Workshop on Tensor Decompositions

Summary of Suggested References

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This document is an attempt to summarize the references suggested by workshop participants. We emphasize that this bibliography is not meant to represent a comprehensive review of the literature on the mathematics, computation and applications of tensors, but rather it provides a sample of the main interests and experiences of workshop participants. In fact, a very extensive bibliographic database on tensors may be found at `http://three-mode.leidenuniv.nl/`

1 Background on Tensors and Decompositions

Several participants remarked that they found the work by De Lathauwer, De Moor and Vandewalle [2], Kolda [3], and Zhang and Golub [5] to be especially useful for an initial introduction to tensor decompositions. These papers, in particular, focus on the basic problem of extending the ideas of rank and SVD for matrices to higher order tensors.

Related work by Kolda [4] should also be considered in this introductory material, as it provides some basic examples of tensor decompositions.

We also recommend the "partial" survey paper by Comon [1], which includes an extensive bibliography and discusses the difficulties of extending the concept of rank to high order tensors. This paper also provides an introduction to a variety of applications where decomposition of high order tensors have been used.

References

[1] P. Comon. Tensor decompositions. In J. G. McWhirter and I. K. Proudler, editors, *Mathematics in Signal Processing V*, pages 1–24. Clarendon Press, Oxford, 2002. Online at http://www.i3s.unice.fr/∼comon/biblio.html#Clarendon02.

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[3] T. G. Kolda. Orthogonal tensor decompositions. *SIAM J. Matrix Anal. Appl.*, 23(1):243–255, 2001.

[4] T. G. Kolda. A counterexample to the possibility of an extension of the Eckart-Young low-rank approximation theorem for the orthogonal rank tensor decomposition. *SIAM J. Matrix Anal. Appl.*, 24(3):762–767, 2002.

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2 Algorithms and Mathematical Issues of Tensor Decompositions

One of the aims of this workshop is to gain a better understanding of the mathematical properties of tensor decompositions, and to provide a framework for notational standards. Such work will help contribute to better algorithms and efficient computational methods. Several workshop participants have worked on these issues, and have suggested the following list of references [1, 2, 3, 4, 5, 6, 7, 8].

References

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3 Applications using Tensor Decompositions

A second aim of the conferences is to identify applications where tensors play an important role, and, in particular, applications which have the potential to substantially benefit from contributions arising from this workshop. Applications of interest include:

- Blind identification, beamforming, and source separation [1, 11, 13].
- Biological applications [2, 3, 4, 5].
- Factor analysis [25, 26, 27, 28, 29, 36].
- Food industry [8].
- Various applications in image processing and computer vision [32, 34, 35, 39, 40, 41, 42, 43].
- Independent component analysis [10, 12, 14, 15, 44].
- Large matrix computations, including Monte Carlo algorithms [16, 17, 18], Kronecker product factorizations, [20, 21, 22, 23, 24, 32, 34, 37, 38], and more general linear operators [6, 19, 31].
- Of course there are many other areas in which tensors play an important role, especially in statistical applications [7, 9, 30, 33].

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