Problems from the AIM Tate Conjecture Workshop
July 23 to July 27, 2007
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1. (C. Hall) We know how to construct points on an elliptic curve E over $\mathbb{F}_q(T)$ when the analytic rank is 1 by using (Drinfel'd) Heegner points. What if X is a hyperelliptic curve over $\mathbb{F}_q(T)$ of higher genus? Is there any way of constructing the points on the Jacobian of X that ought to be there?

   Comments:

   1) We have a lot of explicit examples when the analytic rank is 1. (C. Hall)
   2) If we start with any surface S over $\mathbb{F}_q$ we can, via Lefschetz fibration, make it into a curve over $\mathbb{F}_q(T)$. Typically (since the rank of H^2 is even), one has two Tate classes. One of these is given by hyperplane sections; can we "see" or construct the other one? This question is like the one above, but without requiring C to be hyperelliptic. So, in some sense, the problem is almost as hard as the problem of treating general surfaces. (C. Schoen)

2. (D. Ramakrishnan) Let X be a Hilbert modular surface over $\mathbb{Q}$. All of the Tate classes that arise from modular curves on X are defined over $\mathbb{Q}^{ab}$; however, there also exist Tate classes which are algebraic, but not defined over $\mathbb{Q}^{ab}$. (In fact, they are defined over dihedral extensions of $\mathbb{Q}$.) We know the algebraicity of these classes, but do not know specific representatives.

   Comments:

   1) One possible suggestion for such representatives could be non-congruence curves on X. (D. Ramakrishnan)
   2) If we can find representatives, we can intersect them with the modular curves: this may be a way to find interesting non-Heegner points on the modular curves. (D. Ramakrishnan)

3. (J. Getz) Let X be a Hilbert modular variety. At some level (say, that of cohomology), these look like a product of modular curves. Under what circumstances does X have Tate classes not coming from modular subvarieties? Thinking of these classes as submotives defined by automorphic forms (and thus as Galois representations), do these exotic classes come from CM automorphic forms as in Question 2?

4. (K. Murty) Let F be a real quadratic field, π be an automorphic form over $\mathbb{Q}$, and Π be its base change to F. Then Π contributes to a subspace $H^2(\Pi)$ of $H^2(S_F)$ for some Hilbert modular surface S_F for $GL_2(F)$, while $H^1(\pi)$ sits inside $H^1(M)$ for some modular curve M. As $\mathrm{Gal}(\bar{\mathbb{Q}}/F)$-modules, $H^2(\Pi)$ is $H^1(\pi) \otimes H^1(\pi)$.

   Take two real quadratic fields F_1, F_2 in this way, along with the base changes Π_1, Π_2 of π and the associated surfaces S_{F_1}, S_{F_2}. Inside $H^4(S_{F_1} \times S_{F_2})$ we have $H^2(\Pi_1) \otimes H^2(\Pi_2)$, which is $H^1(\pi)^{\otimes 4}$ as a $\mathrm{Gal}(\bar{\mathbb{Q}}/F_1F_2)$-module. Hence we expect cycles of codimension 2 in $S_{F_1} \times S_{F_2}$ that are defined over F_1F_2. What are they?

5. (D. Ulmer) Consider the curve $C_{d,a} : y^d = x(x-1)(x-a)$ defined over $\mathbb{C}$ (though $\bar{\mathbb{Q}}$ or $\mathbb{F}_q$ will also work). For which values of a does it happen that this curve has CM? (More precisely, for which values of a does the Jacobian of $C_{d,a}$ have endomorphism algebra of dimension $2g$ over $\mathbb{Q}$, where $g = g(C_{a,d}) = d-1$ is the genus?) It happens (for dull reasons) when $a = -1, \frac{1}{2}, 2, \zeta_6, \zeta_6^{-1}$. For a fixed $d > 7$, there are only finitely many values of a for which it happens. Are there infinitely many d such that there's another value of a (besides the trivial ones listed above) for which $C_{d,a}$ has CM?

   Comments:

   1) An application is that, if $C_{d,a}$ does have CM, then one can construct an elliptic curve defined over $\mathbb{C}(T)$ of rank $d+3$. (D. Ulmer)

   2) Two relevant references for this problem are

      i) de Jong, Johan; Noot, Rutger. *Jacobians with complex multiplication.* Arithmetic algebraic geometry (Texel, 1989), 177–192, Progr. Math., 89, Birkhuser Boston, Boston, MA, 1991.

      and

      ii) deJong's course at AWS2002: http://swc.math.arizona.edu/aws/02/02Notes.html

      (D. Ulmer)

   3) If $x(x-1)(x-a)$ is replaced by a cubic polynomial over $\mathbb{C}(T)$ with Galois group S_3 and $d > 1$ is not a power of 3 then the jacobian of the resulting curve is not of CM-type (over an algebraic closure of $\mathbb{C}(T)$). In addition, if d is a power prime then the jacobian does not contain non-zero abelian subvarieties of CM-type (over an algebraic closure of $\mathbb{C}(T)$) if and only if $d = 2$ or d is an odd number that is not divisible by 3. (Y. Zarhin)

   4) People like Wolfart have studied the problem of determining which Jacobians of curves with Belyi parametrizations are CM. Does it help here to reparametrize this equation in the Belyi form, i.e., to express this equation as a cover of $\mathbb{P}^1$ ramified only at $0, 1, \infty$? (D. Ramakrishnan)

6. (J. Getz) Take a product of modular curves $X = \prod_i X_i$. Many smaller products embed diagonally into X. How much of the middle cohomology can be accounted for by these constructions?

   Comments:

   1) Ribet considered the product of two modular curves $X = X_1 \times X_2$; here $NS(X)$ is just the quotient of $\mathrm{Hom}(J(X_1), J(X_2))$ by the pullbacks of divisors on the two factors. He shows it is necessary to consider twisting correspondences, in addition to the Hecke correspondences, to complete the Tate classes. Are there similarly interesting, yet explicit, correspondences for X when we consider higher dimensional factors X_i? (J. Getz, D. Ramakrishnan, D. Ulmer)

   2) For a product of four Shimura curves, there are interesting Tate classes, which are also Hodge classes, which we don't know how to represent by cycles. (D. Ramakrishnan)

7. (K. Murty) A Hilbert modular surface in characteristic p has interesting cycles parametrizing Hilbert-Blumenthal abelian varieties with specified Newton polygon behavior. Are these

contained in the space of special cycles made up of (Hecke translates of) Hirzebruch-Zagier curves? How can we check this?

Comments:

1) Are they pulled back from the Siegel threefold $\mathcal{A}_2$? (Not always.) Work of A. Langer is pertinent to this question. (J. Achter, C. Schoen)

2) Could the Teichmüller curves of McMullen generate these exotic classes? (J. Ellenberg)

3) Material by van der Geer and Moonen is relevant for explicit information on the characteristic p cycles above. (J. Achter)

4) Can we intersect these characteristic p cycles with the reduction mod p of the CM Tate cycles described in Murty's lecture? (J. Ellenberg)

5) How does the involution $\theta \in \mathrm{Gal}(F/\mathbb{Q})$ act on these cycles? (Here F is the real quadratic field for the Hilbert modular surface.) (J. Getz)

8. (J. Ellenberg) In cases where we know all Tate classes are in the image of $C^i \otimes \mathbb{Q}_\ell$, what can we say about the image of $C^i \otimes \mathbb{Z}_\ell$? For instance, for which primes ℓ can we have non-surjectivity?

Comments:

1) Kollár proved that if we take a very general hypersurface in $\mathbb{P}^4$ such that d^3 divides the degree of X, then the image of $C_1(X)$ is contained in $dH^4(X,\mathbb{Z})$. But it is not known if this can happen for X over $\bar{\mathbb{Q}}$. For C_1 of X over $\overline{\mathbb{F}_q}$, the $\mathbb{Q}_\ell$-Tate Conjecture implies the $\mathbb{Z}_\ell$-Tate Conjecture for $\ell \neq p$. (C. Schoen)

2) Over $\mathbb{F}_q$, the $\mathbb{Q}_\ell$-Tate Conjecture implies the $\mathbb{Z}_\ell$-Tate Conjecture for the Weil-étale cohomology, so this becomes a question of comparison between étale and Weil-étale cohomology. (T. Geisser)

9. (J. Milne) Let A be CM abelian variety over $\bar{\mathbb{Q}}$ and let c be an absolute Hodge class. Reduce A to A_0 over $\overline{\mathbb{F}_p}$ and take a Lefschetz class x on A_0 of complementary dimension. Then $x.\bar{c} \in \mathbb{Q}_\ell$; is it in $\mathbb{Q}$ and independent of ℓ?

Comments:

1) The answer is yes, assuming either the Tate Conjecture or the Hodge Conjecture, or also when A_0 is ordinary (since then x can be lifted to characteristic 0). (J. Milne)

2) This would imply the existence of the $\mathbb{Q}$-subalgebras $\mathcal{R}^*$ mentioned described in Milne's lecture. (J. Milne)

3) The question also makes sense for any abelian variety A which has good reduction over $\overline{\mathbb{F}_p}$. (J. Milne)

4) It may be that (A_0, x) still admits a canonical lift, even if we relax the assumption that A_0 is ordinary (see Milne's first comment) . (J. Achter)

10. (C. Schoen) Let J be a general abelian surface (over $\mathbb{C}$, say). Describe the surfaces in $J \times J$.

11. (J. Getz) C. Simpson proved that if X is a variety and if $\rho_X : \pi_1^{geom}(X) \to \mathrm{GL}_2(\mathbb{Q}_\ell)$ is a representation whose algebraic enveloping contains SL_2, then ρ_X factors through $\rho_Y : \pi_1^{geom}(Y) \to \mathrm{GL}_2(\mathbb{Q}_\ell)$, where Y is either a curve or a Shimura variety; the factorization is induced from a map from X to some cover of Y. Is there a version of this result using just π_1 in place of π_1^{geom}?

12. (W. Raskind) Take an abelian variety over a finite extension K of $\mathbb{Q}_p$ with multiplicative reduction. Show the isomorphism

$$\mathrm{End}(A) \otimes \mathbb{Q}_p \xrightarrow{\sim} \mathrm{End}(V_p(A))^{G_K}.$$

Comments:

1) This implies the analogous statement for A over K, with K global, having multiplicative reduction somewhere. (W. Raskind)
2) This should be doable by imitating methods of Serre for the case when A is an elliptic curve. (W. Raskind)
3) This is known for Drinfel'd modular varieties (done by Ito). (W. Raskind)

13. (K. Murty) According to the Tate Conjecture, if one takes the L-function of a variety X over K at the edge of the critical strip, and then increases K, the order of the pole should stay bounded. Can we prove this same statement without assuming the Tate Conjecture?

Comments:

1) This is provable over function fields of finite fields since, by using the work of Lafforgue, we have (T3): the order of the pole equals the rank of the Tate classes. It is also true when X is an abelian variety in characteristic 0. (D. Ramakrishnan)

14. (C. Hall) Let $X(\ell)$ be a modular curve. What can we say about the splitting field of the ℓ-torsion of the Jacobian $J(X(\ell))$?

Comments:

1) This is needed to speak about the L-function of $\mathcal{E} \to X(\ell) \pmod{\ell} \in \mathbb{F}_\ell[T]$. (C. Hall)

15. (J. Ellenberg, D. Ulmer) Let X be a modular curve, which paramterizes elliptic curves with some level structure, and let $\mathcal{E}$ be the universal family of elliptic curves over X. This defines an elliptic surface, typically over $\mathbb{Q}$. It often happens that, for many primes p, the reduction modulo p of this surface acquires new Tate classes, not coming from characteristic zero. Can we find algebraic cycles (living in characteristic p), which account for such Tate classes?

16. (D. Ramakrishnan) Let E be an elliptic curve over $\mathbb{Q}$ with CM by K and with associated Hecke character χ. Then χ^2 is associated to a modular form of weight 3 appearing in $H^2(\mathcal{E})$, $\mathcal{E} \to X$ for a modular curve X (whose level is the norm of the conductor of χ^2 times the discriminant of K). The Tate and Hodge Conjectures predict the existence of a correspondence $E \times E \to \mathcal{E}$ because the same motive appears in H^2 of both sides. Can we exhibit this correspondence explicitly?

Comments:

1) In Shioda's example (level 4), $\mathcal{E}$ is $K3$, and in fact the Kummer surface associated to $E \times E$. (D. Ramakrishnan)
2) This problem is closely related to (the comments from) the previous problem because, when E has supersingular reduction, there are many extra cycles on $E \times E$. (C. Schoen)

17. (W. Raskind) In the situations where the Hodge Conjecture implies the Tate Conjecture, is it also true that the generalized Hodge Conjecture implies the generalized Tate Conjecture, e.g., for CM abelian varieties?

18. (E. Izadi) Take a principally polarized abelian variety A and assume $\text{rank}(NS(A)) = 1$. What is the index of $C^i(A)$ in $H^{2i}(A, \mathbb{Z}(i))$? What is the image? What if we just restrict ourselves to the line spanned by $[\theta]^i$, where θ is the polarization?

19. (D. Ramakrishnan) The following question is due to P. Deligne. Let X be a variety defined over a (finitely-generated) field of characteristic 0. Given a Tate class on c on X, is there a prime p such that, upon reducing X mod p, the reduction of c is algebraic?

20. (M. Flach) Find a natural definition of the Weil-étale topology so that the cohomology of Z vanishes in degrees greater than 3.

    Comments:

    1) The current definition only vanishes for odd degrees and gives infinitely generated groups in even degrees. (M. Flach)
    2) It it not clear how hard this problem is. A better definition might be just around the corner or the only solution might be to truncate the cohomology complex. (M. Flach)

21. (M. Flach) Using the current definition of the Weil-étale topology, define the Weil-étale topos for an arbitrary arithmetic scheme (i.e., a scheme of finite type over $\text{Spec}(\mathbb{Z})$) as a fibred product. Can one compute the cohomology of the sheaf Z on, say, the affine or projective line?