AIM Workshop on Representations of Surface Groups (March 19-23, 2007):
Questions and open problems

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1 Representations into $\text{Sp}(4, \mathbb{R})$

**Question 1.1** (Bill Goldman). Can every maximal representation in $\text{Sp}(4, \mathbb{R})$ be deformed to a proper Zariski closed subgroup?

**Comment 1.2** (Bill Goldman). Every maximal representation of the unitary group can be deformed in this way.

**Comment 1.3** (Anna Wienhard). The problem should be easier for $\text{Sp}(2n, \mathbb{R})$ with $n > 2$ (it is still not known though). For $n = 2$ progress was made at the conference on many components, however there exist $2(g - 2)$ “exotic” components for which nothing is currently known.

2 Geometry of the Hitchin component

**Question 2.1** (John Loftin). Does there exist a mapping class group-invariant Kähler structure on the Hitchin component for $\text{SL}(3, \mathbb{R})$?

**Comment 2.2** (Richard Wentworth). Theorem 1.0.2 of [Lab06] shows that the Hitchin component for $\text{SL}(3, \mathbb{R})$ is given by the bundle of cubic differentials on Teichmüller space. There is a mapping class group-invariant complex structure on this space.
Comment 2.3 (John Loftin). There is evidence for a Kähler structure, since transverse to the fibres there is a Kähler metric (Weil-Petersson), and on the fibres there is a Kähler metric.

Question 2.4 (Richard Wentworth, Francois Labourie). Let $S$ be a closed surface, $\rho : \pi_1(S) \to \text{SL}(n, \mathbb{R})$ a representation in the Hitchin component. Given a complex structure $J$ on $S$ there exists a unique $\rho$-equivariant harmonic map $u : (\tilde{S}, J) \to X = \text{SL}(n, \mathbb{R}) / \text{SO}(n)$ (Corollary 3.5 of [Cor88]). Is there a unique complex structure $J$ such that $u$ is also conformal?

Comment 2.5 (Richard Wentworth). This is true for $n = 2, 3$ (see Theorem 9.3.2 of [Lab06]).

Comment 2.6 (Anna Wienhard). One can ask the same question for maximal representations.

3 Surface Bundles

Question 3.1 (Dieter Kotschick). Fix a closed Riemann surface $B$ of genus $g \geq 3$, and fix $h \geq 2$. There exist at most finitely many non-isotrivial holomorphic genus $h$ fibrations over $B$, $F_h \to X \to B$.

This gives a conjugacy class of representations $\rho : \pi_1(B) \to \text{MCG}(F_h)$. How to characterise the representations which are the holonomy of a holomorphic fibration?

Comment 3.2 (Dieter Kotschick). When the fibration is holomorphic there is a Kähler structure on the surface bundle $X$, and so the cohomology of $X$ satisfies certain constraints from Hodge theory (e.g. $h^1(X)$ is even). These constraints give some restrictions on the representations, but they are not enough to give an “if and only if” statement.

Question 3.3 (Dieter Kotschick). Fix $g, h$ with $h \geq 2$. Does there exist a locally trivial fibration of surfaces $F_h \to X \to B_g$ such that $X$ admits a metric of strictly negative curvature?

Comment 3.4 (Dieter Kotschick). A necessary condition is that the monodromy is pseudo-Anosov. If the curvature is also constant then the signature must be zero and so the Toledo invariant vanishes.

4 Invariants of representations

Question 4.1 (Marc Burger). Let $M$ be a compact manifold foliated by surfaces, with transverse measure $\mu$. Denote the foliation by $\mathcal{F}$. Let $X$ be a Hermitian symmetric space, $G = \text{Isom}(X)^0$ and $\rho : \pi_1(M) \to G$. Given a $\rho$-equivariant map $f : M \to X$ we have $f^* \omega \in H^2(\mathcal{F})$, and let $C$ be a Ruelle-Sullivan cycle. Find bounds on $\langle f^* \omega, C \rangle \in \mathbb{R}$ and study the maximal representations.

Question 4.2 (Oscar Garcia-Prada). Consider a real group $G$ with a symmetric space $X = G/K$ of quaternion-Kähler type. Let $\Omega$ be a 4-form on $X$, $M$ a 4-manifold and $\rho : \pi_1(M) \to G$. Study the Toledo invariant $\langle f^* \Omega, [M] \rangle$. An interesting special case is when $M$ is a Kähler surface.
Comment 4.3 (Dieter Kotschick). It might be interesting to study the case where $M$ is aspherical.

Comment 4.4 (Domingo Toledo). Other interesting special cases are: When $M$ is a complex hyperbolic surface, when $M$ has constant curvature and when $M$ is the product of two surfaces.

**Question 4.5** (Marc Burger). Let $k$ be a field and $V$ a symplectic vector space over $k$. There exists a central extension

$$1 \to W(k) \to \widehat{Sp(V)} \to Sp(V) \to 1$$

where $W(k)$ is the Witt group. Given a representation $\rho : \pi_1(S) \to Sp(V)$, the Toledo invariant $\tau(\rho)$ is an element of $W(k)$. Let $A$ be an algebraic set and consider a map $u : A \to \text{Rep}(\pi_1(S), Sp(2n, \mathbb{R}))$. To this map associate the tautological representation $\rho : \pi_1(S) \to Sp(2n, k(A))$, and define the invariant $\tau(\rho) \in W(k(A))$. Study this.

Comment 4.6 (Marc Burger). When $A$ is a point this is the classical Toledo invariant.

**Question 4.7** (Olivier Guichard). Following on from the previous question, there is also a central extension for $\text{SL}(n)$

$$1 \to K_2(A) \to E(n, A) \to \text{SL}(n, A) \to 1$$

Let $H_n$ be the Hitchin component, and let $A = Q(H_n)$. Given a representation $\rho : \pi_1(S) \to \text{SL}(n, A)$ we obtain $q \in K_2(A)$. There is a map $d\log : K_2(A) \to \Omega^2(H_n)$. Is $d\log(q)$ the Weil-Petersson form?

5 Other questions

**Question 5.1** (Oscar Garcia-Prada). The cyclic group of order $n$ acts on the Hitchin component of $\text{Rep}(\pi_1(S), \text{SL}(n, \mathbb{R}))$ (multiply the Higgs field by roots of unity). Call the subspace of $\mathbb{Z}/n\mathbb{Z}$-invariant Higgs fields "cyclotomic Higgs fields", which are in one-to-one correspondence with $H^0(S, K^n)$. Is this space independent of the complex structure on $S$?

**Question 5.2** (Anna Wienhard). Do there exist embeddings $\text{SL}(2, \mathbb{R}) \hookrightarrow G$ such that the component in $\text{Rep}(\pi_1(S), G)$ containing the representation

\[
\begin{array}{c}
\pi_1(S) \\ \downarrow \text{Fuchsian} \\
\text{SL}(2, \mathbb{R}) \\
\end{array} \rightarrow G
\]

consists entirely of discrete and faithful representations?

Comment 5.3 (Anna Wienhard). For $G$ being a split real form and $\text{SL}(2, \mathbb{R}) \to G$ an irreducible representation this gives the Hitchin component.

For $G$ being of Hermitian type and $\text{SL}(2, \mathbb{R}) \to G$ a tight representation, then this gives one connected component in the space of maximal representations.
**Question 5.4** (Marc Burger). Count rational representations whose image lies in the set of integer points of the symplectic group.

**Question 5.5** (Bill Goldman). Let $S$ be a closed orientable surface with $\chi(S) < 0$, and $G$ a semisimple Lie group. Define $E_\rho : T_S \to \mathbb{R}$ to be the energy function associated to a representation $\rho : \pi_1(S) \to G$ (see [GW05] for more details). What are the most general conditions on $\rho$ for which $E_\rho$ is proper?

**Comment 5.6** (Bill Goldman). If $G$ is split and $\rho$ is a Hitchin representation or maximal symplectic representation then $E_\rho$ is proper (see [Lab05]). If $\rho$ is convex co-compact then $E_\rho$ is proper (see [GW05]).

**References**


