpha \in \Delta_P^+$$

are a basis of $i\mathfrak{p}$.
We define subspaces of $\mathfrak{p}_\mathbb{C}$ by

$$\mathfrak{a} := \sum_{\psi\in\Psi} \mathbb{R}x_{\psi,0} \subset \mathfrak{p}, \qquad \mathfrak{a}_U := \sum_{\psi\in\Psi} \mathbb{R}x_\psi \subset i\mathfrak{p}.$$

They are abelian subalgebras of $\mathfrak{p}_\mathbb{C}$ since the elements of Ψ are pairwise strongly orthogonal. Further they are maximal abelian since Ψ is maximal.
There are groups $A \subset G$ and $A_U \subset G_U$ with Lie algebras $\mathfrak{a}$ respectively $\mathfrak{a}_U$.
For $\alpha \in \Delta$ put

$$\mathfrak{g}_\mathbb{C}[\alpha] := \mathbb{C}H_\alpha + \mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}.$$

By construction this is a 3-dimensional simple subalgebra of $\mathfrak{g}_\mathbb{C}$ and it has real forms

$$\mathfrak{g}[\alpha] = \mathfrak{g} \cap \mathfrak{g}_\mathbb{C}[\alpha] \quad \text{and} \quad \mathfrak{g}_U[\alpha] = \mathfrak{g}_U \cap \mathfrak{g}_\mathbb{C}[\alpha].$$

That defines simple groups $G_{\mathbb{C}}[\alpha]$, $G[\alpha]$ and $G_U[\alpha]$ with Lie algebras $\mathfrak{g}_{\mathbb{C}}[\alpha]$, $\mathfrak{g}[\alpha]$ and $\mathfrak{g}_U[\alpha]$. If $\Gamma \subset \Psi$ we have the direct sums

$$\mathfrak{g}_{\mathbb{C}}[\Gamma] := \sum_{\alpha\in\Gamma} \mathfrak{g}_{\mathbb{C}}[\alpha], \quad \mathfrak{g}[\Gamma] := \sum_{\alpha\in\Gamma} \mathfrak{g}[\alpha], \quad \mathfrak{g}_U[\Gamma] := \sum_{\alpha\in\Gamma} \mathfrak{g}_U[\alpha]$$

with corresponding groups $G_{\mathbb{C}}[\Gamma]$, $G[\Gamma]$ and $G_U[\Gamma]$. By definition of H_α and e_α, we have in $\mathfrak{g}_{\mathbb{C}}[\alpha]$ the relations:

$$[e_\alpha, e_{-\alpha}] = H_\alpha, \quad [e_\alpha, H_\alpha] = -2e_\alpha, \quad [e_{-\alpha}, H_\alpha] = 2e_{-\alpha}.$$

Therefore $\mathfrak{g}_{\mathbb{C}}[\alpha] \simeq \mathfrak{sl}(2,\mathbb{C})$. See Example 2.5.3 for details.

Theorem 2.5.5 (Polydisc Theorem). *Let $\Gamma \subset \Psi$ and $x_0 \in M$ a point fixed by K. Then $G_{\mathbb{C}}[\Gamma]\cdot x_0 = G_U[\Gamma]\cdot x_0$ is a holomorphically embedded submanifold of M, that is a product of $|\Gamma|$ Riemann spheres and $G[\Gamma]\cdot x_0$ is a holomorphically embedded submanifold of M^* that is product of $|\Gamma|$ hemispheres of $G_U[\Gamma]\cdot x_0$. Further we have $M^* = K\cdot G_U[\Psi]$ and $M = K\cdot G[\Psi]$.*

Proof. Example 2.5.3 tells us that $G_U[\alpha]\cdot x_0$ is the Riemann sphere and we can consider $G[\alpha]\cdot x_0$ as a hemisphere in $G_U[\alpha]\cdot x_0$. We know that $G = KAK$ and $G_U = KA_UK$, hence

$$M = G\cdot x_0 = KAK\cdot x_0 = KA\cdot x_0 =\subset K\cdot G[\Psi]\cdot x_0$$

and

$$M^* = G_U\cdot x_0 = KA_UK\cdot x_0 = KA_U\cdot x_0 =\subset K\cdot G_U[\Psi]\cdot x_0$$

□

With the Polydisc Theorem, we can prove the following theorem, which tells us, that we can realize a hermitian symmetric space of non-compact type as a bounded domain.

Theorem 2.5.6. *The map $\phi : P^+ \times K_{\mathbb{C}} \times P^- \to G_{\mathbb{C}}$ given by $(p^+, k, p^-) \mapsto m^+kp^-$ is a complex diffeomorphism onto a dense open subset of $G_{\mathbb{C}}$ that contains G. In particular*

$$\xi : \begin{cases} \mathfrak{p}^+ \to M = G/K_{\mathbb{C}}P^- \\ p \mapsto \exp(p)K_{\mathbb{C}}P^- \end{cases}$$

is a complex diffeomorphism of $\mathfrak{p}^+$ onto a dense open subset of M^* that contains M. Furthermore $\xi^{-1}(M)$ is a bounded domain in $\mathfrak{p}^+$.

Proof. First we prove that ϕ is one-to-one. Assume $p_1^+k_1p_1^- = p_2^+k_2p_2^-$. Then $(p_1^+)^{-1}p_2^+ = k_1p_1^-(k_2p_2^+)^{-1} \in P^+ \cap K_{\mathbb{C}}P^-$. We show in Lemma 2.5.7 that $P^+ \cap K_{\mathbb{C}}P^- = \{e\}$, therefore $p_1^+ = p_2^+$. One can similarly prove that $p_1^- = p_2^-$ and this gives us the injectivity of ϕ.
Since $d\phi : \mathfrak{p}^+ + \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}^- \to \mathfrak{p}^+ + \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}^-$ is an isomorphism, the map ϕ is a local diffeomorphism, hence its image is open.
To show, that the image is dense and that $\xi^{-1}(M)$ is a bounded domain in $\mathfrak{p}^+$, we restrict us to the case where $G_{\mathbb{C}} = G_{\mathbb{C}}[\alpha]$. In the end we will generalize the result using the Polydisc Theorem.
Let $G_{\mathbb{C}} = G_{\mathbb{C}}[\alpha]$. Then $G_{\mathbb{C}} = SL(2,\mathbb{C})/(\pm I)$ and we realize

$$e_\alpha := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_{-\alpha} := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H_\alpha := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In particular

$$x_{\alpha,0} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad x_\alpha = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

A direct calculation shows that

$$\begin{aligned}\exp(tx_{\alpha,0}) &= \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} = \begin{pmatrix} 1 & \tanh t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\cosh t} & 0 \\ 0 & \cosh t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \tanh t & 0 \end{pmatrix} \\ &= \exp(\{\tanh t\}e_\alpha)\exp(\{-\log\cosh t\}H_\alpha)\exp(\{\tanh t\}e_{-\alpha})\end{aligned}$$

and

$$\begin{aligned}\exp(tx_\alpha) &= \begin{pmatrix} \cosh(it) & \sinh(it) \\ \sinh(it) & \cosh(it) \end{pmatrix} = \begin{pmatrix} \cos t & i\sin t \\ i\sin t & \cos t \end{pmatrix} \\ &= \begin{pmatrix} 1 & i\tan t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\cos t} & 0 \\ 0 & \cos t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ i\tan t & 0 \end{pmatrix}.\end{aligned}$$

Note that

$$\xi(\mathfrak{p}^+) = \exp(\mathbb{C}e_\alpha) \cdot K_\mathbb{C}P^- = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \cdot K_\mathbb{C}P^- \big| z \in \mathbb{C} \right\}$$

and $G_U = KA_UK$, where $A_U = \exp(\mathbb{R}x_\alpha)$ is the subgroup generated by $\mathfrak{a} = \mathbb{R}e_\alpha$, since α is the only positive non-compact root. Therefore

$$\xi(\mathfrak{p}^+) = \{k \cdot \exp(tx_\alpha)K_\mathbb{C}P^- | k \in K, t \in \mathbb{R} \ \text{ and } \ \cos t \neq 0\},$$

and this shows that $\xi(\mathfrak{p}^+)$ is dense in M^*. Furthermore we have $M \simeq G/K_\mathbb{C}P^-$. Note that for $\begin{pmatrix} \bar{a} & b \\ \bar{b} & a \end{pmatrix} \in G$ there exists a unique $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in K_\mathbb{C}P^-$ with

$$\begin{pmatrix} \bar{a} & b \\ \bar{b} & a \end{pmatrix} \begin{pmatrix} x & 0 \\ y & z \end{pmatrix} = \begin{pmatrix} 1 & b/a \\ 0 & 1 \end{pmatrix}.$$

Since $|a|^2 - |b|^2 = 1$, we can write $a = w\cosh t$ and $b = \tilde{w}\sinh t$ with $w, \tilde{w} \in U^1$ and $t \in \mathbb{R}$. Hence we have:

$$\begin{aligned}M &= \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} K_\mathbb{C}P^- \big| \ |z| = \tanh t \text{ for some real } t \right\} \\ &= \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} K_\mathbb{C}P^- \big| |z| < 1 \right\} = \xi(\{ze_\alpha | \ |z| < 1\}),\end{aligned}$$

hence $\xi^{-1}(M)$ is a bounded domain in $\mathfrak{p}^+$ and we proved the theorem for $G_\mathbb{C} = G_\mathbb{C}[\alpha]$. The general case follows from the Polydisc Theorem via

$$\begin{aligned}M^* &= \{k\exp(\sum_{\psi\in\Psi} t_\psi x_\psi)K_\mathbb{C}P^+ | k \in K, t_\psi \in \mathbb{R}\} \\ \xi(\mathfrak{p}^+) &= \{k\exp(\sum_{\psi\in\Psi} t_\psi x_\psi)K_\mathbb{C}P^+ | k \in K, t_\psi \in \mathbb{R} \ \text{ and } \ \cos t_\psi \neq 0\} \\ M &= \{k\xi(\sum_{\psi\in\Psi} z_\psi e_\psi) | k \in K \text{ and } |z_\psi| < 1\}.\end{aligned}$$

□

Lemma 2.5.7. *With the notion from above we have $P^+ \cap K_{\mathbb{C}}P^- = \{e\}$.*

Proof. Let $e \neq g \in P^+ \cap K_{\mathbb{C}}P^-$. We know from our discussion above that $\exp : \mathfrak{p}^+ \to P^+$ is a diffeomorphism. Since $g \in P^+$, there exists a unique $x \in \mathfrak{p}^+$ with $\exp x = g$. We can write $x = \sum_{\alpha\in\Phi} c_\alpha e_\alpha$, because of equation (12). Let β be a root with $\alpha - \beta > 0$ if $\alpha \in \Phi$ and $\alpha - \beta$ is a root. Such a β exists. Denoting $\mathfrak{n}^+ = \sum_{\alpha>0} \mathbb{C}e_\alpha$, we have

$$[x, e_{-\beta}] = c_\beta H_\beta + \sum_{\alpha\in\Phi,\alpha\neq\beta} c_\beta [e_\alpha, e_{-\beta}].$$

The sum is in $\mathfrak{n}^+$, since $[e_\alpha, e_{-\beta}] \in \mathfrak{g}_{\alpha-\beta}$ and by our choice of β, either the space $\mathfrak{g}_{\alpha-\beta}$ is zero or $\alpha - \beta$ is a positive root. Therefore

$$\mathrm{ad}(x)e_{-\beta} = [x, e_{-\beta}] \equiv c_\beta h_\beta \mod \mathfrak{n}^+. \tag{13}$$

We know by general theory that $\mathrm{Ad}(g) = \mathrm{Ad}(\exp x) = \exp(\mathrm{ad}(x))$. By the definition of exp as the power series $\exp(A) = \sum_{i=0}^{\infty} \frac{A^i}{i!}$ we get

$$\mathrm{Ad}(g) = \exp(\mathrm{ad}(x))e_{-\beta} \equiv e_{-\beta} + \mathrm{ad}(x)e_{-\beta} + \frac{1}{2}\mathrm{ad}(x)^2 e_{-\beta} + \dots$$

For positive roots α and $\tilde{\alpha}$ we have $[e_\alpha, e_{\tilde{\alpha}}] \subset \mathfrak{g}_{\alpha+\tilde{\alpha}} \subset \mathfrak{n}^+$ and since $H_\beta \in \mathfrak{h}_{\mathbb{C}}$, we have $[H_\beta, e_\alpha] = \alpha(H_\alpha)e_\alpha \in \mathfrak{g}_\alpha \subset \mathfrak{n}^+$, hence $\mathrm{ad}(x)^n e_{-\beta} \in \mathfrak{n}^+$ for $n \geq 2$. Since by equation (13) $\mathrm{ad}(x)e_{-\beta} \equiv c_\beta H_\beta \mod \mathfrak{n}^+$, we get

$$\mathrm{Ad}(g)e_{-\beta} \equiv \underbrace{e_{-\beta}}_{\in\mathfrak{p}^-} + \underbrace{c_\beta H_\beta}_{\in\mathfrak{k}_{\mathbb{C}}} \mod \mathfrak{n}^+.$$

This follows from $g \in P^+$.
But g is also in $K_{\mathbb{C}}P^-$, which normalizes $\mathfrak{p}^-$. Hence $\mathrm{Ad}(g)e_{-\beta} \in \mathfrak{p}^-$ which is a contradiction. Therefore our assumption that there is $e \neq g \in P^+ \cap K_{\mathbb{C}}P^-$ is false and $P^+ \cap K_{\mathbb{C}}P^-$ is trivial. □

Example 2.5.8. We use Example 2.5.3 to explain the Harrish-Chandra embedding. We saw that $\mathfrak{p}^+$ is $\mathbb{C}$. The map $\exp : \mathfrak{p}^+ \to P^+$ which maps $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ to $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ is clearly a diffeomorphism. Since

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x & 0 \\ y & z \end{pmatrix} = \begin{pmatrix} x + ya & za \\ y & z \end{pmatrix},$$

every equivalence class in the image of $\mathfrak{p}^+$ contains one and only one element of the form $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ with $a \in \mathbb{C}$. The image is therefore the sphere without the north-pole. We saw in Example 2.5.3, that one can embed G/K into the 2-sphere as the lower hemisphere. It preimage is the unit disc.

2.6 Filtrations, Gradations and Hodge Structures

In the following part we well need some notions and theory of algebraic groups. See the Appendix for a short overview or [Mi06] for a more detailed introduction.

Let k be a field and V a finite dimensional vector space over k. A $\mathbb{Z}$-gradations of V is a decomposition

$$V = \bigoplus_{i \in \mathbb{Z}} V^i$$

where the V^i are subvectorspaces of V. Given such a gradation we can construct a representation (V, w) of $\mathbb{G}_m$, the multiplicative group of k, via

$$w(\lambda)v := \lambda^i v, \text{ for } v \in V.$$

Conversely, every representation of $(\mathbb{G}_m)^k$ decomposes into representations by characters (maps from $(\mathbb{G}_m)^k$ into $\mathbb{G}_m$). This hold especially for representations of $\mathbb{G}_m$ (see [Mi06, Ch.9]). Since the maps from $\mathbb{G}_m$ to $\mathbb{G}_m$ are all of the form $z \mapsto z^m$, every representation of $\mathbb{G}_m$ defines a gradation as above. Therefore the category of representations of $\mathbb{G}_m$ and the category of graded vector spaces are isomorphic.
Every gradation defines an increasing (resp. decreasing) filtration by

$$W^i(V) = \bigoplus_{j \leq i} V^j$$

$$(\text{resp.} W_i(V) = \bigoplus_{j \geq i} V^j).$$

Let G be a reductive group over a field k of characteristic zero and let $w : \mathbb{G}_m \to G$ be a morphism. For any representation $\varrho : G \to GL(V)$ of G, $\varrho \circ w : \mathbb{G}_m \to GL(V)$ defines an increasing filtration W of V.
The map $w : \mathbb{G}_m \to G$ defines a functor from the category of G representations to the category of $\mathbb{Z}$-graded vector spaces. This functor is compatible with tensorproduct and with taking the dual. Conversely: a functor with these properties is induced by some w. This follows from the Tannaka duality since such a functor is a functor between the representations of G and the representations of $\mathbb{G}_m$ which is by the Tannaka-duality induced by a morphism $w : \mathbb{G}_m \to G$. For more informations to the Tannaka duality see [JS91] or [Mi06, Ch. 24].

Proposition 2.6.1. *Let $n \in \mathbb{Z}$ and H a real finite dimensional vector space. Given one of the following, one can construct the other two.*

i) A bigradation of the complexification[12] *$H_{\mathbb{C}}$:*

$$H_{\mathbb{C}} = \bigoplus_{p+q=n} H^{p,q},$$

such that $H^{q,p}$ is the complex conjugate of $H^{p,q}$.

ii) A finite decreasing filtration F on $H_{\mathbb{C}}$ such that

$$H_{\mathbb{C}} = F^p \oplus \bar{F}^q$$

for $p + q = n + 1$.

[12] $H_{\mathbb{C}} := V \otimes \mathbb{C}$

iii) An action h of the real algebraic group $\mathbb{C}^*$ on H s.t. $x \in \mathbb{R}^* \subset \mathbb{C}^*$ acts as multiplication by x^{-n}.

Proof. a) Given i), we construct ii). Define $F^p := \bigoplus_{i \geq p} H^{i,j}$. This is clearly a decreasing filtration. We have to show that for given p and q with $p+q = n+1$ the complexification $H_\mathbb{C}$ equals $F^p \oplus \bar{F}^q$. Remark, that $i+j = n$ and $H^{l,k}$ is the complex conjugate of $H^{k,l}$.

$$F^p \oplus \bar{F}^q = \left(\bigoplus_{i \geq p} H^{i,j} \right) \oplus \left(\bigoplus_{l \geq q} \bar{H}^{l,k} \right) = \bigoplus_{\substack{i \geq p \\ l \geq n+1-p}} H^{i,j} \oplus H^{k,l} = \bigoplus_{p+q=n} H^{p,q} = H_\mathbb{C}.$$

b) Given ii), we construct i). Define $H^{p,q} := F^p \cap \bar{F}^q$. Note that $H^{q,p} = \bar{F}^p \cap F^q$ is the complex conjugate of $H^{p,q}$. Since F is a filtration we have

$$H_\mathbb{C} = \ldots \supset F^p \supset F^{p+1} \supset \ldots.$$

In additional we have $H_\mathbb{C} = F^{p+1} \oplus \bar{F}^q$ if $p+q = n$. Therefore we have $H^{p,q} = F^p \cap \bar{F}^q = F^p/F^{p+1}$. Since the dimension of $H_\mathbb{C}$ is finite and F is a filtration we have $H_\mathbb{C} = \bigoplus_{p+q=n} F^p/F^{p+1} = \bigoplus_{p+q=n} H^{p,q}$.

iii) Given i) we construct iii) via

$$h(z)v = z^{-p}\bar{z}^{-q}v$$

for $z \in \mathbb{C}^*$ and $v \in H^{p,q}$.

iv) Given iii). Every representation of $\mathbb{C}^*$ in an algebraic group is multiplication with $z^{-p}\bar{z}^{-q}$.

□

Definition 2.6.2. Let H be a vector space. Given i), ii) or iii) of Proposition 2.6.1 is a *Hodge structure* on H. The filtration in ii) is called *Hodge filtration.*
Let H be a vector space equipped with a Hodge structure. A *polarization* ψ on H is a bilinear form invariant under $h(U^1)$, such that $\psi(x, h(-i)y)$ is symmetric and positive definite.

Example 2.6.3. Let H be a real vector with a complex structure J. The complex structure gives an action of $\mathbb{C}^*$ on H, via $(a+ib) \cdot v := (a+bJ)v$ for $a+ib \in \mathbb{C}^*$ and $v \in H$. Denote by $\bar{H}$ the set that equals H as a vector space equipped with the action of $\mathbb{C}^*$ via $(a+ib) \cdot v := (a-bJ)b$. By Proposition 2.6.1 this gives a Hogde structure on H with $H^{-1,0} := H$ and $H^{0,-1} := \bar{H}$. With the notations above we have $F^0 = H^{0,-1}$ and $F^{-1} = H^{0,-1} \oplus H^{-1,0}$. $H \to H^\mathbb{C}/F^0(H_\mathbb{C})$ is a $\mathbb{C}$-linear isomorphism.
In this case a polarization is nothing but the imaginary part of a positive definite hermitian form. Hermitian means here, that $\phi(\lambda x, \mu y) = \lambda\phi(x,y)\bar{\mu}$, for $x, y \in H$ and $\lambda, \mu \in \mathbb{C}$. Since it is non-degenerate, we can find a basis $\{e_1, \ldots, e_m\}$ of H which is orthonormal with respect to ϕ. By assumption $\phi(e_i, e_i)$ is bigger than zero, hence real. If $x = \sum x_i e_i$ and $y = \sum y_i e_i$ we have

$$\phi(x,y) = \sum x_i \bar{y}_i \phi(e_i, e_i) = \sum \overline{y_i \bar{x}_i \phi(e_i, e_i)} = \overline{\phi(y,x)} = \phi(\bar{y}, \bar{x}).$$

Denote by ϑ the real part of ϕ and by ψ its imaginary part. We write zx instead of $h(z)x$. For $z \in U^1$ we have

$$\phi(zx, zy) = z\bar{z}\phi(x,y) = \phi(x,y),$$

that means ϕ and therefore ψ are invariant under $h(U^1)$. We have

$$\phi(x,-iy)=\phi(iy,x)=\phi(y,-ix),$$

hence $\phi(x,-iy)$ is symmetric. The last thing to check is the positive definiteness. Since ϕ is positive definite (and therefore real) we have $\phi(x,x)=\vartheta(x,x)>0$ and

$$\vartheta(x,-iy)+i\psi(x,-iy)=\phi(x,-iy)=i\phi(x,y)=i\vartheta(x,y)-\psi(x,y),$$

comparison of real and imaginary part of both sides shows that $\psi(x,-iy)=\vartheta(x,y)$ and therefore it is positive definite.
For the converse on can define $\phi(x,y):=\psi(x,-iy)+i\psi(x,y)$.

Remark 2.6.4. Let $w:\mathbb{G}_m\to G$ be a morphism. It defines for every representation (V,ϱ) of G a filtration. We look at the case where this representation is the adjoint representation of G on $\mathfrak{g}=Lie(G)$. We use the following notations: $w(\mathbb{G}_m)$ is abelian, hence contained in a maximal torus T. We denote by $\mathfrak{w}$ the Lie algebra of $w(\mathbb{G}_m)$ and by $\mathfrak{t}$ the Lie algebra of T. For $i\in\mathbb{Z}$ let

$$\mathfrak{g}^i:=\{x\in\mathfrak{g}|\mathrm{Ad}(w(\lambda))x=\lambda^i x,\ \forall\lambda\in\mathbb{G}_m\}$$

and for $\alpha:\mathfrak{t}\to\mathbb{R}$ linear

$$\mathfrak{g}_\alpha:=\{x\in\mathfrak{g}|\mathrm{ad}(h)x=\alpha(h)x,\ \forall h\in\mathfrak{t}\}.$$

One sees immediately, that the $\mathfrak{g}^i$ and the $\mathfrak{g}_\alpha$ are similar. They are both eigenspaces to some common eigenvalues. The non-empty $\mathfrak{g}_\alpha$ are rootspaces of $\mathfrak{g}$. We define a positive root system. $\mathfrak{w}$ is one-dimensional, hence the value of every root α on $\mathfrak{w}$ is determined by an arbitrary non-zero vector $a\in\mathfrak{w}$. Roots α which have the property that the restriction to the one-dimensional subspace $\mathfrak{w}$ is multiplication with a negative scalar, are now called positive. Let α be a positive root and $i:=\alpha(a)$. Since $\exp(\alpha(a))=\exp(\mathrm{ad}(a))=\mathrm{Ad}(\exp(a))$ with $\exp(a)\in w(\mathbb{G}_m)$ there exists a $\lambda\in\mathbb{G}_m$ with $w(\lambda)=\exp(a)$. Since $\exp(\alpha(a))=\mathrm{Ad}(\exp(a))=\lambda^i$ we have $i\in\mathbb{Z}^-$ and $\mathfrak{g}_\alpha\subset\mathfrak{g}^i$ for some $i\in\mathbb{Z}^-$. Therefore $\mathfrak{g}_\alpha\subset\mathfrak{g}^i$ for $i\leq 0$ and therefore $\mathfrak{g}_\alpha\subset W_0(\mathfrak{g})$. This shows that $W_0(\mathfrak{g})$ is a parabolic Lie algebra. It generates a parabolic subgroup $W_0(G)$ of G, which respects the filtration W on every representation of G.

Remark 2.6.5. Let M be a Hermitian symmetric space and G the identity component of its group of automorphisms. For $x\in M$ we have by Remark 1.4.9 a morphism $u_x:U^1\to G$ sending z to the automorphism of M that fixes and acts as multiplication by z on T_xM. For $g\in G$ the maps u_{gx} and gu_xg^{-1} act both as multiplication by z on $T_{gx}M$ and leave gx fixed. Therefore they are equal by Lemma A.1.5. We saw in Proposition 1.4.10 that the centralizer of u_x is equal to the stabilizer of x. Therefore the mapping $x\mapsto u_x$ identifies M with the set of conjugates of u_x.

Definition 2.6.6. Let V be a n-dimensional vector space. The set of d-dimensional subvector spaces of V, denoted by $G_d(V)$ is called *Grassmann variety*.

Let $\mathbf{d}=(d_1,\ldots,d_r)$ with $0<d_1<\ldots<d_r<n$. A *flag* F of type $\mathbf{d}$ is a sequence of decreasing vector spaces V^i

$$F:V\subset V^1\subset\ldots\subset V^r\subset 0$$

with $\dim V^i=d_i$. The set of flags of type $\mathbf{d}$ is called *flag variety*.

The group $GL(V)$ acts transitively on the set of bases of V, hence it acts transitively on $G_d(V)$. Let W be in $G_d(V)$ and denote by $P(W)$ the closed subset of $GL(V)$ which stabilizes W. Since the action of $GL(V)$ is transitive, we have

$$GL(V)/P(W) \simeq G_d(W).$$

Hence $G_d(V)$ is a manifold. The tangent space to $G_d(V)$ at W is

$$T_W(G_d(V)) \simeq \operatorname{Hom}(W, V/W).$$

See [Mi04, p. 13f.].

The map $F \mapsto (V^i)$ from $G_{\mathbf{d}}(V)$ to $\prod_i G_{d_i}(V)$ realizes $G_{\mathbf{d}}(V)$ as a closed subset of $\prod_i G_{d_i}(V)$. The proof that $G_{\mathbf{d}}(V)$ is a manifold works like the proof for $G_d(V)$. The tangent space of $G_{\mathbf{d}}(V)$ at a point F consists of homomorphisms $\varphi^i : V^i \to V/V^i$ with $\varphi^i|_{V^{i+1}} \equiv \varphi^{i+1} \mod V^{i+1}$.

Definition 2.6.7. Let $V^{p,q}$ be a real Hodge decomposition for the real vector space V. A *Hodge tensor* is a multilinear map (r-tensor)

$$t : \underbrace{V \times \ldots \times V}_{r-\text{times}} \to \mathbb{R}$$

which is left invariant by the action of h.

Now we will discuss *Variations of Hodge structures* and we will see that they are parametrized by Hermitian symmetric spaces.

Fix a real vector space V and let S be a connected complex manifold. A family of Hodge structures $(V_s^{p,q})$ parametrized by $s \in S$ is said to be *continuous* if, for each (p,q) the $V_s^{p,q}$ vary continuously with s, i.e. $d(p,q) = \dim V^{p,q}$ is constant and $s \mapsto V_s^{p,q} \in G_{d(p,q)}(V)$ is continuous. Let $\mathbf{d} = (\ldots, d(p), \ldots)$ with $d(p) = \sum_{r \geq p} d(r,s)$. A continuous family of Hodge structures $(V_s^{p,q})$ is *holomorphic* if the Hodge filtration F_s vary holomorphically, i.e.

$$\psi : s \mapsto F_s \in G_{\mathbf{d}}(V)$$

is holomorphic. The differential of ψ at s is $\mathbb{C}$-linear:

$$d\psi_s : T_s S \to T_{F_s}(G_{\mathbf{d}}(V)) \subset \oplus_p \operatorname{hom}(F_s^p, V/F_s^p).$$

If the image of $d\phi_s$ is contained in

$$\oplus_p \operatorname{hom}(F_s^p, F_s^{p-1}/F_s^p)$$

for all s, then the holomorphic family is called a *variation of Hodge structures on S*.

Let V be a real vector space and let $T = \{t_i\}$ be a finite family of tensors with t_0 a non-degenerate bilinear form on V. Let $d : \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$ be such that

- $d(p,q) = 0$ for almost all p, q;
- $d(q,p) = d(p,q)$;
- $d(p,q) = 0$ unless $p + q = n$.

Let $S(d,T)$ be the set of all Hodge structure $(V^{p,q})$ on V such that

- $dim V^{p,q} = d(p,q)$ for all p,q;
- each $t \in T$ is a Hodge tensor of $V^{p,q}$;
- t_0 is a polarization of $V^{p,q}$.

The set $S(d,T)$ is a subspace of $\prod_{d(p,q)\neq 0} F_{d(p,q)}(V)$. Let $S(d,T)^+$ be a connected component of $S(d,T)$.

The next theorem is the reason why we introduced Hodge structures in the context of symmetric spaces.

Theorem 2.6.8. *(a) If nonempty, $S(d,T)^+$ has a unique structure of a complex manifold for which $(V, V_S^{p,q})$ is a holomorphic family of Hodge structures.*

(b) With this complex structure, $S(d,T)^+$ is a hermitian symmetric domain if $(V, V_s^{p,q})$ is a variation of Hodge structures.

(c) Every irreducible hermitian symmetric domain is of the form $S(d,T)$ for a suitable V, d and T.

Proof. To show (a), first of all one shows that S^+ can be written as a homogeneous space, namely as a quotient of the smallest subgroup of $GL(V)$ which contains $h(\mathbb{S})$ for all $h \in S^+$ and a point stabilizer. The complex structure on S^+ comes from an embedding into $G_{\mathbf{d}}(V)$.

For part (b) see [De79]. The proof uses Remark 1.4.14.

Given an irreducible Hermitian symmetric domain. We are searching for a vector space V, a suitable set of tensors T and a map d. The vector space V arises as a self dual representation $G \to GL(V)$ where G the algebraic subgroup of $GL(\mathfrak{h})$ for $\mathfrak{h} = \mathrm{Lie}(Hol(M)_+)$ with $G(\mathbb{R})^+ = Hol(M)^+$. One can show that there exists a non-degenerate bilinear form t_0 on V fixed by G. Furthermore one can find a set of tensors such that G is the subgroup of $GL(V)$ which fixes T. Let $x \in D$ and let $h_x : \mathbb{S} \to U^1 \to u_x$ be the map which maps z to $u_x(z/\bar{z})$. This h_x defines a Hodge structure for which the $t \in T$ are Hodge tensors and t_0 is a polarisation.

See [De79, 1.3] and [Mi04, Ch. 2] for details

2.7 Symmetric Cones and Jordan Algebras

In this chapter we will give a short introduction to symmetric cones and Jordan algebras and their correspondence. A good reference is [FK94].

2.7.1 Cones

Let V be a finite dimensional real Euclidean vector space with scalar product $(\cdot|\cdot)$. A subset $C \subset V$ is called *cone* if it is non-empty and if for all $\lambda > 0$ and $c \in C$ the vector $\lambda v \in C$. A cone is convex if and only if $\lambda, \mu > 0$ and $x, y \in C$ imply $x\lambda x + \mu y \in C$. A cone C is *proper* if $C \cap (-C) = \{0\}$. The *dual open cone* of a convex cone C is defined by

$$C^* := \{y \in Y | (x|y) > 0 \; \forall x \in \bar{C} \setminus \{0\}\}.$$

An open convex cone C is said to be *self dual* if $C = C^*$. Such a cone is proper since for a non-zero $x \in C \cap (-C)$ we have $-x \in C$ and from $(x| -x) < 0$ we deduce see that $C = C^*$ is impossible.

The automorphism group $G(C)$ of an open convex cone C is defined by

$$G(C) = \{g \in GL(V) | gC = C\}.$$

It is a closed subgroup of $GL(V)$ hence a Lie group. The open cone C is said to be *homogeneous* if $G(C)$ acts transitively on it. In this case we can apply Theorem A.1.4. A cone is said to be *symmetric* if it is homogeneous and self dual.

Example 2.7.1. In Example 1.2.4 we discussed $P = P(n, \mathbb{R})$, the set of positive definite matrices in $Sym(n, \mathbb{R})$. It is clearly a cone in $\mathbb{R}^{n\times n}$. The closure $\bar{P}$ of P is the set of positive matrices. We want to prove that P is a symmetric cone. We know from the discussion in Example 1.2.4 that the group $GL(n, \mathbb{R})$ acts via

$$g \cdot x := gxg^\top$$

linearly on P and that this action is transitive. It remains to show that P is self-dual. To show that we need a criterion for $y \in Sym(n, \mathbb{R})$ to be positive definite. An inner product on $Sym(n, \mathbb{R})$ is given by

$$(x|y) = \operatorname{tr}(xy) = \sum_{i,j} x_{ij} y_{ij}.$$

The quadratic form Q associated to the symmetric matrix x is give by

$$Q(\xi) = \sum_{i,j} x_{i,j} \xi_i \xi_j$$

for $\xi \in \mathbb{R}^n$. Therefore we have

$$Q(\xi) = (x|\xi\xi^\top).$$

Back to our problem: let $y \in Q^*$. For a non-zero $\xi \in \mathbb{R}^n$ the matrix $x = \xi\xi^\top$ is positive and symmetric hence $x \in \bar{P} \setminus \{0\}$. By the definition of P^* we get

$$\sum_{j=1}^{k} y_{ij} \xi_i \xi_j = (y|x) > 0.$$

The vector ξ was arbitrary, hence $y \in P$.

Any element $x \in \bar{p} \setminus \{0\}$ can be written as

$$x = \sum_{j=1}^{k} \alpha_j \alpha_j^\top$$

where the α_j are independent vectors in $\mathbb{R}^n$ and $k \geq 1$. Therefore, if $y \in P$:

$$(y|x) = \sum_{j=1}^{k} (y|\alpha_j \alpha_j^\top) > 0$$

since all $(y|\alpha_j\alpha_j^\top) = Q(\alpha_j) > 0$. Hence $P \subset P^*$. This shows $P = P^*$, therefore P is a symmetric cone.

One can turn a symmetric cone C into a Riemannian symmetric space like we did it with bounded domains in $\mathbb{C}^n$. First we introduce a metric on C. This works like in the case of the bounded domains. Define

$$\varphi(x) := \int_{C^*} e^{-(x|y)} dy.$$

This integral exists, it is uniformly convergent for x in any compact subset and it is C^∞. For all $g \in G(C)$ we have

$$\varphi(gx) = |\det g|^{-1} \varphi(x).$$

The map $x \mapsto x^* := \nabla \log \varphi(x)$ is a bijection of C onto C^*. It has a unique fixed point.

We define for $x \in C$ and $u, v \in V$

$$G_x(u, v) := \frac{\partial^2}{\partial u \partial v} \log \varphi(x).$$

Where One can show (see [FK94, Ch. I.4]) that is in fact a metric, which is invariant under $G(C)$. The map $x \mapsto x^*$ becomes an isometry, hence C is a Riemannian symmetric space.

2.7.2 Jordan Algebras

A *Jordan algebra* is a vector space V with a bilinear (not necessarily associative!) product $V \times V \to V,\ (x, y) \mapsto xy$ with

$$\begin{aligned} yx &= xy \\ x(x^2 y) &= x^2(xy). \end{aligned}$$

Define for $L(x)$ the map $L(x) : V \to V$ which maps y to xy. The second property says that $L(x)$ and $L(x^2)$ commute.

Example 2.7.2. Let V be the algebra of $n \times n$ matrices over a field F. For $x, y \in V$ put

$$x \circ y := \frac{1}{2}(xy + yx).$$

This defines a non-associative Jordan algebra as one can easily check by a direct calculation.

A Jordan algebra is said to *Euclidian* if there exists a positive definite bilinear form $(\cdot|\cdot)$ such that $(L(x)u|v) = (u|L(x)|v)$ for all $x, u, v \in V$.

2.7.3 Correspondence between Cones and Jordan Algebras

Let V be an Euclidian Jordan algebra. Let

$$Q := \{x^2 | x \in V\}.$$

Clearly, Q is a cone. The interior C of Q is a symmetric cone. This is not obvious and the proof is very technical. It can be found in [FK94, p. 46ff]. The cone C is the connected component of e in the set of invertible elements.

Let now C be a symmetric cone in a Euclidian space V. We denote by $G(C)$ its automorphism group, by G the identity component of $G(C)$ and by K the subgroup $G \cap O(V)$. One can show that there exists a point $e \in C$ whose stabilizer is K. We write $\mathfrak{g}$ for the Lie algebra of G and $\mathfrak{k}$ for the Lie algebra of K. An element X of $\mathfrak{g}$ belongs to $\mathfrak{k}$ if an only if $X \cdot e = 0$. We have $G \cdot e = C$ and one can show that $\mathfrak{g} \cdot e = V$. Therefore we have a subspace $\mathfrak{p}$ of $\mathfrak{g}$ which is isomorphic to V via $X \mapsto X \cdot e$. We denote by L its inverse: for $x \in V$, $L(x)$ is the unique element in $\mathfrak{p}$ such that $L(x) \cdot e = x$.

Defining

$$xy := L(x)y$$

gives V the structure of a Jordan algebra with identity element e. We have $xy - yx = [L(x), L(y)] \cdot e = 0$, since $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ and $\mathfrak{k} \cdot e = 0$. The proof of the second property of a Jordan algebra is again very technical and therefore omitted. It can be found in [FK94, p. 49f].

A Appendix

A.1 Riemannian geometry

Theorem A.1.1 (Hopf-Rinow)**.** *Let M be a Riemannian manifold. The following three conditions are equivalent*

(i) M is geodesically complete;

(ii) M is complete;

(iii) Each bounded subset $U \subset M$ is relatively compact.

These three conditions imply

(iv) Two points in M can be joined by a geodesic.

Proof. This proof is due to de Rham in [dRh52].
"(iii) implies (ii)" is trivial, "(ii) implies (i)" is easy, since every non-closed geodesic which can not be continued to infinity contains a non-convergent Cauchy-sequence. Therefore it suffices to show that (i) implies (iii) and (iv).
Fix $a \in V$. An initial vector of a geodesic arc γ from a to x is a tangent vector of γ with the same length and direction. By assumption i) for every vector in T_aM there exits a geodesic arc starting in a in the direction of this vector and with the same length.
For $r \geq 0$ let S_r be the set of points $x \in M$ with $d(x, a) \leq r$. Denote by E_r the subset of S_r which contains the points that can be joined with a by a geodesic. E_r is compact. To see this take a sequence x_k in E_r. Every x_k can be joined with a by a geodesic γ_k, parameterized such that $\gamma_k(0) = a$ and $\gamma_k(1) = x_k$. Therefore $\dot{\gamma}_k$ is contained in the ball with radius r and center 0 in T_aM, which is compact. Hence $\dot{\gamma}_k$ has an accumulation point X. X generates a geodesic γ_X and $\gamma_X(1)$ is an accumulation point of x_k. Therefore E_r is compact. We will show that $E_r = S_r$ for all $r \geq 0$. This relation is true for $r = 0$. If it is true for $r = R > 0$, then it is also true for $r < R$. Conversely if it holds for every $r < R$, it also holds for $r = R$, since E_R is closed. We will show that if $E_R = S_R$, then there exists $s > 0$ such that $E_{R+s} = S_{R+s}$.
First note, that the metric $d(x, y)$ is defined as the infimum of the length' of pathes joining x and y. Hence for all $n \in \mathbb{N}$ we can find for all $y \in M$ a path (not necessarily a geodesic) from a to y such that its length is smaller than $d(a, y) + h^{-1}$. Denote by x_n the last point of this path which is contained in $E_R = S_R$. Then we have $d(a, x_n) = R$ and

$$d(x_n, y) \leq d(a, y) - R + n^{-1}.$$

From the triangle inequality we get

$$d(a, y) - R = d(a, y) - d(a, x_n) \leq d(x_n, y).$$

Since $(x_n) \subset E_R$ it has an accumulation point x and by the inequalities above we get $d(a, y) = d(x, y) + R$.
On M there exists a continuous function $s : M \to \mathbb{R}$, such that if $d(x, y) \leq s(x)$, then x and y can be joined by a unique geodesic of length $d(x, y)$. Every point in M has a normal neighborhood, hence s can be chosen to be positive. Since E_R is compact, s has a minimum on it, denote it by $s(> 0)$. s depends continuously of R. Let $y \in M$ such that $R < d(a, y) \leq R + s$. There exists $x \in E_R$ such that $d(a, x) = R$ and $d(x, y) = d(a, y) - R \leq s$.

Hence x and y respectively x and a can be joined by geodesics. The path between a and y has length $d(a,y)$ and is therefore a geodesic. Hence $y \in E_{R+s}$. This shows that if $E_R = S_R$ then $E_{R+s(R)} = S_{R+s(R)}$ hence $E_{R+s(R)+s(R+s(R))} = S_{R+s(R)+s(R+s(R))}$ and so on. If the sequence $R_{n+1} := R_n + s(R_n)$ converges to $R \in \mathbb{R}$, then

$$R = \lim_{n\to\infty} R_n = \lim_{n\to\infty}(R_n + s(R_n)) = R + s(R),$$

therefore $s(R) = 0$. Since s is positive, R_n can not converge. Hence $S_R = E_R$ for all R. □

Theorem A.1.2. *The stabilizer K of a point $p \in M$ in the group of isometries G is compact.*

Proof. G is equipped with the compact open topology, which is generated by sets of the form $W(C,U) := \{f \in G | f(C) \subset U\}$, where $C, U \subset G$ with C compact and U open. K is a closed subset of $W(\{p\}, U)$. The last has compact closure, hence K is compact. A more detailed proof can be found in [He78, I.3]. □

Theorem A.1.3. *The automorphisms of a symmetric space form a finite dimensional Lie group G.*

Proof. If G is a Lie group, it has a Lie algebra $\mathfrak{g}$. The stabilizer of a point p is a compact subgroup, which can be embedded in $O(T_pM)$ via $k \mapsto dk$, hence it is a Lie (sub-)group with Lie algebra $\mathfrak{k}$. Therefore $\mathfrak{g}$ decomposes into $\mathfrak{k} \oplus \mathfrak{p}$. If we can determine $\mathfrak{p}$ we can generate G in a neighborhood of e via exp and with some technical stuff one can in fact show, that G is a Lie group. The transformations $\tau(\gamma, u)$, defined in Corollary 1.1.11 with $\gamma(0) = p$ are elements of G. They are generated by elements of T_pM. In fact one can show (see [He78, I.3]) that they do the job. □

Theorem A.1.4. *Let G be a locally compact group with a countable base. Suppose G is a transitive topological transformation group of a locally compact Hausdorff space M. Let p be any point in M and K the subgroup of G which leaves p fixed. Then K is closed and the mapping $gH \mapsto g \cdot p$ is a homeomorphism of G/H onto M.*

Proof. Since K is the stabilizer of p it is clear that the map $gH \mapsto g \cdot p$ is welldefined and bijectiv. See [He78, Thm. IV.3.3] for details. □

Lemma A.1.5. *Let M be a Riemannian manifold, φ and ψ two isometries of M onto itself. Suppose there exists a point $x \in M$ for which $\varphi(x) = \psi(x)$ and $d\varphi_x = d\psi_x$. Then $\varphi = \psi$.*

Proof. We may assume $\varphi(x) = x$ and $d\varphi_x$ is the identity. In an arbitrary normal neighborhood of x all points are left fixed by φ because φ is an isometry. Since M is connected each point $y \in M$ can be connected to x by a chain of overlapping normal neighborhoods and hence $\varphi(y) = y$. □

Lemma A.1.6. *If K is a compact Lie subgroup of a Lie group G, then the matrices of $ad(\mathfrak{k})$ are all skew-symmetric.*

Proof. $\mathrm{Ad}(K)$ leaves (on $\mathfrak{g}$) the positive non-degenerate bilinear form

$$< a, b >= \int_K (\mathrm{Ad}(k) \cdot a, \mathrm{Ad}(k) \cdot b) dk,$$

invariant. $(\cdot,\cdot)$ was an arbitrary positive definite bilinear form on $\mathfrak{g}$. Therefore $\mathrm{Ad}(K)$ is a closed subgroup of the orthogonal group that leaves $<\cdot,\cdot>$ fixed, which is compact. Hence the matrices in $\mathrm{Lie}(\mathrm{Ad}(K))$ are skew-symmetric. Since for $k \in \mathfrak{k}$ we have $\exp(k) \in K$ and

$$\exp \circ \mathrm{ad}(k) = \mathrm{Ad} \circ \exp(k) \in \mathrm{Ad}(K).$$

Therefore $\mathrm{ad}(k) \in \mathrm{Lie}(\mathrm{Ad}(K))$ and hence $\mathrm{ad}(\mathfrak{k})$ consists only of skew-symmetric matrices. □

Lemma A.1.7. *An skew-symmetric real matrix A is diagonalisable and all eigenvalues are of the form it with $t \in \mathbb{R}$.*

Proof. A skew-symmetric matrix over $\mathbb{R}$ can be seen as a hermitian matrix over $\mathbb{C}$. If it is diagonalisable, its eigenvalues must have the form it with $t \in \mathbb{R}$.
We know from linear algebra, that a symmetric matrix is diagonalisable. A^2 is symmetric and hence diagonalisable. Let $\lambda \in \mathbb{R}$ be an eigenvalue of A^2 and $e \neq 0$ an eigenvector with respect to λ. The space X spanned by e and Ae is invariant under A because Ae and $A^2e = \lambda e$ are contained in X. Now we have two possibilities. The dimension of X is either 1 or 2. In the first case $Ae = \tilde{\lambda}e$ with $\tilde{\lambda}^2 = \lambda$. Therefore A can be written as

$$\begin{pmatrix} \tilde{A} & 0 \\ 0 & \tilde{\lambda} \end{pmatrix}$$

with $\tilde{A}$ a $(n-1) \times (n-1)$-matrix and since A is skew-symmetric, λ and $\tilde{\lambda}$ are zero.
If $\lambda = 0$ and $Ae = \tilde{\lambda}e$, we get $0 = A^2e = \tilde{\lambda}^2 e$ and since $e \neq 0$ by assumption we get $\tilde{\lambda} = 0$ and therefore $\dim X = 1$. Therefore if $\dim X = 2$ we have $\lambda \neq 0$. The matrix of A restricted to X with respect to the basis $\{e, Ae\}$ is

$$\begin{pmatrix} 0 & \lambda \\ 1 & 0 \end{pmatrix}$$

Since $\lambda \neq 0$, this matrix is diagonalisable, because it has two distinct eigenvalues $\pm\sqrt{\lambda}$. □

Remark A.1.8. We can use the same argumentation to show that a matrix A is diagonalizable if its square is diagonalizable with non-zero entries on the diagonal. To see that, take an eigenvector e and look at the space $X = \{e, Ae\}$. If it is one-dimensional we have $Ae = \lambda e$. If the dimension is 2, A acts as matrix

$$\begin{pmatrix} 0 & \lambda \\ 1 & 0 \end{pmatrix}$$

which is diagonalizable since λ is non-zero. This works especially for involutions and complex structures on vector spaces.

A.2 Roots

A good introduction to roots is [Kn96]. But everything we need can also be found in [He78, Ch.III.3].

Definition A.2.1. Let $\mathfrak{g}$ be a Lie algebra. A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called *Cartan subalgebra* if for all $h \in \mathfrak{h}$ the map $\mathrm{ad}(h)$ is semi-simple, i.e. every invariant subspace has an invariant complement.

Theorem A.2.2. *Every semi-simple Lie algebra over $\mathbb{C}$ has a Cartan subalgebra.*

Definition A.2.3. Let $\alpha : \mathfrak{h} \to \mathbb{C}$ be a linear functional. We put

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} | \mathrm{ad}(H)X = \alpha(H)X \forall H \in \mathfrak{h}\}.$$

If $\mathfrak{g}_\alpha \neq 0$, the functional α is called *root* and $\mathfrak{g}_\alpha$ is the *root space.*

Theorem A.2.4. *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. We denote by Δ the set of non-zero roots of $\mathfrak{g}$. Then*

(i) $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus \mathfrak{g}_\alpha$*;*

(ii) $\dim_{\mathbb{C}} \mathfrak{g}_\alpha = 1$*;*

(iii) if α and β are roots with $\alpha + \beta \neq 0$, then $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal with respect to $B_{\mathfrak{g}}$;

(iv) $B|_{\mathfrak{h}\times\mathfrak{h}}$ is non-degenerate and for every linear functional α on $\mathfrak{h}$ exists a unique $H_\alpha \in \mathfrak{h}$ with

$$\alpha(H) = 2\frac{B(H, H_\alpha)}{B(H_\alpha, H_\alpha)} \qquad \forall H \in \mathfrak{h}$$

and $\alpha(H_\alpha) \neq 0$;

(v) if $\alpha \in \Delta$ then $-\alpha \in \Delta$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{C}H_\alpha$;

(vi) if $\alpha + \beta \neq 0$ then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

A.3 Algebraic groups

We follow [Mi06].
Let k be a field. An algebraic group is a group defined by polynomials. Examples are $SL(V)$, it is defined as $\{A \in M^{n\times n}(k) | \det(A) - 1 = 0\}$ or $GL(V)$, which is defined as $\{(A, t) | A \in M^{n\times n}(k), t \in k : \det(A)t - 1 = 0\}$, where $M^{n\times n}(k)$ denotes the vector space of $n \times n$-matrices over k. Since a polynomial over k is also a polynomial over a k-algebra A we need a concretization of the definition:

Definition A.3.1. Let G be a functor of k-algebras into groups. G is an affine algebraic group if there exists a finitely generated k-algebra A such that

$$G(R) = \mathrm{Hom}_{k-\text{algebra}}(A, R)$$

is functorially in R.

To see why this definition makes sense, G be a group defined by the sets of zeros of polynomials $f_1, \ldots, f_n$ and A be the ideal in $k[X_1, \ldots, X_n]$ generated by the f_i. Then $G(R) = \mathrm{Hom}(k[X_1, \ldots, X_n]/A, R)$.
By [Mi06, Ch. 3] one can embed every algebraic group in the general linear group over a vector space V.

Example A.3.2. (i) $\mathbb{G}_a$ is defined by the set of zeros of the zero-polynomial. $\mathbb{G}_a(k) = k$ as an additive group. In the above notation it is $\mathrm{Hom}_k(k[X], R)$.

(ii) $\mathbb{G}_m$ is defined as the set of zeros of the polynomial $XY - 1$. $\mathbb{G}_m(k) = k^*$ is the multiplicative group of k. We have $\mathbb{G}_m(R) = \mathrm{Hom}_k(k[X, X^{-1}], R)$.

Definition A.3.3. (i) An algebraic group is a *split torus* if it is isomorphic to a product of copies of $\mathbb{G}_m$, and it is a *torus* if it becomes a split torus over the algebraic closure of k.

(ii) Let R be a ring with 1. $x \in R$ is called *unipotent*, if $1 - x$ is nilpotent[13]

(iii) A group G is *solvable* if there exists a chain of subgroups

$$G = G_1 \supset G_2 \supset \ldots \supset G_n = 1,$$

such that G_i is normal in G_{i+1} and G_i/G_{i+1} is abelian.

(iv) The *radical* of an algebraic group G is the largest[14] normal solvable subgroup. It is denoted by RG.

(v) The *unipotent radical* is the intersection of RG and the set of unipotent elements of G. It is denoted by R_uG.

(vi) A smooth connected algebraic group $G \neq 1$ is *semisimple* if it has no smooth connected normal commutative subgroup other than the identity, and it is *reductive* if the only such subgroups are tori.

Proposition A.3.4. *Let k be a field of characteristic zero. Every algebraic group over k has a radical and a unipotent radical.*

Proof. Let H and N be normal algebraic solvable subgroups of G. Then HN is normal and solvable. (See [Mi06, p. 94]) □

Remark A.3.5. A group is reductive if and only if its unipotent radical is zero. A reductive group has the property that every representation decomposes in irreducible representations.

Definition A.3.6. The *derived group* G^{der} is the intersection of the normal algebraic subgroups N of G such that G/N is commutative.

Theorem A.3.7. *If G is reductive, then the derived group G^{der} of G is semisimple, the connected center Z° of G is a torus, and $Z^\circ \cap G^{der}$ is the (finite) center of G^{der}; moreover, $Z^\circ \cdot G^{der} = G$.*

Definition A.3.8. A *Levi-subgroup* H has the property $G = R_uG \rtimes H$.

Definition A.3.9. (i) A *parabolic subgroup* P is a subgroup, such that G/P is projective.

(ii) A *Borel subgroup* is a maximal connected solvable algebraic subgroup.

Proposition A.3.10. *A subgroup P is parabolic if and only if it is contained in a Borel subgroup. Every parabolic subgroup can be identified with a stabilizer of a (not necessarily complete) flag.*

Lemma A.3.11. *(Schur's lemma). Let G be a group and (V, ϱ) a irreducible complex representation of G, then the only automorphism of this representation, i.e. vectorspace-morphisms which commute with $\varrho(G)$ are complex multiples of the identity matrix.*

[13] i.e. there exists a $n \in \mathbb{N}$ with $(1 - x)^n = 0$.

[14] Largest means that it contains every other normal solvable subgroup.

A.4 Details

Proposition A.4.1. *The centralizer of a torus in a connected compact Lie group is connected.*

This proposition is Corollary 4.51 in [Kn96]. The proof needs two lemmas:

Lemma A.4.2. *Let A be a compact abelian Lie group. Denote by A_0 its identity component. Assume that A/A_0 is cyclic. Then A has an element whose powers are dense in A.*

Proof. Since A_0 is connected, closed and abelian it is a compact torus, hence isomorphic to $S^1 \times \ldots \times S^1$. Therefore there exists $a_0 \in A_0$ such that the powers are dense in A_0. Since A is compact, the factor group A/A_0 is finite (say of order N) and we can choose by assumtion a representative b of a generating coset. Clearly $b^N \in A_0$. Since A_0 is a torus we can find $c \in A_0$ such that $b^N c^N = a_0$. The closure of the powers of bc contains A_0 and a representative of each coset in A/A_0, hence it contains A. □

Lemma A.4.3. *Let G be a compact connected Lie group and let S be a torus of G. If $g \in G$ centralizes S, then there exists a torus S' in G containing both S and g.*

Proof. Let A be the closure of $\bigcup_{n=-\infty}^{\infty} g^n S$. This is a subgroup of G, since for $g^n s, g^m t \in A$ we have $g^n s(g^m t)^{-1} = g^{m-n} st^{-1} \in A$. The identity component of A_0 is closed, abelian and connected, hence a torus. Since it is also open, the set $\bigcup_{n=-\infty}^{\infty} g^n A_0$ is open in A and it contains $\bigcup_{n=-\infty}^{\infty} g^n S$. Therefore $\bigcup_{n=-\infty}^{n} g^n A_0 = A$. By compactness of A_0 some nonzero power of g is in A_0. Therefore A/A_0 is cyclic and by Lemma [??] we can find $a \in A$ whose powers are dense in A. Since A is compact, we can find an element X of the Lie algebra $\mathfrak{a}$ of A such that $a = \exp X$. The closure of $\{\exp tX | t \in \mathbb{R}\}$ is a torus S' containing A, and therefore S and g. □

Proof of Proposition A.4.1*:* The centralizer of S is by the lemma above the union of tori, hence connected.

Proposition A.4.4. *Let D and D' be domains, $F : D \to D'$ an isomorphism and denote by j_F the Jacobi determinant of F. The map $\sigma : \mathcal{H}^2(D') \to \mathcal{H}^2(D)$ given by $f \mapsto (f \circ F) j_F$ is a Hilbert space isomorphism.*

Proof. The linearity is obvious. The inverse of σ is give by $f \mapsto (f \circ F^{-1}) j_F^{-1}$. It exists since F is an isomorphism, which implies the existence of F^{-1} and j_F^{-1}. The last thing to check is that σ leaves the scalar product invariant. Let $f, g \in \mathcal{H}^2(D)$:

$$((f \circ F) j_F | (g \circ F) j_F) = \int_D f(F(z)) \overline{g(F(z))} j_F(z) \overline{j_F(z)} d\mu(z) = \int_{D'} f(z') \overline{g(z')} d\mu(z) = (f|g).$$

In the last step we used the fact that $|j_F|^2$ is the Jacobi determinant of F regarded as a map $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$. This can be seen by writing $\frac{\partial}{\partial z} = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y}$ and using the Cauchy-Riemann differential equations. □

Proposition A.4.5. *For the Bergman kernels K of D and K' of D' we have*

$$K'(F(z), F(w)) j_F(z) \overline{j_F(w)} = K(z, w).$$

Proof. Let $f : D' \to \mathbb{C}$ in $\mathcal{H}^2(D)$ and $w \in D$. We have

$$\begin{aligned} f(F(w))j_F(w) &= (f|K_{F(w)})j_F(w) = ((f \circ F)j_F|(K'_{F(w)} \circ F)j_F)j_F(w) \\ &= ((f \circ F)j_F|(K'_{F(w)} \circ F)j_F\overline{j_F(w)}), \end{aligned}$$

where we applied in the first step the defrinitio of K' on $f : D' \to \mathbb{C}$. In the second step we used Proposition A.4.4. One the other hand, if one applies the definition of K to $(f \circ F)j_F : D \to \mathbb{C}$, we get

$$f(F(w))j_F(w) = ((f \circ F)j_F(w)|K_w).$$

□

Example A.4.6.

$$\begin{aligned}
G = SU(1,1) &= \left\{ \begin{pmatrix} a & \bar{b} \\ b & \bar{a} \end{pmatrix} \Big| a, b \in \mathbb{C},\ |a|^2 - |b|^2 = 1 \right\}; \\
\mathfrak{g} = \mathfrak{su}(1,1) &= \left\{ \begin{pmatrix} ix & y + iz \\ y - iz & -ix \end{pmatrix} \Big| x, y, z \in \mathbb{R} \right\}; \\
\mathfrak{k} &= \left\{ \begin{pmatrix} ix & 0 \\ 0 & -ix \end{pmatrix} \Big| x \in \mathbb{R} \right\}; \\
\mathfrak{p} &= \left\{ \begin{pmatrix} 0 & y + iz \\ y - iz & 0 \end{pmatrix} \Big| x, y \in \mathbb{R} \right\}; \\
K &= \left\{ \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} \Big| a \in \mathbb{C},\ |a|^2 = 1 \right\}; \\
\mathfrak{g}_U &= \left\{ \begin{pmatrix} ix & iy - z \\ iy + z & -ix \end{pmatrix} \Big| x, y, z \in \mathbb{R} \right\}; \\
G_U &= \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \Big| a, b \in \mathbb{C},\ |a|^2 + |b|^2 = 1 \right\}; \\
\mathfrak{g}_{\mathbb{C}} &= \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \Big| a, b, c \in \mathbb{C} \right\}; \\
G_{\mathbb{C}} &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Big| a, b, c, d \in \mathbb{C},\ ad - bc = 1 \right\}; \\
\mathfrak{k}_{\mathbb{C}} &= \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \Big| a \in \mathbb{C} \right\}; \\
K_{\mathbb{C}}P^- &= \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \Big| a, c, d \in \mathbb{C},\ ad = 1 \right\};
\end{aligned}$$

Proposition A.4.7. *The matrices in* $SU(1,1)$ *are exactly those of the form*

$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$$

with $a, b \in \mathbb{C}$ *and* $|a|^2 - |b|^2 = 1$.

Proof. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix in $SU(1,1)$. By definition we have

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a\bar{a} - c\bar{c} & \bar{a}b - \bar{c}d \\ a\bar{b} - c\bar{d} & b\bar{b} - d\bar{d} \end{pmatrix},$$

hence

$$a\bar{a} - c\bar{c} = 1 \tag{14}$$
$$d\bar{d} - b\bar{b} = 1 \tag{15}$$
$$a\bar{b} = c\bar{d} \tag{16}$$
$$ad - bc = 1. \tag{17}$$

We have to show that $c = \bar{b}$ and $d = \bar{a}$. We multiply (14) and (15), their product is equal to 1 and we get:

$$\begin{aligned} &(a\bar{a} - c\bar{c})(d\bar{d} - b\bar{b}) = a\bar{a}d\bar{d} - c\bar{c}d\bar{d} - a\bar{a}b\bar{b} + c\bar{c}b\bar{b} = a\bar{a}d\bar{d} - 2c\bar{c}d\bar{d} + c\bar{c}b\bar{b} \\ =&(a\bar{a} - c\bar{c})d\bar{d} - c\bar{c}(d\bar{d} - b\bar{b}) = d\bar{d} - c\bar{c} = 1. \end{aligned}$$

We used (16) in the second step and (14) and (15) in the last step. Again with (14) and (15) we see with $d\bar{d} - c\bar{c} = 1$ that $b\bar{b} = c\bar{c}$ and $d\bar{d} = a\bar{a}$. Therefore the absolute values of a and d respectively b and c are equal and there exist two complex numbers s_1 and s_2 in the unit circle such that $b = s_1 c$ and $d = s_2 a$. With (14) and (17) we have

$$ad - bc = as_2a - cs_1c = 1 = a\bar{a} - c\bar{c}$$

and therefore

$$a(s_2a - \bar{a}) = c(s_1c - \bar{c}).$$

With (16) we get $0 = a\bar{b} - c\bar{d} = a\bar{c}\bar{s}_1 - c\bar{a}\bar{s}_2$. Adding $ac - ac$ gives

$$c(\bar{a}\bar{s}_2 - a) = a(\bar{c}\bar{s}_1 - c).$$

(14) can be written as $|a|^2 - |c|^2 = 1$, hence $a \neq 0$. Now assume $cs_2 - \bar{c} \neq 0$. Since a is non-zero the right-hand side of the above equation is non-zere, hence the left-hand side and we can divide by all factors of the equations. This yields

$$\frac{a}{c} = \frac{s_1c - \bar{c}}{s_2a - \bar{a}} = \frac{\bar{c}}{\bar{a}},$$

hence $a\bar{a} = c\bar{c}$ with is a contradiction to (14). Therefore $b = s_1c = \bar{c}$. If c is non-zero, $d = \bar{a}$ follows from (16), if c is zero, it follows from 17 and we are done. □

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