Teichmüller theory

(F)

three different but related realisations of $\mathcal{T}(S)$.

(1) $\mathcal{T}(S)$ is a « space of curves » (in $\mathbb{RP}^1$)

(2) $\mathcal{T}(S)$ is a « space of functions »

(3) $\mathcal{T}(S)$ is a « space of flows » on a certain 3-manifold.

Extend this extends to Hitchin representations in $SL(n,\mathbb{R})$

$(n \to \infty)$

Fock-Goncharov coordinates (and cluster algebra) for open surfaces.

I] Boundary at ∞^{ty} of $\pi_1(S)$

(E)

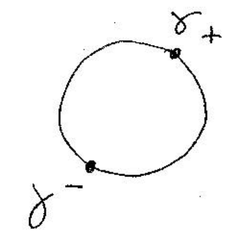
$\pi_1(S) \hookrightarrow \partial_\infty \pi_1(S)$ equipped with an action of $\pi_1(S)$

(i) $\partial_\infty \pi_1(S)$ is hölder homeo to S^1

(ii) $\pi_1(S)$ acts by hölder homeomorphisms

(iii) each orbit of $\pi_1(S)$ is dense

(iv)



each $\gamma \neq id$ has two fixed points

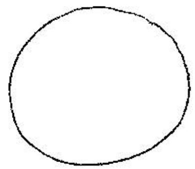
one attractive, one repulsive.

$\Big\{$ choice of a hyperbolic metric

$\Downarrow$

a realisation of $\partial_\infty \pi_1(S)$

Indeed



$\partial_\infty \mathbb{H}^2 = \mathbb{RP}^1 \circlearrowleft PSL(2,\mathbb{R})$

F

hyperbolic metric g

$\rightarrow$ a ρ-equivariant identification of $\partial_\infty \pi_1(S)$ with $\mathbb{RP}^1$.

(1) $\mathcal{T}(S) = \{$ space of curves in $\mathbb{RP}^1\}$

$= \{(f,\rho) : \rho : \pi_1(S) \hookrightarrow PSL(2,\mathbb{R})$, $f : \partial_\infty \pi_1(S) \to \mathbb{RP}^1\}$

(2) space of functions

crossratio on $\mathbb{RP}^1$

$$b(x,y,z,t) = \frac{x-y}{x-t} \cdot \frac{z-t}{z-y}.$$

satisfies some rules.

G

(a) $b(x,y,z,t) = b(z,t,x,y)$

(b) $b(x,y,z,t) \cdot b(z,y,w,t) = b(x,y,w,t)$.

(c) $b(x,y,x,t) = 1$

$b(x,x,y,t) = 0$

(R2) $1 - b(x,y,z,t) = b(t,y,z,x)$

Fact Let b on $(S^1)^4$ such that it satisfies (a)(b)(c) R(2)

prop $\mathcal{T}(S)$ is the space of invariant crossratio on $(\partial_\infty \pi_1 S)^4$ satisfying (R2)

II | Dynamics of the geodesic flows

hyperbolic metric g

$$\to \; U_g S,\ \{\varphi^g_t\}_{t\in\mathbb{R}}$$

the geodesic flow on the unitary tangent bundle.

Stability (the geodesic flow does not depend on the metric).

Given two metric g, h, there exist a homeomorphism

$$\phi : U_g S \to U_h S$$

which is an orbit equivalence:

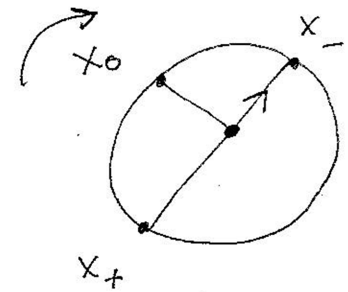
$$\forall x \forall t \exists s \quad \phi(\varphi^g_t(x)) = \varphi^g_s(\phi(x))$$

Invariant description.

$$(\partial_\infty \pi_1 S)^{3+} = \{\text{space of distinct triples}\}.$$

given a metric g:

$$(\partial_\infty \pi_1 S)^{3+} \longrightarrow U\mathbb{H}^2.$$



this map is $\pi_1(S)$-equivariant.

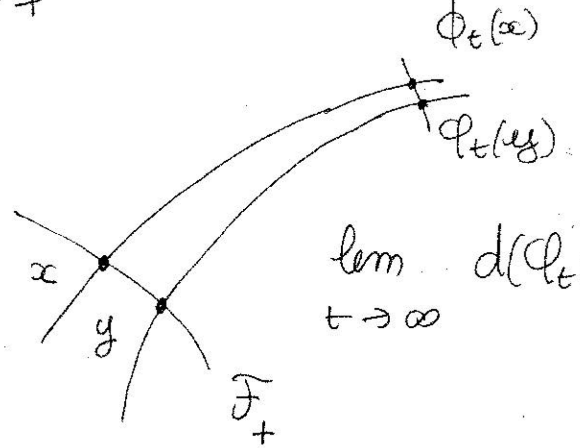
$\rightsquigarrow$ the tree foliations of $(\partial_\infty \pi_1(S))^{3+}/\pi_1(S)$.

$\mathcal{L}$: (x_+, x_-) are cst

$\rightsquigarrow$ the orbits of the geodesic flow.

$\mathcal{F}_+$: (x_0, x_-) are cst.

$\mathcal{F}_+$ is the stable manifold (F)



$$\lim_{t\to\infty} d(\varphi_t(x),\varphi_t(y)) = 0$$

$\mathcal{F}_-$ is the unstable manifold.

$$\lim_{t\to-\infty} d(\varphi_t(x),\varphi_t(y)) = 0$$

Anosov structure on $\partial_\infty \pi_1(S)^3 / \pi_1(S)$

(3) $\tau(S)$ = choice of a parametrisation of $\mathcal{L}$

= « space of flows ».

III | Hitchin representations

Def
A) Fuchsian representation
B) Hitchin if it can be deformed to a Hitchin Fuchsian representation.

A | Curves in $\mathbb{RP}^{n-1}$

$\xi : S^1 \to \mathbb{RP}^{n-1}$ is hyperconvex.
If $\forall x_1, \dots, x_n$ $\xi(x_1) + \dots + \xi(x_n) = \mathbb{R}^n$

ex $n=2$, $n=3$, Veronese embedding.

Thm If ρ is Hitchin, then there exist a ρ-equivariant hyperconvex curve in $\mathbb{RP}^{n-1}$

Thm (Guichard)
Conversely $\to$ ρ is Hitchin.

Re 1) $n=3$ due to Choi-Goldman.
2) it says all H-rep are « quasi fuchsian »

3) osculating flags

hyperconvexity $\longleftrightarrow$ positi of triple of flags.

FG. generalize (Th) to all real split groups. $g \to \infty$

B| Crossratios

Thm (L) | $\{\text{Hitchin}\} \longleftrightarrow \left\{\begin{array}{l}\text{crossratios} \\ \text{satisfying } R(n)\end{array}\right\}$

$$\frac{\lambda_{max}\,\rho(\gamma)}{\lambda_{min}\,\rho(\gamma)} = b(\gamma^+, y, \gamma^-, \gamma y)$$

what is $R(n)$ ...

$\xi : \partial_\infty \pi_1(S) \to \mathbb{RP}^n$

$\xi^* \ \partial_\infty \pi_1(S) \to (\mathbb{RP}^n)^*$

$$b(x,y,z,t) = \frac{\langle \xi(x) | \xi^*(y) \rangle}{\langle \xi(x) | \xi^*(t) \rangle} \frac{\langle \xi(y) | \xi^*(t) \rangle}{\langle \xi(y) | \xi^*(t) \rangle}$$

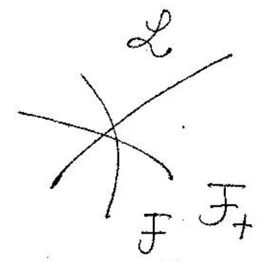
$0 \neq \hat{\xi}(\alpha) \in \xi(\alpha)$

$n \to \infty$

relations with Higgs Bundle

C| Dynamics

$$M^3 = \left(\partial_\infty \pi_1(S)\right)^3 / \pi_1(S)$$



Let $\rho : \pi_1(S) \to SL(n, \mathbb{R})$.

Let E be the associated bundle.

Def || ρ is <u>Anosov</u> if

$$E = E_1 \oplus \cdots \oplus E_n$$

such that:

(i) $rk(E_i) = 1$.

(ii) E_i is // along $\mathcal{L}$.

(iii) F_i^+ is // " $\mathcal{F}_+$

(v) F_i^- is // " $\mathcal{F}_-$

the action of $\mathcal{L}$ is contracting

on $E_i \otimes E_j^*$ $i > j$ in the future

$i < j$ in the past.

F

Stability of hyperbolicity of dia $\Rightarrow$ {Anosov representations} is open.

(cf QF-case in $SL(2,\mathbb{C})$)

- ρ Anosov $\longrightarrow$

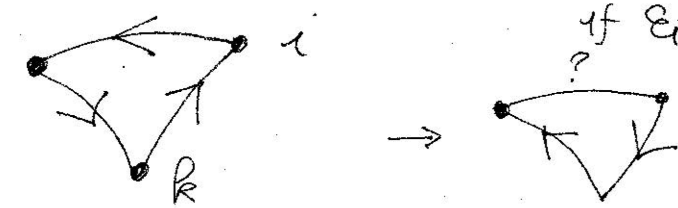
indeed E_1 : on $(\partial_\infty \pi_1 S)^{3+}$ is constant along $\mathcal{F}_+$ and $\mathcal{L}$.

$$\begin{tikzcd} (\partial_\infty \pi_1(S))^{3+} \arrow[r] \arrow[rr, bend right, "E"'] & \partial_\infty \pi_1(S) \arrow[r, "\xi"] & \mathbb{RP}^n \end{tikzcd}$$

Fock-Goncharov Coordinates and cluster-algebras.

mutation at a vertex k.

$$\varepsilon'_{ij} = \begin{cases} -\varepsilon_{ij} & \text{if } k=i \text{ or } k=j \\ \varepsilon_{ij} & \text{if } \varepsilon_{ik}\varepsilon_{ij} \leq 0 \\ \varepsilon_{ij} + |\varepsilon_{ik}| \cdot \varepsilon_{kj} & \text{if } \varepsilon_{ik}\varepsilon_{ij} \geq 0 \end{cases}$$



transformation of graphs

transformations of algebra

$$X'_{ij} \; ; \; X'_i \to \begin{cases} X_i(1+X_k)^{-\varepsilon_{ik}} & \varepsilon_{ik} > \\ X_i(1+X_k^{-1})^{-\varepsilon_{ik}} & \varepsilon_{ik} < 0 \end{cases}$$

explanation of the mutations rules: the map is a Poisson algebra isomorphism.