Frequency Domain Integral Equations for Acoustic and Electromagnetic Scattering Problems

Christophe Geuzaine, U of Liege

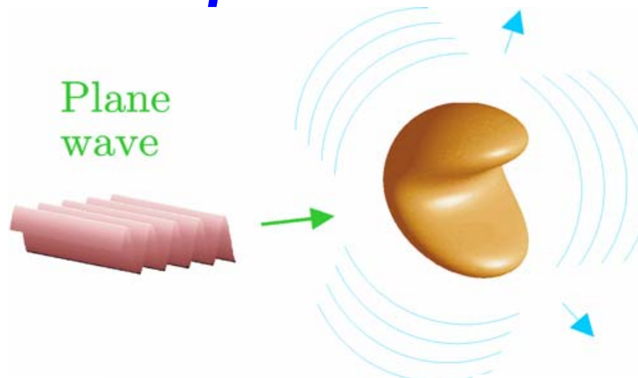
Fernando Reitich, U of Minnesota

Catalin Turc, UNC Charlotte

Outline

- *Integral equations formulations for acoustic and electromagnetic problems - Layer potentials*
- *Combined Field Integral Equations: Direct and Indirect Methods*
- *Numerical Methods – Medium range frequency: variational formulations (BEM) and Nystrom methods*
  - *Spectra of the layer potentials (bounded & smooth manifolds) → Regularization; Geometric singularities: Lipschitz manifolds, open surfaces*
  - *Accelerated computations: FMM, FFT based acceleration techniques, equivalent sources*
- *High-frequency Scattering Problems: Ansatz formulation, localization techniques; Single scattering configurations; Multiple scattering configurations*

Time-Harmonic Acoustic and Maxwell Equations



Ω

Acoustic

$$u = u^i + u^s$$

$$\Delta u + k^2 u = 0$$

$$\lim_{|\mathbf{x}| \to \infty} |\mathbf{x}| \left(\frac{\partial u^s}{\partial |\mathbf{x}|} - iku^s \right) = 0$$

$$BC \;\; sound \;\; soft: u = 0 \;\; on \;\; \Gamma$$

$$sound \;\; hard: \frac{\partial u}{\partial \mathbf{n}} = 0 \;\; on \;\; \Gamma$$

Electromagnetic

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}^i + \vec{\mathbf{E}}^s, \; \vec{\mathbf{H}} = \vec{\mathbf{H}}^i + \vec{\mathbf{H}}^s$$

$$\begin{cases} \text{curl } \vec{\mathbf{E}} - ik\vec{\mathbf{H}} = \mathbf{0} \\ \text{curl } \vec{\mathbf{H}} + ik\vec{\mathbf{E}} = \mathbf{0} \end{cases}$$

$$\lim_{|\mathbf{x}| \to \infty} \left| \left(\vec{\mathbf{H}}^s \times \mathbf{x} \right) + |\mathbf{x}| \; \vec{\mathbf{E}}^s \right| = 0$$

$$BC \;\; PEC: \mathbf{n} \times \mathbf{E} = \mathbf{0} \;\; on \;\; \Gamma$$

Scattering Simulations
Basic challenges

Fields oscillate on a scale set by the wavelength $\lambda = \frac{2\pi}{k}$ *of radiation* &

Noncoercive operators (Coercive+ Compact) discretization step proportional to wavelength $h \approx \frac{\lambda}{10}(\mathrm{FEM}),\ \frac{\lambda}{6}(\mathrm{BEM})$

$\Rightarrow$
- Computational cost
- Memory

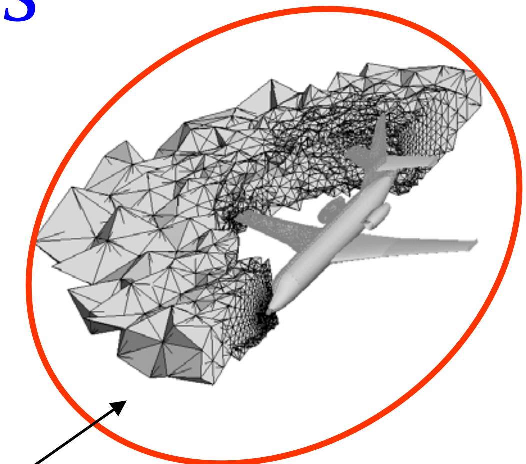
Order $10^p\ \lambda^{-p}$

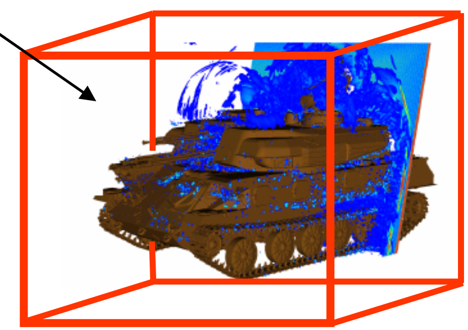
Scattering Simulations
Numerical Methods

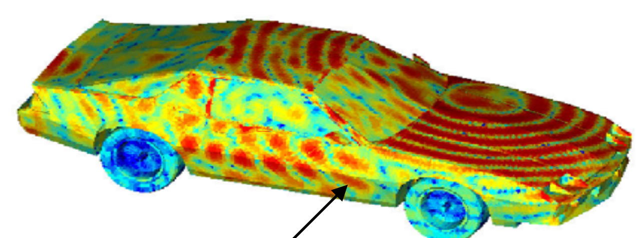
• ***Variational Methods***
(MoM, Finite Element Methods, Finite Volume Methods, etc)

• ***Differential Equation Methods***
(Finite Differences, etc)

• ***Integral Equation Methods*** *reduce the dimensions*; radiation condition *outgoing Green's function*; *the advent of fast methods*; but *can be plagued by spurious resonances* → CFIE→ *poor spectral properties* → *Regularization Techniques*

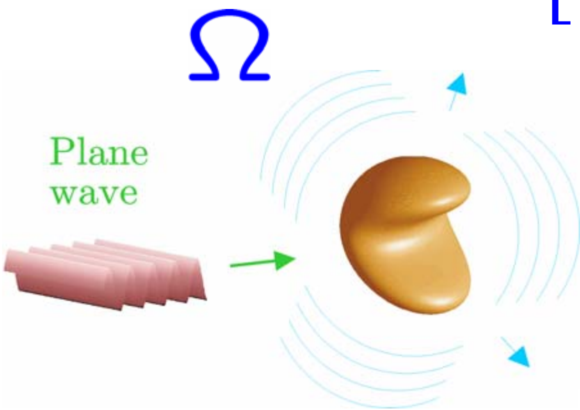






Layer Potentials Acoustics

[*Verchota, Kenig, Costabel, Dauge*]



$$G_k(\mathbf{x}-\mathbf{y}) = e^{ik|\mathbf{x}-\mathbf{y}|}/(4\pi|\mathbf{x}-\mathbf{y}|)$$

$$0 = \int_\Gamma \left(\frac{\partial G_k(\mathbf{z}-\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} u^i(\mathbf{y}) - G_k(\mathbf{z}-\mathbf{y}) \frac{\partial u^i(\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} \right) d\sigma(\mathbf{y}), \;\; \mathbf{z} \in \Omega$$

$$u^s(\mathbf{z}) = \underbrace{\int_\Gamma \frac{\partial G_k(\mathbf{z}-\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} u^s(\mathbf{y}) d\sigma(\mathbf{y})}_{(U_{\mathrm{DL}}\ u^s)(\mathbf{z})} - \underbrace{\int_\Gamma G_k(\mathbf{z}-\mathbf{y}) \frac{\partial u^s(\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} d\sigma(\mathbf{y})}_{(U_{\mathrm{SL}}\ \frac{\partial u^s}{\partial \mathbf{n}})(\mathbf{z})}, \;\; \mathbf{z} \in \Omega$$

$$\gamma_D : H^1_{\mathrm{loc}}(\Omega) \to H^{\frac{1}{2}}(\Gamma) \qquad \gamma_N : H_{\mathrm{loc}}(\Delta, \Omega) \to H^{-\frac{1}{2}}(\Gamma)$$

$$S\ \phi = \{\gamma_D U_{\mathrm{SL}}(\phi)\}_\Gamma \qquad K^*\ \phi = \{\gamma_N U_{\mathrm{SL}}(\phi)\}_\Gamma$$

$$K\ \phi = \{\gamma_D U_{\mathrm{DL}}(\phi)\}_\Gamma \qquad N\ \phi = \{\gamma_N u_{\mathrm{DL}}(\phi)\}_\Gamma$$

$$S : H^s(\Gamma) \to H^{s+1}(\Gamma),\ -1 \le s \le 0 \qquad K^* : H^s(\Gamma) \to H^s(\Gamma),\ \ -1 \le s \le 0$$

$$K : H^s(\Gamma) \to H^s(\Gamma),\ \ 0 \le s \le 1 \qquad N : H^s(\Gamma) \to H^{s-1}(\Gamma),\ 0 \le s \le 1$$

Acoustic Integral Equations (Closed Manifolds)

Sound - soft

$$\left(S\,\frac{\partial u}{\partial \mathbf{n}}\right)(\mathbf{x}) = u^i(\mathbf{x}),\;\; \mathbf{x}\in\Gamma$$

$$\frac{1}{2}\frac{\partial u}{\partial \mathbf{n}} + \left(K^*\,\frac{\partial u}{\partial \mathbf{n}}\right)(\mathbf{x}) = \frac{\partial u^i(\mathbf{x})}{\partial \mathbf{n}(\mathbf{x})},\;\; \mathbf{x}\in\Gamma$$

Sound - hard

$$(N\;u)(\mathbf{x}) = -\frac{\partial u^i(\mathbf{x})}{\partial \mathbf{n}(\mathbf{x})},\;\; \mathbf{x}\in\Gamma$$

$$\frac{1}{2}u - (K\;u)(\mathbf{x}) = u^i(\mathbf{x}),\;\; \mathbf{x}\in\Gamma$$

Neither is uniquely solvable when $-k^2$ is an eigenvalue of the interior Neumann or Dirichlet Laplacian !

Calderon Projectors [Calderon '54]

$$C_{ext} = \begin{pmatrix} -K & S \\ -N & K^* \end{pmatrix},\quad C_{ext}^2 = C_{ext}$$

$$(K^*)^2 - N\;S = \frac{I}{4},\;\; K^2 - S\;N = \frac{I}{4}$$

Electromagnetic Integral Equations (Closed Manifolds)

$$\begin{pmatrix} -\frac{I}{2} - \mathcal{K} & \mathcal{T} \\ -\mathcal{T} & -\frac{I}{2} - \mathcal{K} \end{pmatrix} \begin{pmatrix} \mathbf{n} \times \vec{\mathbf{E}}^s \\ \mathbf{n} \times \vec{\mathbf{H}}^s \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

$$(\mathcal{K}\vec{\mathbf{a}})(\mathbf{x}) = \mathbf{n}(\mathbf{x}) \times \int_\Gamma \nabla_{\mathbf{y}} G_k(\mathbf{x}-\mathbf{y}) \times \vec{\mathbf{a}}(\mathbf{y}) d\sigma(\mathbf{y}) \qquad (\vec{\mathbf{a}} \cdot \mathbf{n} = 0)$$

$$\begin{aligned} (\mathcal{T}\vec{\mathbf{a}})(\mathbf{x}) &= ik\mathbf{n}(\mathbf{x}) \times \int_\Gamma G_k(\mathbf{x}-\mathbf{y})\vec{\mathbf{a}}(\mathbf{y}) d\sigma(\mathbf{y}) \\ &+ \frac{i}{k}\mathbf{n}(\mathbf{x}) \times \mathrm{PV} \int_\Gamma \nabla_{\mathbf{x}} G_k(\mathbf{x}-\mathbf{y}) \mathrm{div}_\Gamma \vec{\mathbf{a}}(\mathbf{y}) \mathrm{d}\sigma(\mathbf{y}) \end{aligned}$$

$$\mathcal{K} : H_{\mathrm{div}}^{-\frac{1}{2}}(\Gamma) \longrightarrow H_{\mathrm{div}}^{\frac{1}{2}}(\Gamma), \;\; \mathbf{n} \times \mathcal{T} : H_{\mathrm{div}}^{-\frac{1}{2}}(\Gamma) \longrightarrow H_{\mathrm{curl}}^{-\frac{1}{2}}(\Gamma)$$

$$\vec{\mathbf{J}} = \mathbf{n} \times \vec{\mathbf{H}} \qquad \begin{pmatrix} \frac{I}{2} - \mathcal{K} & \mathcal{T} \\ -\mathcal{T} & \frac{I}{2} - \mathcal{K} \end{pmatrix} \begin{pmatrix} \mathbf{0} \\ \vec{\mathbf{J}} \end{pmatrix} = \begin{pmatrix} \mathbf{n} \times \vec{\mathbf{E}}^s \\ \mathbf{n} \times \vec{\mathbf{H}}^s \end{pmatrix}$$

MFIE $\frac{\vec{\mathbf{J}}}{2} + \mathcal{K}\vec{\mathbf{J}} = \mathbf{n} \times \mathbf{H}^i$

EFIE $\mathcal{T}\vec{\mathbf{J}} = -\mathbf{n} \times \mathbf{E}^i$

Unique Solvability Acoustics: CFIE (Closed Manifolds)

[*Brackhage & Werner, Panich, Burton & Miller*]

Sound - Soft $u^s(\mathbf{z}) = (u_{\mathrm{DL}}\phi)(\mathbf{z}) - i\gamma(u_{\mathrm{SL}}\phi)(\mathbf{z})$

ICFIE $$\frac{1}{2}\phi(\mathbf{x}) + (K\ \phi)(\mathbf{x}) - i\gamma(S\ \phi)(\mathbf{x}) = -u^i(\mathbf{x}),\ \ \mathbf{x} \in \Gamma$$

CFIE $$\frac{1}{2}\frac{\partial u}{\partial n}(\mathbf{x}) + \left(K^*\ \frac{\partial u}{\partial n}\right)(\mathbf{x}) - i\gamma\left(S\ \frac{\partial u}{\partial n}\right)(\mathbf{x}) = \frac{\partial u^i}{\partial \mathbf{n}}(\mathbf{x}) - i\gamma u^i(\mathbf{x})$$

Sound - Hard $u^s(\mathbf{z}) = (u_{\mathrm{SL}}\psi)(\mathbf{z}) + i\gamma(u_{\mathrm{DL}}\psi)(\mathbf{z})$

ICFIE $$\frac{1}{2}\psi(\mathbf{x}) + (K^*\ \psi)(\mathbf{x}) + i\gamma(N\ \psi)(\mathbf{x}) = -\frac{\partial u^i(\mathbf{x})}{\partial \mathbf{n}(\mathbf{x})},\ \ \mathbf{x} \in \Gamma$$

CFIE $$\frac{1}{2}u(\mathbf{x}) - (K\ u)(\mathbf{x}) + i\gamma(N\ u)(\mathbf{x}) = u^i(\mathbf{x}) - i\gamma\frac{\partial u^i(\mathbf{x})}{\partial \mathbf{n}(\mathbf{x})},\ \ \mathbf{x} \in \Gamma$$

Fredholmness ensured if $K,\ K^* : L^2(\Gamma) \to L^2(\Gamma)$ *compact* : Γ *smooth*

Unique Solvability Electromagnetics: CFIE (Closed Manifolds)

[Harrington & Mautz, Kress]

CFIE $\quad \dfrac{\vec{\mathbf{J}}}{2} + \mathcal{K}\vec{\mathbf{J}} + \eta_1(\mathbf{n}\times\mathcal{T})(\vec{\mathbf{J}}) = \mathbf{n}\times\mathbf{H}^i - \eta_1\ \mathbf{n}\times\mathbf{E}^i$

$$\begin{aligned}\vec{\mathbf{E}}(\mathbf{z}) &= (\mathcal{M}\vec{\mathbf{a}} + i\eta_2\mathcal{E}(\mathbf{n}\times\vec{\mathbf{a}}))(\mathbf{z}) = \mathrm{curl}\ \int_\Gamma G_k(\mathbf{z}-\mathbf{y})\vec{\mathbf{a}}(\mathbf{y})d\sigma(\mathbf{y}) \\ &+ \ i\eta_2\ \mathrm{curl}\ \mathrm{curl}\ \int_\Gamma G_k(\mathbf{z}-\mathbf{y})(\mathbf{n}(\mathbf{y})\times\vec{\mathbf{a}}(\mathbf{y}))d\sigma(\mathbf{y})\end{aligned}$$

ICFIE $\quad \dfrac{\vec{\mathbf{a}}}{2} - \mathcal{K}\vec{\mathbf{a}} + \eta_2 k\mathcal{T}(\mathbf{n}\times\vec{\mathbf{a}}) = -\mathbf{n}\times\vec{\mathbf{E}}^i$

Fredholmness ensured if $\mathcal{K} : H_{\mathrm{div}}^{-\frac{1}{2}} \to H_{\mathrm{div}}^{-\frac{1}{2}}$ *compact* : Γ *smooth*

Variational Formulations Acoustics (BEM)

Sound - Soft

$$\left(S_k \frac{\partial u}{\partial \mathbf{n}}\right)(\mathbf{x}) = u^i(\mathbf{x}), \ \mathbf{x} \in \Gamma$$

$$\frac{1}{4\pi}\int_\Gamma\int_\Gamma \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}\frac{\partial u}{\partial \mathbf{n}}(\mathbf{y})\psi(\mathbf{x})d\sigma(\mathbf{x})d\sigma(\mathbf{y}) = \int_\Gamma u^i(\mathbf{x})\psi(\mathbf{x})d\sigma(\mathbf{x}), \ \forall\psi \in H^{-\frac{1}{2}}(\Gamma)$$

Sound - Hard

$$(N_k\, u)(\mathbf{x}) = -\frac{\partial u^i(\mathbf{x})}{\partial \mathbf{n}(\mathbf{x})}, \ \mathbf{x} \in \Gamma$$

$$\frac{1}{4\pi}\int_\Gamma\int_\Gamma \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}(\overrightarrow{\mathrm{curl}}_\Gamma u(\mathbf{y}) \cdot \overrightarrow{\mathrm{curl}}\psi(\mathbf{x}))d\sigma(\mathbf{x})d\sigma(\mathbf{y}) -$$

$$\frac{k^2}{4\pi}\int_\Gamma\int_\Gamma \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}u(\mathbf{y})\psi(\mathbf{x})(\mathbf{n}(\mathbf{y})\cdot\mathbf{n}(\mathbf{x}))d\sigma(\mathbf{x})d\sigma(\mathbf{y}) = \int_\Gamma \frac{\partial u^i(\mathbf{x})}{\partial \mathbf{n}(\mathbf{x})}\psi(\mathbf{x})d\sigma(\mathbf{x}), \ \forall\psi \in H^{\frac{1}{2}}(\Gamma)$$

Existence + uniqueness $-k^2$ not an eigenvalue of the int Dirichlet/Neumann Laplacian: Gårding Inequality

$S_0 : H^{-\frac{1}{2}}(\Gamma) \longrightarrow H^{\frac{1}{2}}(\Gamma)$ *coercive* $\int_\Gamma (S_0\psi)\bar{\psi}d\sigma \geq \alpha||\psi||^2_{H^{-\frac{1}{2}}(\Gamma)}, \ \alpha > 0$

$N_0 : H^{\frac{1}{2}}(\Gamma)/\mathbf{R} \longrightarrow H^{-\frac{1}{2},0}(\Gamma)$ $\int_\Gamma\int_\Gamma \frac{(\overrightarrow{\mathrm{curl}}_\Gamma\psi(\mathbf{y}) \cdot \overrightarrow{\mathrm{curl}}\psi(\mathbf{x}))}{4\pi|\mathbf{x}-\mathbf{y}|}d\sigma(\mathbf{x})d\sigma(\mathbf{y}) \geq \alpha||\psi||^2_{H^{\frac{1}{2}}(\Gamma)/\mathbf{R}}$

$$S_k - S_0 : H^{-\frac{1}{2}}(\Gamma) \to H^1(\Gamma) \hookrightarrow H^{\frac{1}{2}}(\Gamma)$$

CFIE cannot use *natural trace spaces* $\rightarrow$ *weak form in* $L^2(\Gamma)$

Variational Formulations EM (BEM)

$$\mathbf{n} \times \mathcal{T} : H_{\mathrm{div}}^{-\frac{1}{2}}(\Gamma) \longrightarrow H_{\mathrm{curl}}^{-\frac{1}{2}}(\Gamma),\ \ duality\ H_{\mathrm{div}}^{-\frac{1}{2}}(\Gamma)\ and\ H_{\mathrm{curl}}^{-\frac{1}{2}}(\Gamma)$$

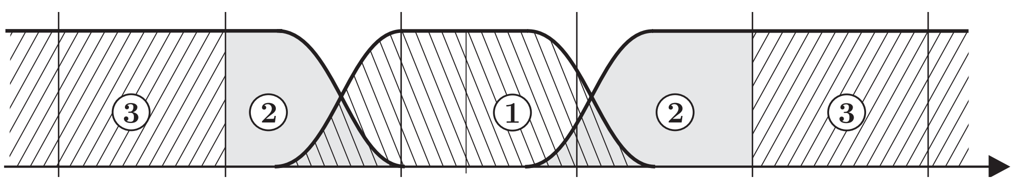
$$Rumsey : ik \int_\Gamma \int_\Gamma G_k(\mathbf{x}-\mathbf{y})(\mathbf{J}(\mathbf{y}) \cdot \mathbf{j}(\mathbf{x}))d\sigma(\mathbf{y})d\sigma(\mathbf{x})$$
$$-\frac{i}{k} \int_\Gamma \int_\Gamma G_k(\mathbf{x}-\mathbf{y})\mathrm{div}_\Gamma \mathbf{J}(\mathbf{y})\mathrm{div}_\Gamma \mathbf{j}(\mathbf{x})d\sigma(\mathbf{y})d\sigma(\mathbf{x}) =$$
$$-\int_\Gamma \mathbf{E}^i(\mathbf{x}) \cdot \mathbf{j}(\mathbf{x})d\sigma(\mathbf{x}),\ \forall\ \mathbf{j} \in H_{\mathrm{div}}^{-\frac{1}{2}}(\Gamma)$$

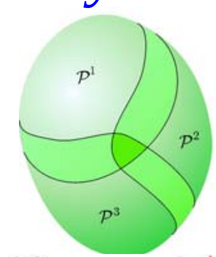
- *Fredholm alternative available* [*Bendali, Nedelec*]- *inf-sup conditions*
- *Saddle point (mixed) formulation:* $\mathbf{j} = \mathbf{g} + \overrightarrow{\mathrm{curl}}_\Gamma \mathbf{p}$
- *Use the Helmholtz decomposition, saddle point (mixed) formulation* [*De La Bourdonnaye*]
- *Discrete problem: conforming elements (Raviart-Thomas) in* H_{div} [*Bendali, Nedelec*]; *discrete inf-sup conditions*

High-order Numerical Methods: Nystrom Methods

Challenge: the singularity of the Green's function

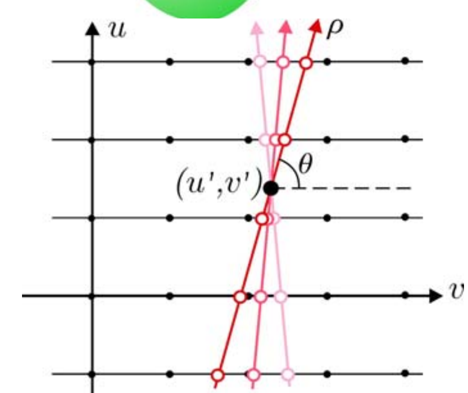
- *Homeomorphisms to Unit Sphere and use of Spherical harmonics* [*Kress, Wienert, Ganesh, Graham*]
- *AIM* [*Bleszynski et al.*], *FFT based* [*Catedra et al., Philips*]
- *Multipole based* [*Rokhlin, Wandzura et al., Chew et al.*]
- *Partitions of Unity and polar coordinate cov for weakly singular kernels to treat the adjancent interactions*[*Bruno & Kunyansky*]





$$\int_{\partial\mathcal{D}} ...ds = \sum_j \int_{\mathcal{P}^j} ...w_j(u_j,v_j)du_jdv_j$$

$$L(u',v',\theta) = \int_{-r_1}^{r_1} f_k^*(\rho,\theta)\frac{|\rho|}{|\mathbf{R}|}\cos k\,|\mathbf{R}|\,\frac{\mathbf{R}\cdot\boldsymbol{\nu}(r)}{\mathbf{R}^2}d\rho$$



High-order Nystrom Methods

[Bruno & Kunyansky '01]

Scattering by a sphere of radius 2.7 λ

Patches	Unknowns	Discretization density	Max Error	RMS
$6 \times 17 \times 17$	1350	3 per 1λ	0.1	2.9×10^{-2}
$6 \times 33 \times 33$	5766	6 per 1λ	9.0×10^{-4}	1.8×10^{-4}
$6 \times 65 \times 65$	23790	12 per 1λ	3.6×10^{-6}	1.4×10^{-6}
$6 \times 129 \times 129$	93726	24 per 1λ	1.6×10^{-8}	5.6×10^{-9}

Doubling the discretization density improves the accuracy by 200 to 300 times!

Algorithm	Rad.	Time	Unknowns	RMS Error
Nystrom [1]	2.7λ	$1953s$ (setup)	5400	2.3%
Galerkin [1]	2.7λ	$38803s$ (setup)	5400	0.48%
Br.& Kun.	2.7λ	$294s$	2526	0.068%
Br.& Kun.	2.7λ	$1430s$	5430	0.0025%

[1] *Wandzura et al. '98*

High-order Nystrom Methods

Challenge: hypersingular operators, smooth manifolds

$$(N\phi)(\mathbf{x}) = k^2\phi(\mathbf{x})\int_\Gamma G_k(\mathbf{x}-\mathbf{y})(\mathbf{n}(\mathbf{x})\cdot\mathbf{n}(\mathbf{y}))d\sigma(\mathbf{y}) + PV\int_\Gamma \frac{\partial^2 G_k(\mathbf{x}-\mathbf{y})}{\partial\mathbf{n}(\mathbf{x})\,\partial\mathbf{n}(\mathbf{y})}(\phi(\mathbf{y})-\phi(\mathbf{x}))d\sigma(\mathbf{y})$$

polar cov + h.o. quadrature for Hilbert transforms [2D Kress '95, Bruno & Lintner '07, Bruno & Turc '07]

$$\begin{aligned}(\mathcal{T}\vec{\mathbf{a}})(\mathbf{x}) &= ik\mathbf{n}(\mathbf{x})\times\int_\Gamma G_k(\mathbf{x}-\mathbf{y})\vec{\mathbf{a}}(\mathbf{y})d\sigma(\mathbf{y})\\ &\quad -\frac{i}{k}\int_\Gamma(\mathbf{n}(\mathbf{y})-\mathbf{n}(\mathbf{x}))\times\nabla_{\mathbf{x}}G_k(\mathbf{x}-\mathbf{y})\mathrm{div}_\Gamma\vec{\mathbf{a}}(\mathbf{y})d\sigma(\mathbf{y})\\ &\quad -\frac{i}{k}\int_\Gamma G_k(\mathbf{x}-\mathbf{y})\overrightarrow{\mathrm{curl}}_\Gamma\,\mathrm{div}_\Gamma\vec{\mathbf{a}}(\mathbf{y})d\sigma(\mathbf{y})\end{aligned}$$

$$\begin{aligned}(\mathcal{H}\phi)(\mathbf{x}) = \sum_{p=1}^{P}\int_{\mathcal{P}^p} & G(\mathbf{x},\mathbf{x}^p(u^p,v^p),\mathbf{n}^p(u^p,v^p))w^p(\mathbf{x}(u^p,v^p))J^p(u^p,v^p)\\ & \cdot\Big(\sum_{0\le\alpha+\beta\le 2} c_{\alpha\beta}(u^p,v^p)\partial^\alpha_{u^p}\partial^\beta_{v^p}\phi^p(u^p,v^p)\Big)du^p\,dv^p\end{aligned}$$

the derivatives of the densities ϕ^p through derivatives of *periodic* functions only ! [Bruno & Turc '07]

High-Order Numerical Methods (Sound-hard)

Challenge: spectral distribution of the hyper-singular operators in CFIE not suitable for iterative solvers

$$N = OPS(1);\, \Gamma = \mathbf{S}^2 : \quad N\; Y_n^m = -ik^3(j_n^(k))'(h_n^{(1)}(k))' = \mathcal{O}(n)$$

*Use high-frequency local approximations of the Neumann—to—Dirichlet operator in the illuminated region **only*** [*Levadoux ‘01, Antoine et al. ’05*]

$$\partial_{\mathbf{n}} u^s = \left(\frac{I}{2} - K^*\right)\partial_{\mathbf{n}} u^s - N u^s \;\; on\; \Gamma$$

$$\partial_{\mathbf{n}} u^s|_\Gamma \to u^s|_\Gamma = (ND)\partial_{\mathbf{n}} u^s|_\Gamma,\;\; ND : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$$

- *local approximations of ND operator require straightening of the curved boundary and (local) high-frequency approximations or high-order OSRC; valid only for convex obstacles*

- *Nedelec and Christiansen ’02 : precondition N with S in a BEM framework; non SPD problem, the spectral condition number of the preconditioned matrix – bounded independently of the mesh refinement, when uniform inf-sup conditions hold on the Galerkin spaces; does not remove the spurious resonances*

High-Order Numerical Methods (Sound-hard)

Smooth Closed Manifolds: Regularized CFIE [*Kress '85, Amini '90 , Bruno & Turc '07*]

$$u^s(\mathbf{z}) = (u_{\mathrm{SL}}\ \phi)(\mathbf{z}) + i\eta(u_{\mathrm{DL}} \circ \mathcal{R}\phi)(\mathbf{z})$$

RICFIE $$\frac{1}{2}\phi + (K^*\phi) + i\eta(N \circ \mathcal{R})\phi = -u^i$$

$\mathcal{R} = S_{ik_1},\ S_0^2,$ *coercive* $\rightarrow$ Fredholm Op of index 0 in RICFIE;

Choice of the coupling parameter and wavenumber k_1 in $\mathcal{R}$

Use the exact eigenvalues for the sphere:

- *minimize the condition number of the resulting diagonal matrix: Kress '85, Amini '90*
- *minimize the number of GMRES iterations for the diagonal matrix*

Number of iterations to 10^{-4} GMRES residual and far field max errors (off-centered point-source inside; app 6 points/wavelength)

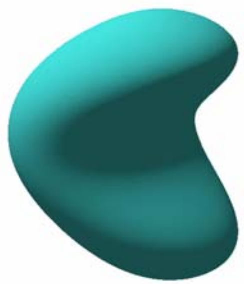
Unit Sphere

Size	Unknowns	ICFIE		RICFIE	
		Iter.	ϵ_∞	Iter.	ϵ_∞
2.5λ	5766	9	2.5e-05	5	1.7e-05
5.1λ	18150	7	1.7e-05	4	2.2e-05

Ellipsoid $a = 1, b = 0.25, c = 0.375$

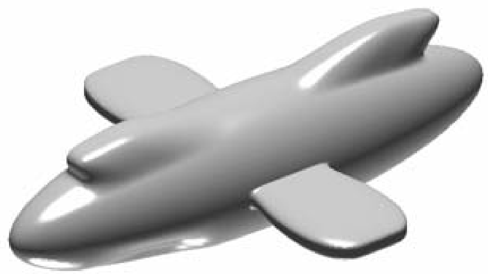
Size	Unknowns	ICFIE		RICFIE	
		Iter.	ϵ_∞	Iter.	ϵ_∞
2.5λ	5766	29	3.4e-05	11	2.5e-05
5.1λ	18150	26	1.8e-05	12	1.4e-05

Bean



Size	Unknowns	ICFIE		RICFIE	
		Iter.	ϵ_∞	Iter.	ϵ_∞
2.5λ	5766	23	6.6e-05	10	5.3e-05
5.1λ	18150	25	1.6e-05	11	1.4e-05

Airplane



Size	Unknowns	ICFIE		RICFIE	
		Iter.	ϵ_∞	Iter.	ϵ_∞
2.5λ	24200	119	3.5e-03	14	5.6e-04
5.1λ	24200	75	4.1e-03	18	2.3e-04

*One RCFIE iteration is **only 1.2** times more expansive than the CFIE !*

High-Order Numerical Methods: EM

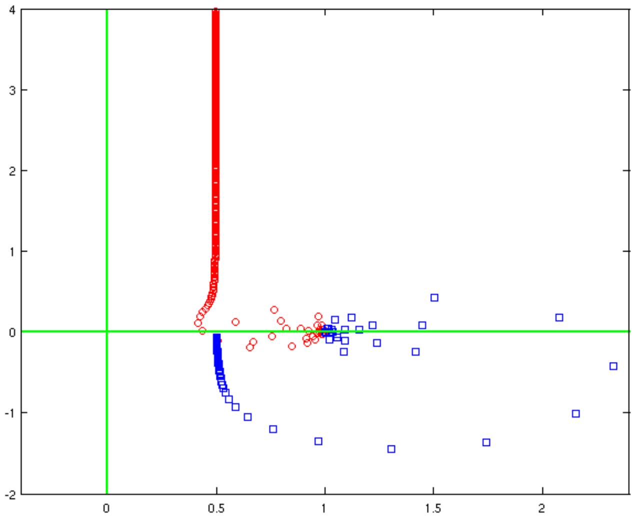
The electric operator T [Bendali '84, etc]

$$\mathbf{n} \times \mathcal{T} = ik(-\mathcal{T}_1 + \mathcal{T}_2 + S_0\Pi_H) + \text{Compact}$$

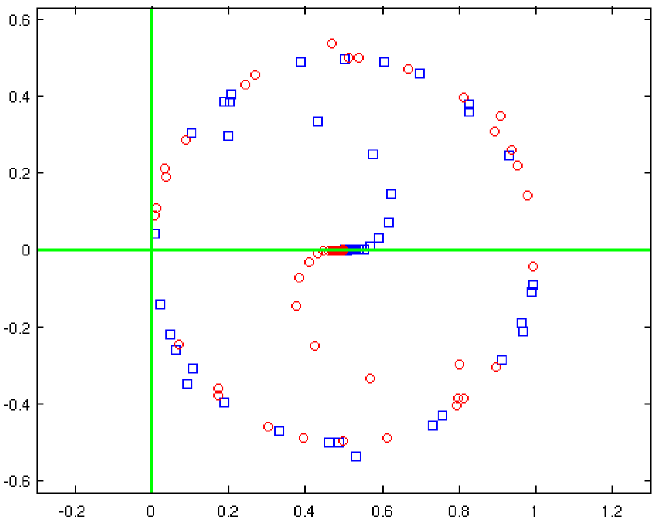
$$\mathcal{T}_1 = \frac{1}{k^2}(-\nabla_\Gamma)S_0\mathrm{div}_\Gamma \qquad \mathcal{T}_2 = \overrightarrow{\mathrm{curl}}\Delta_\Gamma^{-1}\mathrm{curl}_\Gamma\vec{\mathrm{S}}_0\overrightarrow{\mathrm{curl}}_\Gamma\Delta_\Gamma^{-1}\mathrm{curl}_\Gamma$$

CFIE Spectral Properties Unit Sphere [Kress '85]

Eigenvalues CFIE , k=32



Eigenvalues MFIE , k=32



$$\left(\frac{I}{2} + \mathcal{K} + \eta_1 \mathbf{n} \times \mathcal{T}\right)(\nabla_S Y_n^m) = \mathrm{CFIE}_n^{(1)} \nabla_S Y_n^m,$$

$$\mathrm{CFIE}_{\mathrm{n}}^{(1)} = \frac{1}{2} + \lambda_n - \eta_1 \Lambda_n^{(2)}$$

$$\left(\frac{I}{2} + \mathcal{K} + \eta_1 \mathbf{n} \times \mathcal{T}\right)(\overrightarrow{\mathrm{curl}}_S Y_n^m) = \mathrm{CFIE}_{\mathrm{n}}^{(2)} \overrightarrow{\mathrm{curl}}_S Y_n^m,$$

$$\mathrm{CFIE}_{\mathrm{n}}^{(2)} = \frac{1}{2} - \lambda_n + \eta_1 \Lambda_n^{(1)}$$

High-Order Numerical Methods:EM

*Use high-frequency approximations of the admittance operator in the illuminated region **only** (Adams '01, Levadoux '05)*

$$\begin{pmatrix} -\frac{I}{2} - \mathcal{K} & \mathcal{T} \\ -\mathcal{T} & -\frac{I}{2} - \mathcal{K} \end{pmatrix} \begin{pmatrix} \mathbf{n} \times \vec{\mathbf{E}}^s \\ \mathbf{n} \times \vec{\mathbf{H}}^s \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

$$\mathbf{n} \times \vec{\mathbf{H}}^s = \mathcal{C}(\mathbf{n} \times \vec{\mathbf{E}}^s)$$

- *local approximations of $\mathcal{C}$ require straightening of the curved boundary and (local) high-frequency approximations; the composition of the operators is expensive!*

- *a global approximation to the admittance operator would require Fourier-Airy integral operators to treat shadow boundaries and GO solutions for non-convex geometries*

Mollify the effect of T: Calderon Projectors

[Calderon '54, Povzner & Sukharevskii '60]

$$\begin{pmatrix} -\frac{I}{2}-\mathcal{K} & \mathcal{T} \\ -\mathcal{T} & -\frac{I}{2}-\mathcal{K} \end{pmatrix} \begin{pmatrix} \mathbf{n}\times\vec{\mathbf{E}}^s \\ \mathbf{n}\times\vec{\mathbf{H}}^s \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

$$\begin{pmatrix} \frac{I}{2}-\mathcal{K} & \mathcal{T} \\ -\mathcal{T} & \frac{I}{2}-\mathcal{K} \end{pmatrix} \begin{pmatrix} \mathbf{0} \\ \vec{\mathbf{J}} \end{pmatrix} = \begin{pmatrix} \mathbf{n}\times\vec{\mathbf{E}}^s \\ \mathbf{n}\times\vec{\mathbf{H}}^s \end{pmatrix}$$

$$\mathcal{T}^2 = \frac{I}{4} - \mathcal{K}^2$$

- *Nedelec and Christiansen '02: precondition EFIE with itself in a BEM framework; does not remove the spurious resonances: the effect of resonances studied by Christiansen '04*

- *convergence achieved only on a subspace of the Raviart-Thomas elements*

Regularized CFIE

[Rokhlin et al. '02, Bruno & Turc '07]

$$\vec{\mathbf{E}}^s(\mathbf{z}) = \mathrm{curl}\int_\Gamma G_k(\mathbf{z}-\mathbf{y})\vec{\mathbf{a}}(\mathbf{y})d\sigma(\mathbf{y}) \; + i\xi\;\mathrm{curl}\;\mathrm{curl}\int_\Gamma G_k(\mathbf{z}-\mathbf{y})(\mathbf{n}(\mathbf{y})\times(\mathcal{R}\vec{\mathbf{a}})(\mathbf{y}))d\sigma(\mathbf{y})$$

ICFIE-R
$$\frac{\vec{\mathbf{a}}}{2} + \mathcal{K}\vec{\mathbf{a}} + \xi k \mathcal{T}(\mathbf{n}\times(\mathcal{R}\vec{\mathbf{a}})) = -\mathbf{n}\times\vec{\mathbf{E}}^i$$

DCFIE-R
$$\frac{\vec{\mathbf{J}}}{2} - \mathcal{K}\vec{\mathbf{J}} + \xi k\;(\mathbf{n}\times\mathcal{R})\circ\mathcal{T}\vec{\mathbf{J}} = \mathbf{n}\times\vec{\mathbf{H}}^i - \xi k\;(\mathbf{n}\times\mathcal{R})(\mathbf{n}\times\vec{\mathbf{E}}^i)$$

Choose $\mathcal{R}$ appropriately → Fredholm operator in **ICFIE_R**; *coercivity - index 0*

Rokhlin et al. : in DCFIE-R choose n x $\mathcal{R}$ = $\mathcal{T}$(ik); results presented for low frequency only

Bruno & Turc: use the properties of the single-layer operator - spectral distribution is well suited for iterative solution;

$$(\mathbf{n}\times\mathbf{S}_k\vec{\mathbf{a}})(\mathbf{x}) = \mathbf{n}(\mathbf{x})\times\int_\Gamma G_k(\mathbf{x}-\mathbf{y})\vec{\mathbf{a}}(\mathbf{y})d\sigma(\mathbf{x})$$

$$\mathcal{R}_{ik_1,\alpha} = \mathbf{S}_{ik_1}(\mathbf{I}+\alpha\nabla_\Gamma \mathrm{div}_\Gamma),\;\; \mathrm{k}_1 \geq 0,\;\; \alpha \leq 0$$

Choices of the Regularizing Operators

$$\mathrm{CFIE} = \frac{I}{2} + \mathcal{K} + \eta\, \mathbf{n} \times \mathcal{T}$$

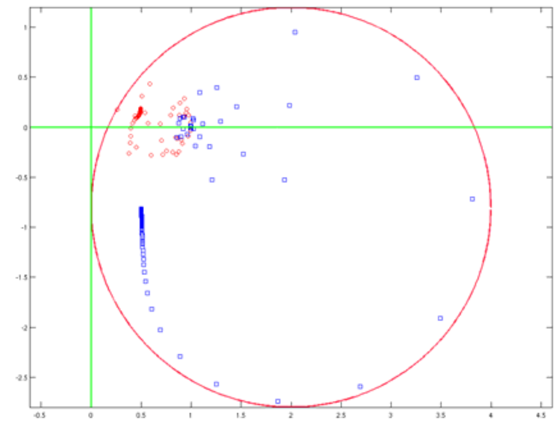
$$\mathrm{ICFIE} - \mathrm{R}_0 = \frac{I}{2} + \mathcal{K} + \xi k \mathcal{T} \left(\mathbf{n} \times \mathcal{R}^{(1)}_{0,-\frac{1}{k^2}} \right)$$

$$\mathrm{ICFIE} - \mathrm{R}_1 = \frac{I}{2} + \mathcal{K} + \xi k \mathcal{T} \left(\mathbf{n} \times \mathcal{R}^{(1)}_{ik_1,-\frac{1}{k_1^2}} \right)$$

$$\mathrm{DCFIE} - \mathrm{R}_1 = \frac{I}{2} - \mathcal{K} + \xi k \left(\mathbf{n} \times \mathcal{R}^{(1)}_{ik_1,-\frac{1}{k_1^2}} \right) \circ \mathcal{T}$$

$$\mathrm{CFIE}_{[1]} = \frac{I}{2} - \mathcal{K} + ik^2 \mathcal{T}(ik) \circ \mathcal{T}(ik)$$

Eigenvalue clustering
ICFIE-R_1 , k=32



[1] : *Rokhlin et al. : Well conditioned boundary integral equations for three dimensional electromagnetic scattering '02*

Coupling parameter: use the exact eigenvalues for the sphere to minimize the number of GMRES iterations

Number of iterations to 10^{-4} GMRES residual and far field max errors (plane-wave incidence; used app 6 points/wavelength)

Unit Sphere

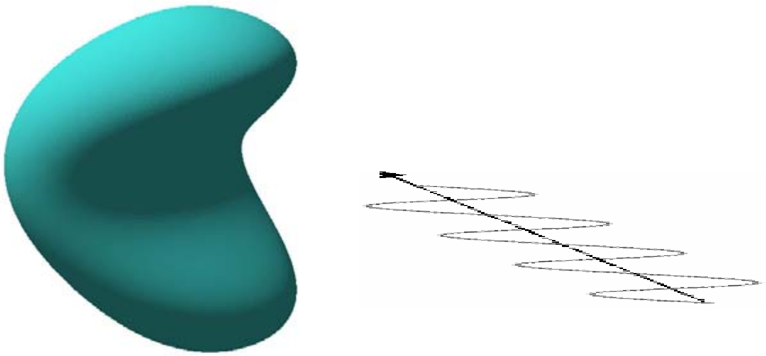
Size	Unknowns	ICFIE − R_0		ICFIE − R_1		DCFIE − R_1		CFIE[1]		CFIE	
		Iter.	ϵ_∞	Iter.	ϵ_∞	Iter.	ϵ_∞	Iter.	ϵ_∞	Iter.	ϵ_∞
2.5λ	11532	11	4.2e-04	9	4.7e-04	9	5.8e-04	11	3.4e-03	26	5.6e-04
5.1λ	36300	17	2.4e-04	10	3.1e-04	10	2.7e-04	13	2.1e-03	27	2.3e-04
10.2λ	76800	24	1.7e-04	12	2.9e-04	12	3.7e-04	17	1.9e-03	33	2.9e-04

Elongated ellipsoid $a = 1, b = 0.25, c = 0.375$

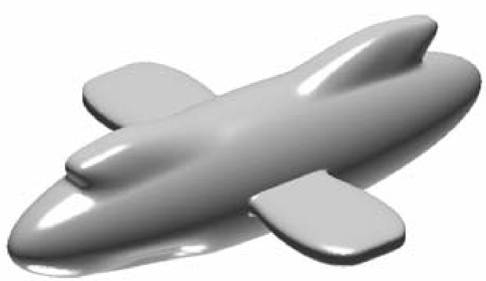
Size	Unknowns	ICFIE − R_0		ICFIE − R_1		DCFIE − R_1		CFIE[1]		CFIE	
		Iter.	ϵ_∞	Iter.	ϵ_∞	Iter.	ϵ_∞	Iter.	ϵ_∞	Iter.	ϵ_∞
2.5λ	11532	7	6.4e-05	11	1.2e-04	9	1.9e-04	20	1.8e-03	52	2.8e-04
5.1λ	36300	12	4.6e-05	12	7.9e-05	13	8.6e-05	27	5.9e-04	59	9.9e-05
10.2λ	76800	13	8.9e-05	12	9.9e-05	11	9.9e-05	16	4.0e-04	93	9.2e-05

*One CFIE-R iteration is **only 1.3** more expansive than the CFIE one and **1.5 less expensive** that CFIE[1]!*

Number of iterations to 10^{-4} GMRES residual and far field max errors



Size	Unknowns	ICFIE – R_0		ICFIE – R_1		DCFIE – R_1		$CFIE_{[1]}$		CFIE	
		Iter.	ϵ_∞	Iter.	ϵ_∞	Iter.	ϵ_∞	Iter.	ϵ_∞	Iter.	ϵ_∞
2.5λ	11532	15	1.5e-04	12	2.6e-04	11	3.3e-04	17	1.9e-03	38	3.8e-04
5.1λ	36300	18	1.1e-04	11	2.9e-04	10	2.1e-04	16	4.0e-04	46	1.8e-04
10.2λ	76800	25	1.3e-04	12	2.6e-04	12	2.0e-04	16	1.2e-03	53	1.5e-04



Size	Unknowns	ICFIE – R_0		ICFIE – R_1		DCFIE – R_1		$CFIE_{[1]}$		CFIE	
		Iter.	ϵ_∞	Iter.	ϵ_∞	Iter.	ϵ_∞	Iter.	ϵ_∞	Iter.	ϵ_∞
2.5λ	48400	28	2.1e-02	27	2.5e-02	42	1.4e-02	151	8.0e-02	> 300	1.8e-02
5.1λ	48400	48	3.2e-03	33	7.9e-03	33	7.2e-03	179	4.5e-02	> 300	1.0e-02

Closed Lipschitz Manifolds

$Classical\ CFIE: \ K, K^* : L^2(\Gamma) \rightarrow L^2(\Gamma)\ not\ compact$

[*Fabes, Jodeit, Lewis, Verchota, Kenig, Coiffman, Meyer, Jerison*]

Regularized CFIE Ac and EM (polygons) [*Buffa & Hiptmair '06*]

$$u^s(\mathbf{z}) = u_{\mathrm{DL}}(\mathcal{R}\ \phi)(\mathbf{z}) - i\gamma(u_{\mathrm{SL}}\phi)(\mathbf{z})$$

$$\mathcal{R} : H^{-\frac{1}{2}}(\Gamma) \rightarrow H^{\frac{1}{2}}(\Gamma)\ compact\ +\ coercive$$

RICFIE

$$\frac{1}{2}\mathcal{R} \circ \phi + (K \circ \mathcal{R})\phi - i\gamma S\ \phi = -u^i$$

$$(\nabla_\Gamma \mathcal{R}\phi, \nabla_\Gamma v) = (\phi, v),\ \forall v \in H_0^1(\Gamma_1) \times \cdots H_0^1(\Gamma_n),\ \phi \in H^{-1}(\Gamma)$$

High-Order Nystrom: Edge effects [*Bruno & Paffenroth '05*]

K, K, $\mathcal{K}$ are (PV) Calderon-Zygmund op + Densities are singular*

- *Acoustic: use the Holder continuity of the solution; rewrite CFIE with terms that are uniformly integrable across edges*
- *EM: extract the leading order singularity in the current; solve eigenvalue problem*

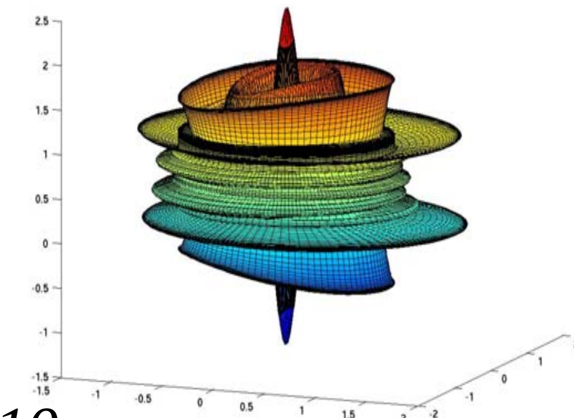
High-Order Nystrom: Closed Manifolds Edges

Cylinder of radius 1 and height 1

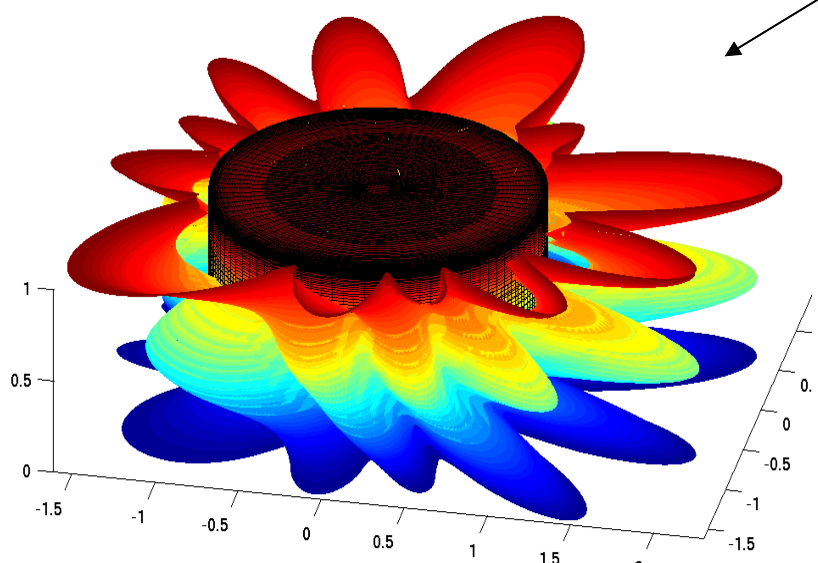
$$k = 10,\ 20$$

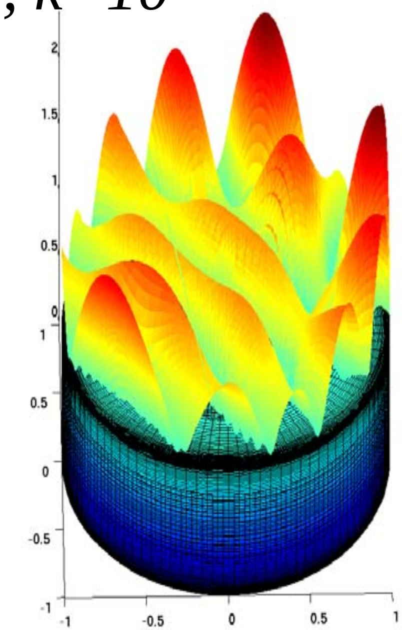
Discretization	k	Normalized Max. Error
$10 \times 65 \times 65$	10	$1.7 \cdot 10^{-4}$
$10 \times 65 \times 65$	20	$2.0 \cdot 10^{-3}$

a) Point source, k=10



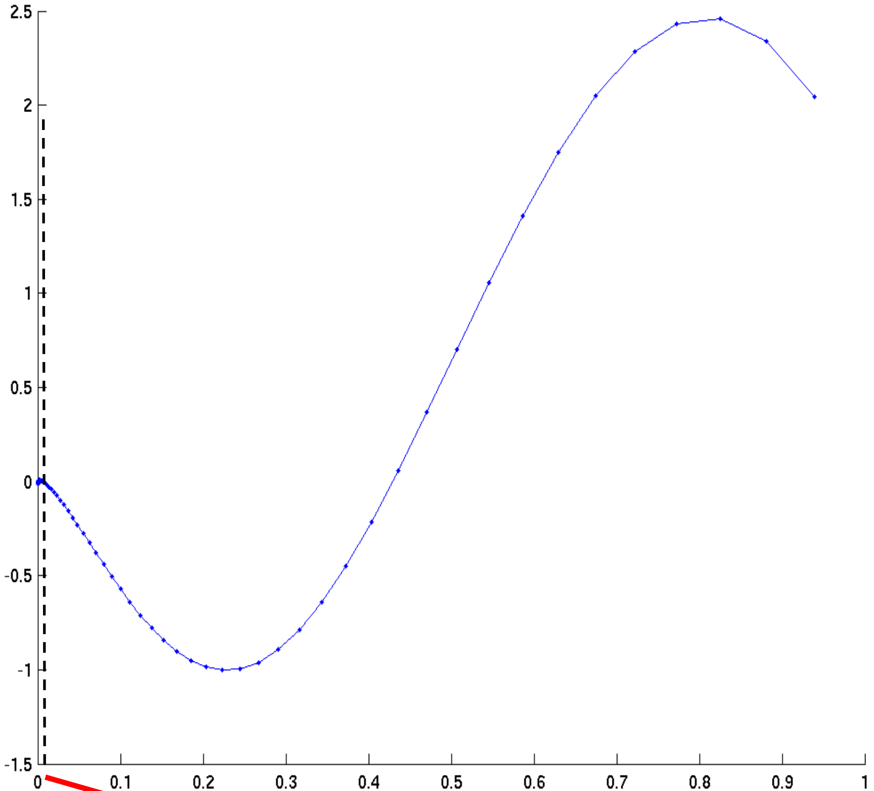
b) Plane wave incidence, k=10

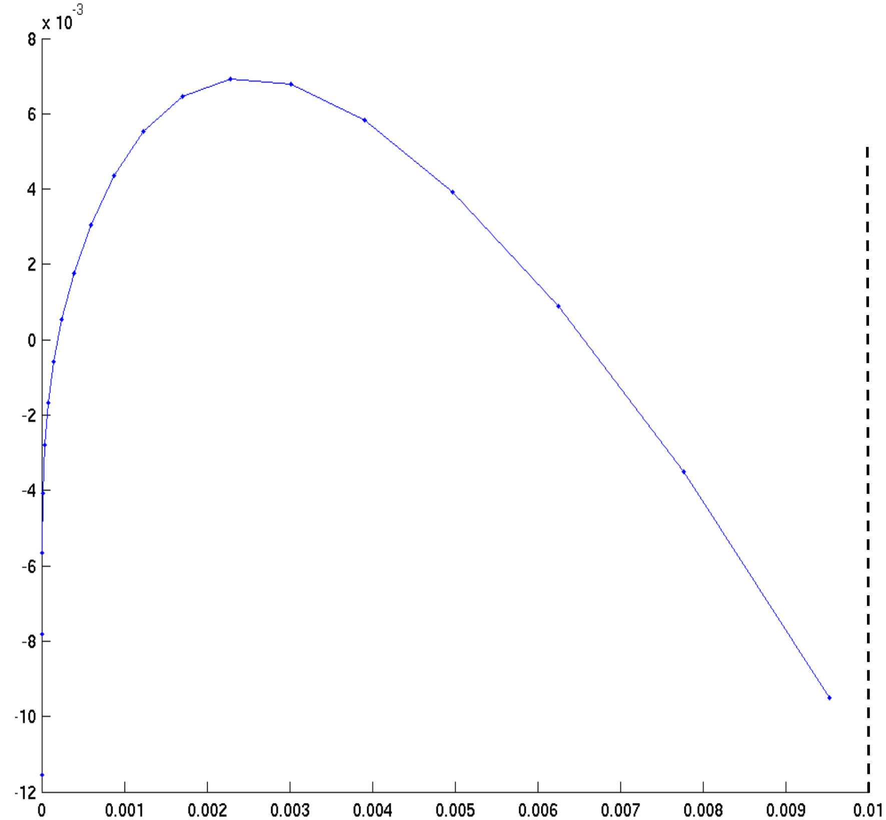




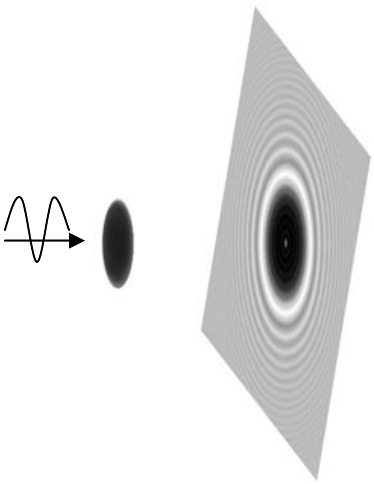
High-Order Nystrom: Closed Manifolds Edges

Current along a radius – focus near the edge





Open Surfaces: Boundary Value Problem



$$S\ \varphi = -u^i \quad N\ \psi = -\frac{\partial u^i}{\partial \mathbf{n}}$$

- *Ill Posed… Solutions exist and are unique [Stephan '87]*
- *In the neighborhood of the edge, the densities are singular* $\varphi \sim \frac{1}{\sqrt{d}} \quad \psi \sim \sqrt{d}$
- *Calderon relations are not valid*

Singularities+First Kind Equations = A significant computational challenge

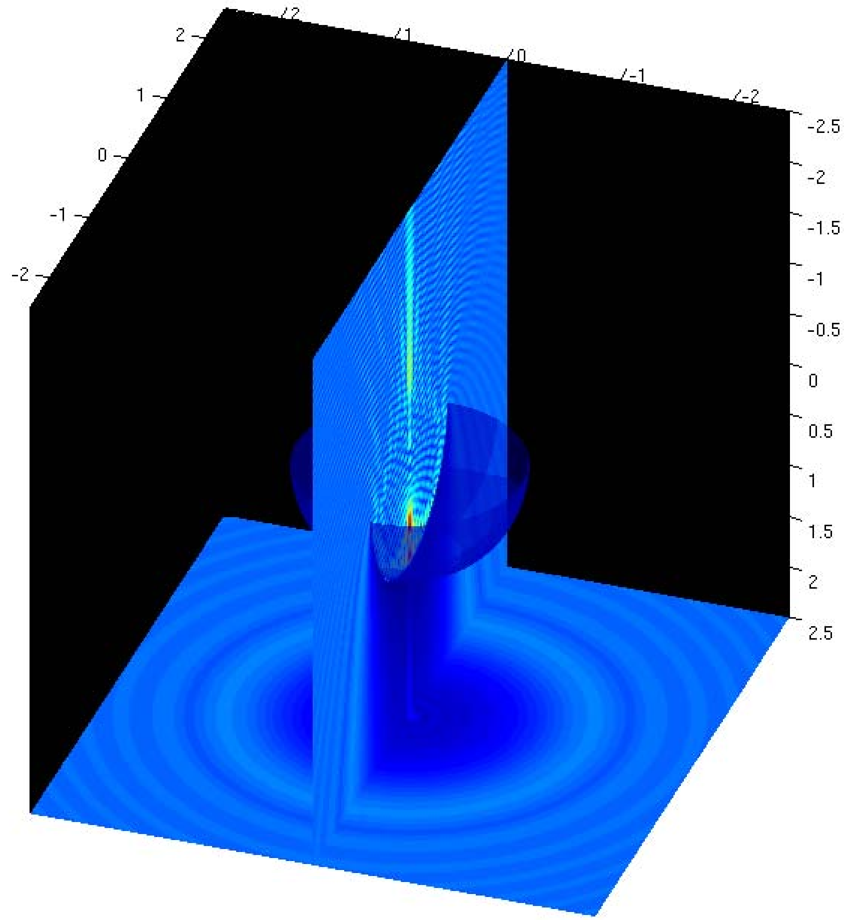
Bruno & Lintner '07: New "Regular second-kind (RSK) equations" + high-order numerical solution

$\omega(y) \sim \sqrt{d(y)}$, $d(y)$ the distance to the edge

$$N_\omega(\psi)(x) = \frac{\partial}{\partial n(x)} \int_\Gamma \frac{\partial G_k(x,y)}{\partial n(y)} \psi(y)\omega(y) dS_y \quad S_\omega(\varphi)(x) = \int_\Gamma G_k(x,y) \frac{\varphi(y)}{\omega(y)} dS_y$$

These operators map smooth functions to smooth functions [Costabel, Dauge, Duduchava '01]

3D Open Surfaces: Precondition N_ω by S_ω

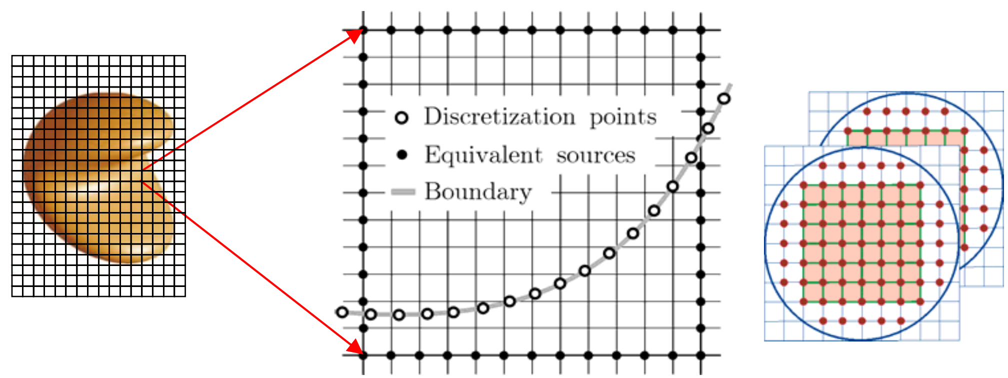


kD	Unknowns	RFK Iters.	RSK Iters.	Far Field Error
2	64x64x5	7	7	$\leq 1. \times 10^{-5}$
20	64x64x5	20	10	$\leq 1. \times 10^{-5}$
40	64x64x5	51	13	$\leq 1. \times 10^{-4}$
80	64x64x5	130	14	$\leq 1. \times 10^{-3}$

Fast Methods : Reducing $O(N^2)$

- *Translation-invariant Kernels: use FFTs $\rightarrow$ $O(N^{3/2} \log N)$ [Bleszynski, Bojarski]; do not exhibit high-order convergence*
- *FMM : interactions between well-separated regions is app of low rank [Rokhlin]; panel clustering [Hackbusch & Nowak], H-matrix framework [Hackbusch et al]; FMM—subwavelength breakdown*
- *HF-FMM: for larger frequencies, use far-field signatures; uses partial wave expansions, far field signatures, exp expansions, filtering and interpolated spherical harmonics [Rokhlin et al. '06]*

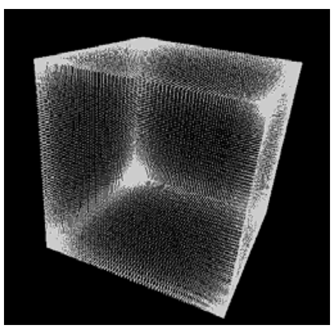
Non-adjacent interactions:
Equivalent Sources
[Bruno et al.]



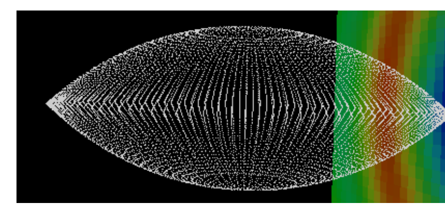
- *High-Order, Fast, Stable, Accurate $O(N^{6/5}\log(N))$ - $O(N^{4/3}\log(N))$ operations*
- *Acceleration strategy does not lead to accuracy breakdowns*

Large Spheres
(comparison w/ O(N log(N)) FISC)

Algorithm	Diameter	Time	RAM	Unknowns	RMS Error	Computer
FISC	120λ	$32 \times 14.5h$	$26.7Gb$	9,633,792	4.6%	SGI Origin 2000 (32 proc.)
Br. & Kun.	80λ	$55h$	$2.5Gb$	1,500,000	0.005%	AMD 1.4GHz (1 proc.)
Br. & Kun.	100λ	$68h$	$2.5Gb$	1,500,000	0.03%	AMD 1.4GHz (1 proc.)



Singular Scatterers



Geometry	Diameters	Time	Unknowns	RMS Error	Computer
Cube (Br. & Kun.)	$10\lambda \times 10\lambda \times 10\lambda$	$21h$	96,774	0.049%	AMD 1.4GHz (1 proc.)
Flying Saucer (Br. & Kun.)	$42\lambda \times 42\lambda \times 17\lambda$	$53h$	290,874	0.0045% (1 proc.)	AMD 1.4GHz

High-Frequency Scattering Simulations

Goal: Reduce number points/wavelength; Use an Ansatz (Phase Extraction)

$$\textit{Sound Soft}: \frac{1}{2}\underbrace{\frac{\partial u(\mathbf{x})}{\partial \mathbf{n}(\mathbf{x})}} = \frac{\partial e^{ik\vec{\mathbf{d}}\cdot\mathbf{x}}}{\partial \mathbf{n}(\mathbf{x})} - \int_S \underbrace{\frac{\partial u(\mathbf{x}')}{\partial \mathbf{n}(\mathbf{x}')}} \frac{\partial G_k(\mathbf{x}-\mathbf{x}')}{\partial \mathbf{n}(\mathbf{x})}\, d\sigma(\mathbf{x}')$$

$$e^{ik\vec{d}\cdot\mathbf{x}}|_S \rightarrow \partial_{\mathbf{n}} u|_S = (DN)e^{ik\vec{d}\cdot\mathbf{x}}|_S = e^{ik\vec{d}\cdot\mathbf{x}}\ \mu_{slow}(k,\mathbf{x})\ ?$$

[*Keller, Fok, Ludwig, Taylor, Melrose, Balabane, etc*]

Convex Case: Phase extraction, refinement around shadow boundaries; use the ansatz in BEM

$[\mathcal{O}(k^{\frac{1}{3}})\ \textit{Nedelec et al.}\ '94;\ \textit{Gillad \& Keller}\ '04:\ \textit{creeping rays}]$

$[\textit{Langdon \&Chandler}-\textit{Wilde}\ '05:\ \textit{polygons} \rightarrow \textit{corner diffraction};\ \mathcal{O}(k^{\frac{1}{9}})\ \textit{Graham et al.}\ 06]$

An $\mathcal{O}(1)$ Convergent High-Order High-Frequency Nystrom Approach

[*Bruno, Geuzeine, Reitich ‘04; Reitich,Turc ’05, ‘06; Anand, Boubendir, Ecevit, Reitich ’06, ‘07*]

A Convergent High-Frequency Approach

I. Ansatz   II. Localized integration

Highly oscillatory!

$$\frac{1}{2}\mu_{slow}(\mathbf{x}) + \int_S \left[\frac{\partial G_k(\mathbf{x},\mathbf{x}')}{\partial \mathbf{n}(\mathbf{x})} e^{i\vec{\mathbf{k}}\cdot(\mathbf{x}'-\mathbf{x})}\right] \mu_{slow}(\mathbf{x}')\, d\sigma(\mathbf{x}') = i\vec{\mathbf{k}}\cdot\mathbf{n}(\mathbf{x})$$

$$= \int_S e^{ik\underbrace{\left(|\mathbf{x}-\mathbf{x}'|+\frac{\vec{\mathbf{k}}}{k}\cdot\mathbf{x}'\right)}_{\varphi(\mathbf{x}')}} F(\mathbf{x}',k)\, d\sigma(\mathbf{x}')$$

$\Rightarrow$ Most significant contributions from "*critical points*"

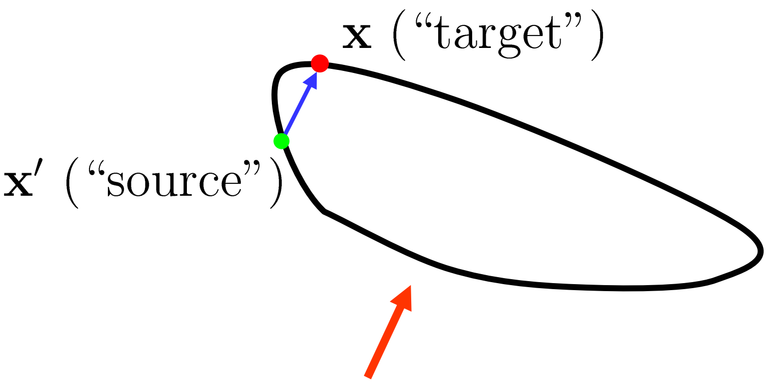
- ~~*Boundary points*~~
- *Singular points* $(\mathbf{x}' = \mathbf{x})$
- *Stationary points* $(\nabla_S\, \varphi = 0)$

A Convergent High-Frequency Approach

II. Localized Integration

$$\mathcal{I}(\mathbf{x},k) \equiv \int_S \left[\frac{\partial G_k(\mathbf{x},\mathbf{x}')}{\partial \mathbf{n}(\mathbf{x})} e^{i\vec{\mathbf{k}}\cdot(\mathbf{x}'-\mathbf{x})} \right] \mu_{slow}(\mathbf{x}')\, d\sigma(\mathbf{x}') :$$

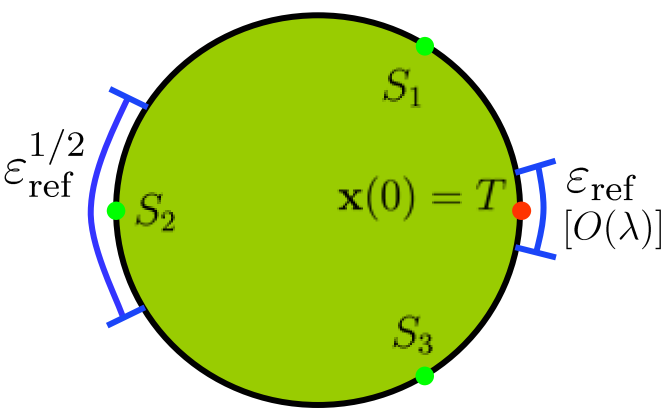
- *Singular points* $(\mathbf{x}' = \mathbf{x})$
- *Stationary points* $(\nabla_S\, \varphi = 0)$



Localized Integration

$$\int_{-A}^{A} f_A(t) e^{ikt^2} = \int_{-\varepsilon}^{\varepsilon} f_\varepsilon(t) e^{ikt^2} + \mathcal{O}((k\varepsilon^2)^{-n})$$

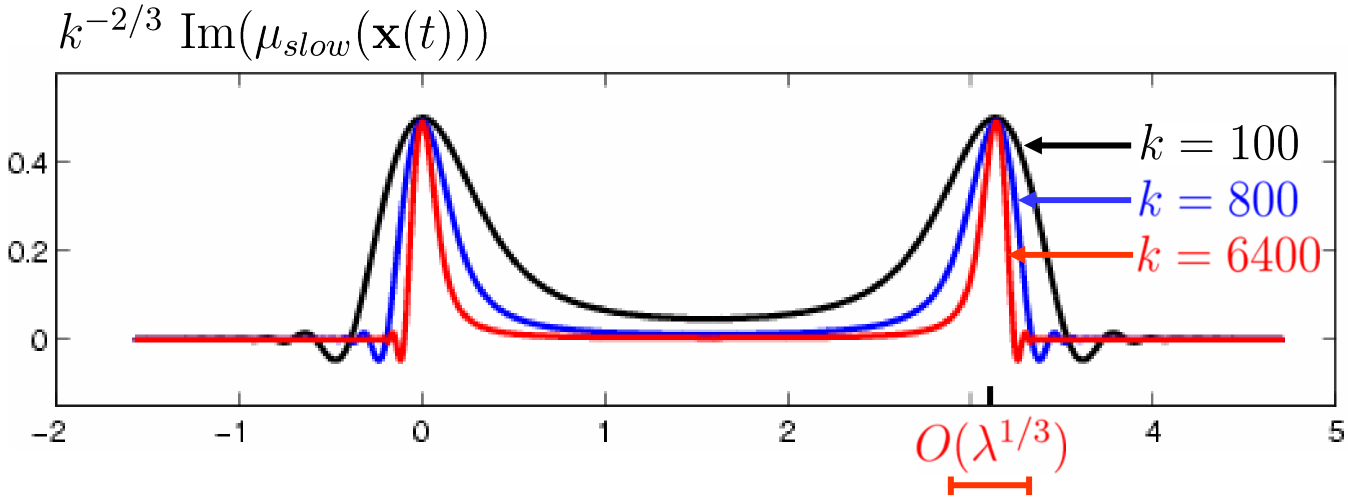
$$Example: \int_0^{2\pi} \Phi_k(\mathbf{x}(t),\mathbf{x}(0)) e^{i\mathbf{k}\cdot(\mathbf{x}(t)-\mathbf{x}(0))} \cos(t)\, dt$$

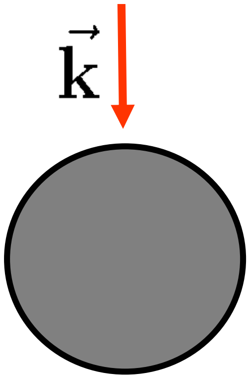


k	Nr. pts.	ε	Error
1000	2100	1.0	1.5e-6
2000	2100	0.5	4.8e-8
4000	2100	0.25	1.2e-7
8000	2100	0.125	9.8e-7
16000	2100	0.0625	1.5e-6

A Convergent High-Frequency Approach

III. Shadow Boundaries

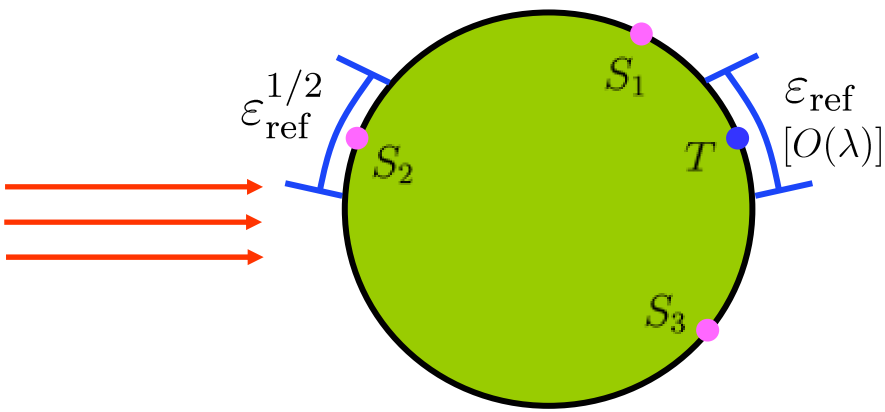




<u>Approach:</u> *separate and fully solve (highly oscillatory nature of) density at narrow (order $\lambda^{1/3}$) shadow-boundary regions*

A Convergent High-Frequency Approach
Single Scattering Configurations

- ***Frequency-independent cost***



ka	GMRES iterations	Error	CPU time
25 unknowns, $\varepsilon = \varepsilon_{\mathrm{ref}}$			
1	9	$1.0e{-}12$	< 1s
10	11	$1.6e{-}4$	< 1s
100	13	$9.3e{-}4$	3s
1000	13	$8.3e{-}3$	5s
10000	15	$1.0e{-}2$	6s
100000	14	$1.1e{-}2$	6s
100 unknowns, $\varepsilon = 5\varepsilon_{\mathrm{ref}}$			
1	9	$1.0e{-}12$	< 1s
10	17	$3.0e{-}11$	5s
100	22	$1.5e{-}5$	11s
1000	25	$3.1e{-}5$	$2m30s$
10000	27	$8.4e{-}5$	$3m12s$
100000	30	$8.8e{-}5$	$3m43s$

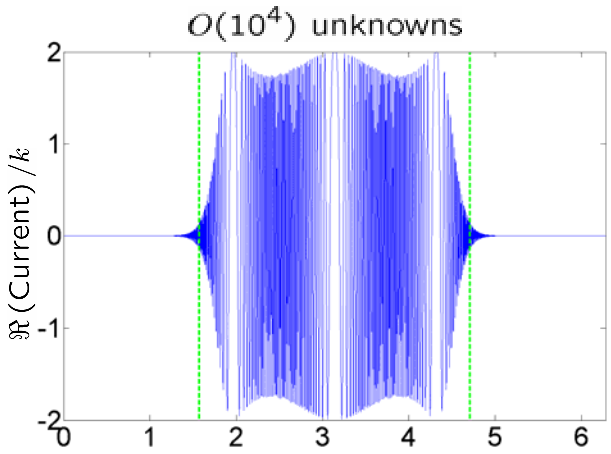
Would require over half a million unknowns to resolve with a classical method

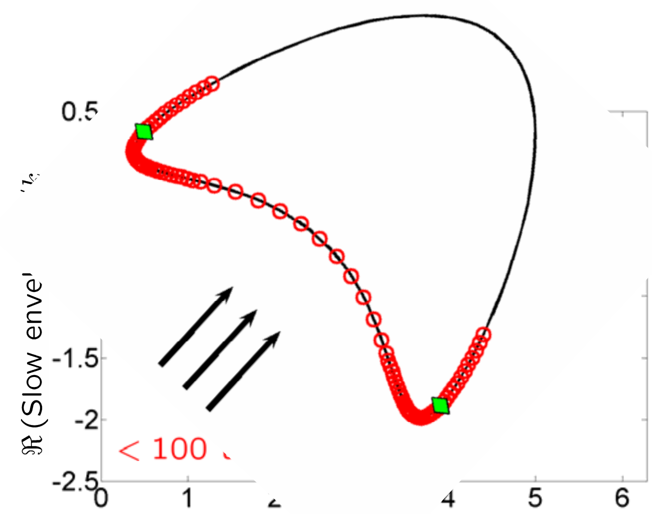
[*Bruno, Geuzaine & Reitich*, ’04-’06]

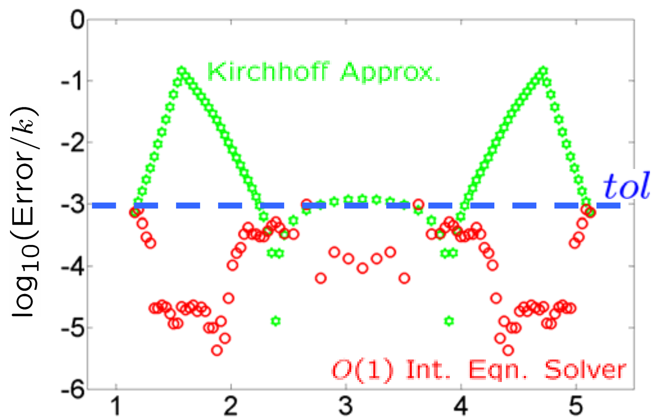
A Convergent High-Frequency Approach
Single Scattering Configurations

- ***Error control***

Perimeter/$\lambda \approx 900$:





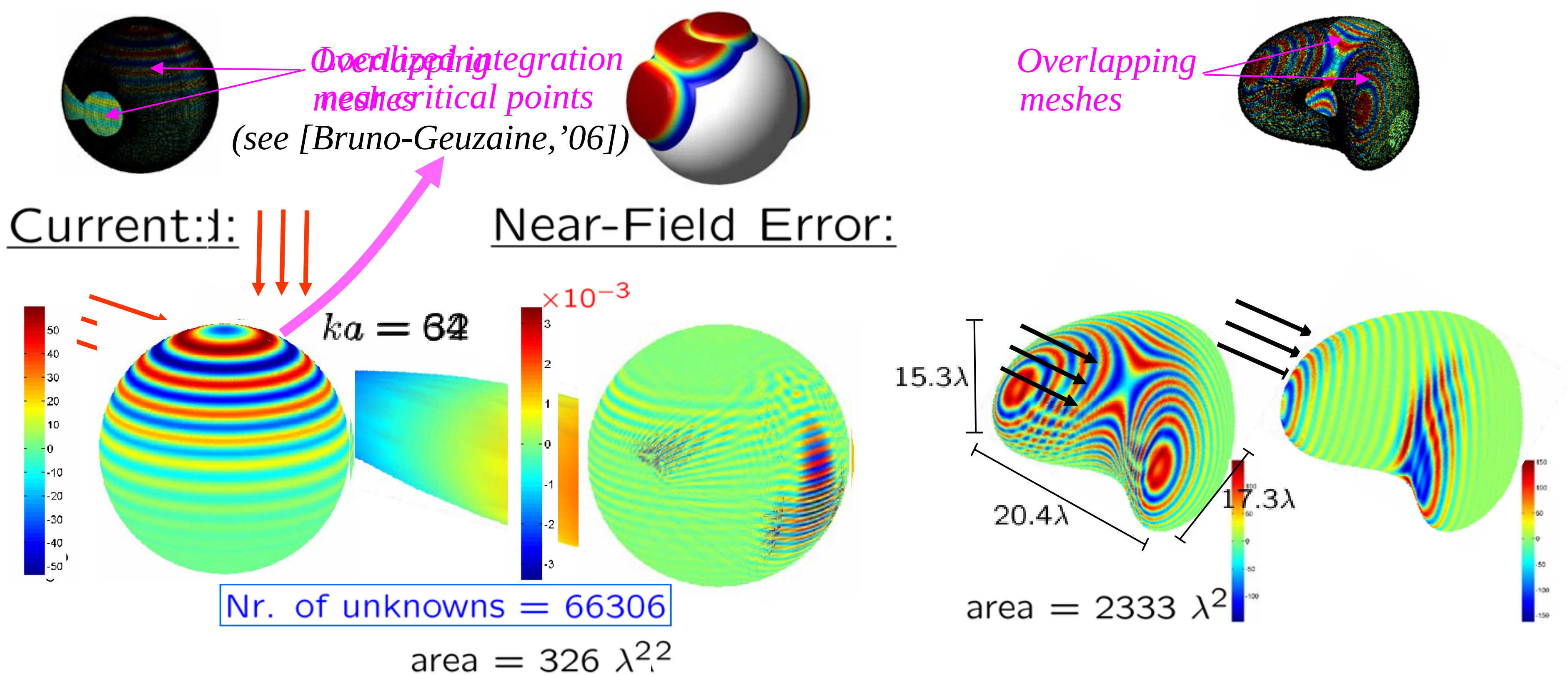


[*Bruno, Geuzaine & Reitich*, '04-'06]

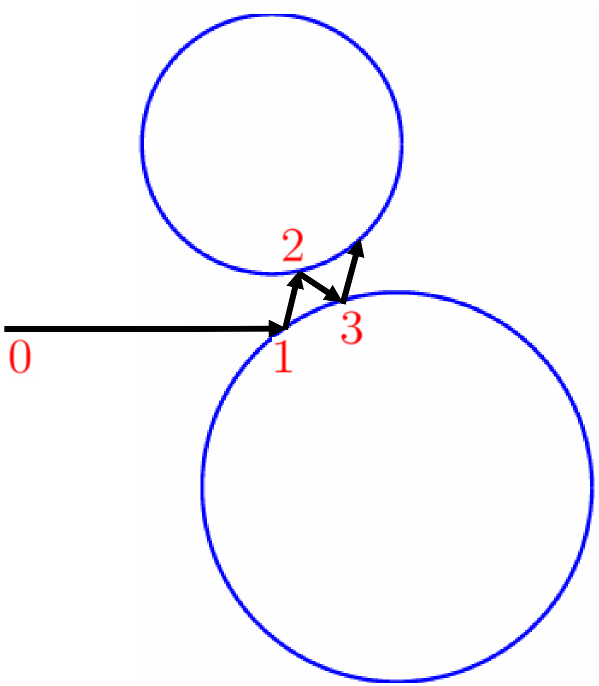
A Convergent High-Frequency Approach

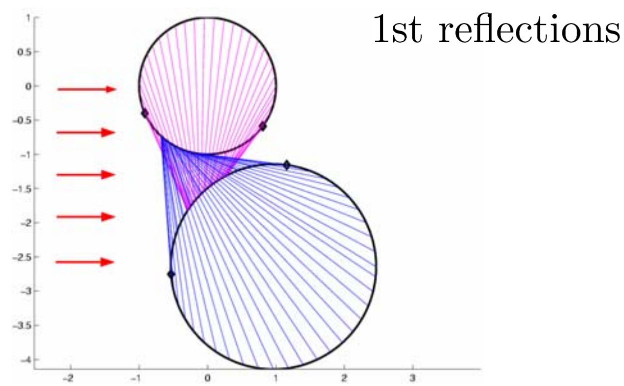
3D Implementation (in progress)

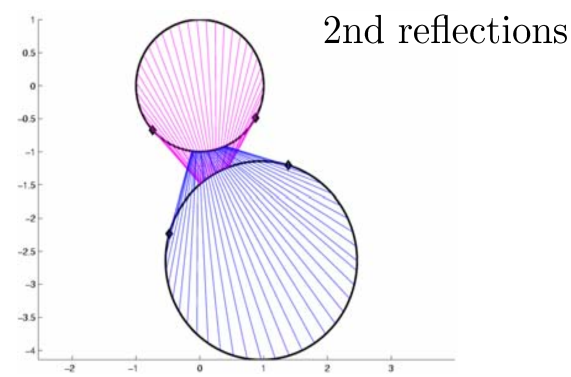
- *Integration:* fast, high-order method [Bruno-Kunyansky,'01]
- precursor of 3D high-frequency integrator

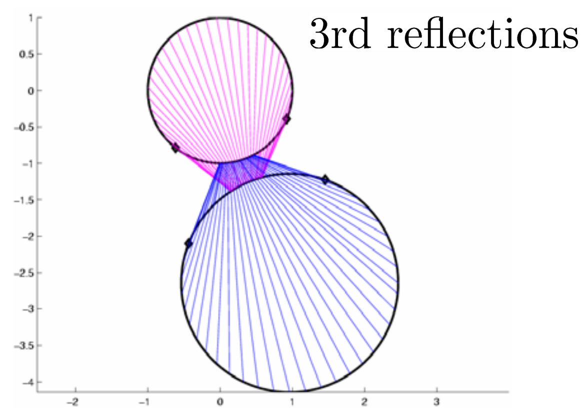


A Convergent High-Frequency Approach
IV. Multiple Scattering









. . .

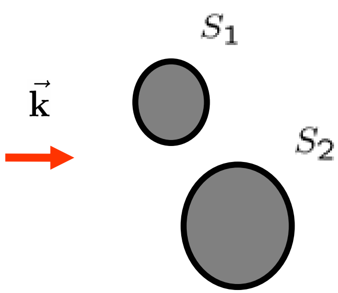
Approach: construct an appropriate *ansatz* using classical ray-tracing (GO) techniques

A Convergent High-Frequency Approach

IV. Implementation: Multiple reflections

$$\mu_1(t) + \int_{S_1} dt'\, G_k(\mathbf{x}(t), \mathbf{x}(t'))\, \mu_1(t') \quad + \int_{S_2} d\tau'\, G_k(\mathbf{x}(t), \mathbf{z}(\tau'))\, \mu_2(\tau') = e^{ik\phi^0(t)}$$

$$\mu_2(\tau) + \int_{S_2} d\tau'\, G_k(\mathbf{z}(\tau), \mathbf{z}(\tau'))\, \mu_2(\tau') \, + \int_{S_1} dt'\, G_k(\mathbf{z}(\tau), \mathbf{x}(t'))\, \mu_1(t') = e^{ik\nu^0(\tau)}$$



$$\begin{bmatrix} I - T_{11} & -R_{12} \\ -R_{21} & I - T_{22} \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} e^{ik\phi^0(t)} \\ e^{ik\nu^0(\tau)} \end{bmatrix}$$

Multiple reflections $\longleftrightarrow$ Neumann series

$$\underbrace{\begin{bmatrix} I & (I - T_{11})^{-1}(-R_{12}) \\ (I - T_{22})^{-1}(-R_{21}) & I \end{bmatrix}}_{I - A} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \underbrace{\begin{bmatrix} (I - T_{11})^{-1} e^{ik\phi^0(t)} \\ (I - T_{22})^{-1} e^{ik\nu^0(\tau)} \end{bmatrix}}_{\begin{bmatrix} \mu^0_{1,slow} e^{ik\phi^0} \\ \mu^0_{2,slow} e^{ik\nu^0} \end{bmatrix}}$$

$$\Longrightarrow \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \mu^0_{1,slow} e^{ik\phi^0} \\ \mu^0_{2,slow} e^{ik\nu^0} \end{bmatrix}}_{\text{Isolated Obstacles}} + A \underbrace{\begin{bmatrix} \mu^0_{1,slow} e^{ik\phi^0} \\ \mu^0_{2,slow} e^{ik\nu^0} \end{bmatrix}}_{\text{First Reflections}} + A^2 \underbrace{\begin{bmatrix} \mu^0_{1,slow} e^{ik\phi^0} \\ \mu^0_{2,slow} e^{ik\nu^0} \end{bmatrix}}_{\text{Second Reflections}} + \cdots \quad \cdots$$

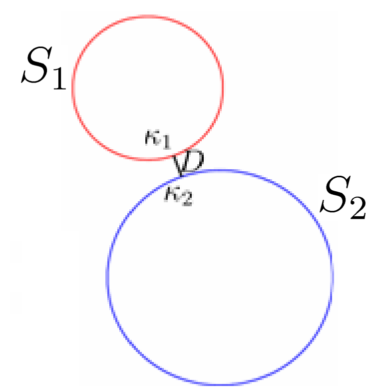
A Convergent High-Frequency Approach

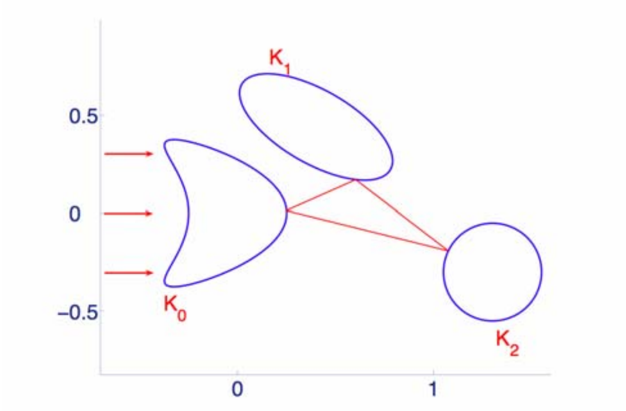
Convergence of Iterations [Reitich & Ecevit '06]

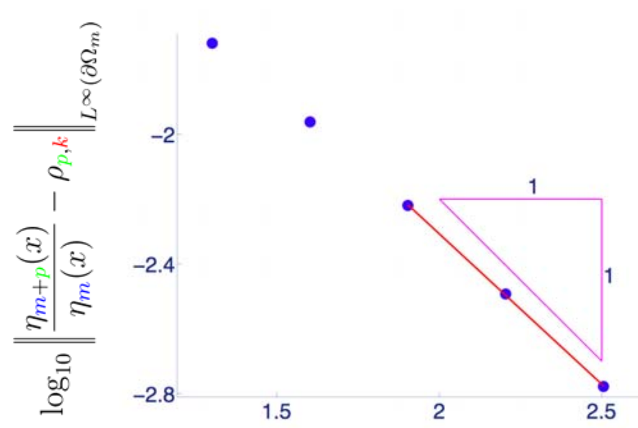
$$\mu^n_{1,slow} + \int_{S_1} dt'\, G_k(\mathbf{x}(t), \mathbf{x}(t'))\ \mu^n_{1,slow}(t') e^{ik(\phi^n(t') - \phi^n(t))}$$
$$= -e^{-ik\phi^n(t)} \int_{S_2} d\tau'\, G_k\ \mu^{n-1}_{2,slow} e^{ik\nu^{n-1}}$$

$$k \gg 1 \quad \Rightarrow \quad \frac{\max\left|\mu^n_{2,slow}\right|^2}{\max\left|\mu^{n+1}_{1,slow}\right|^2} \sim \mathcal{R}_2 \equiv (1+\kappa_2 D) + \sqrt{(1+\kappa_2 D)^2 - \frac{1+\kappa_2 D}{1+\kappa_1 D}}$$

$$\Rightarrow \quad \text{Rate of conv.} = \overbrace{[\mathcal{R}_1\mathcal{R}_2]^{-1/2}}^{r_A(<1)} + \mathcal{O}\left(\frac{1}{k^2}\right)$$

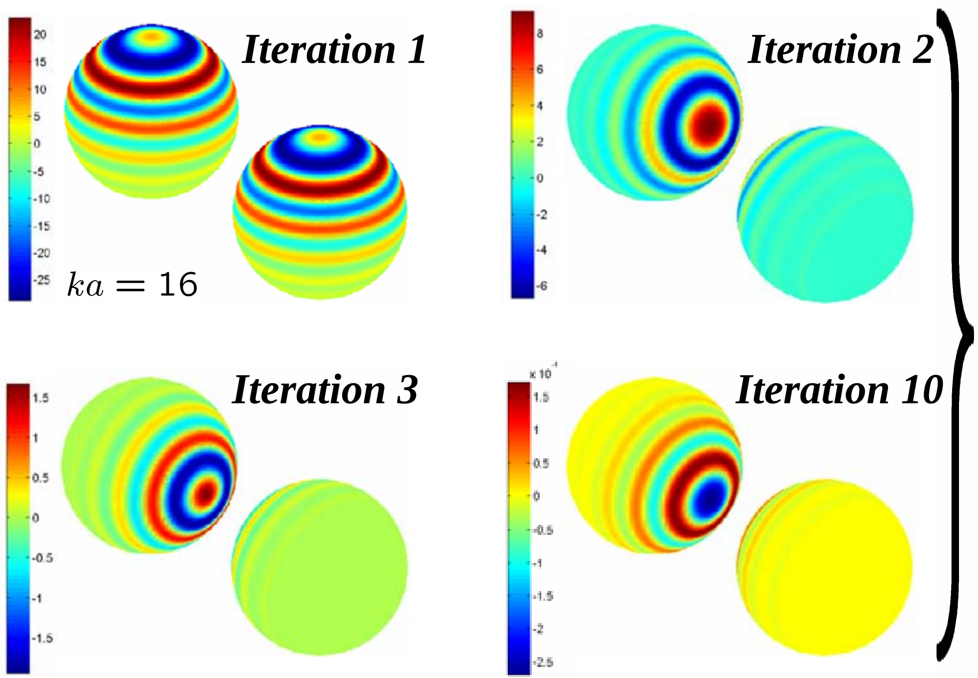


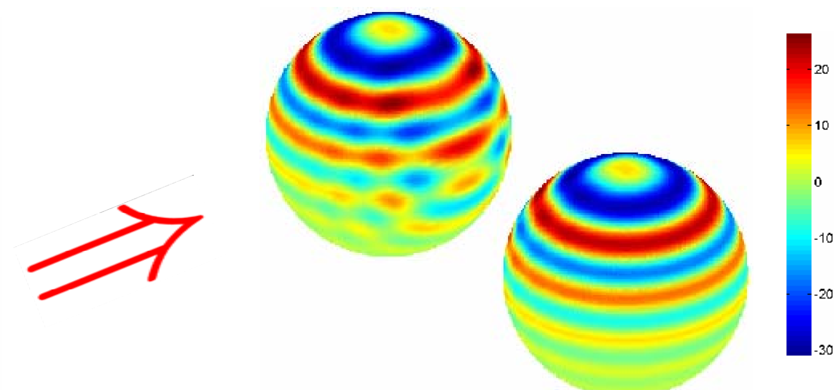




A Convergent High-Frequency Approach
3D Implementation (in progress)

- *Multiple scattering:* *iterative solution*
- *Converges with theoretically predicted rate*





Conv. Rate:

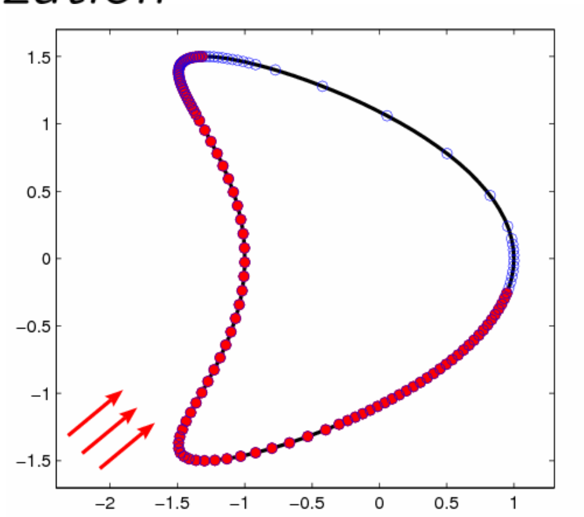
Exact ($ka=\infty$)	$ka = 4$	$ka = 8$	$ka = 16$
0.0718	0.0785	0.0740	0.0724

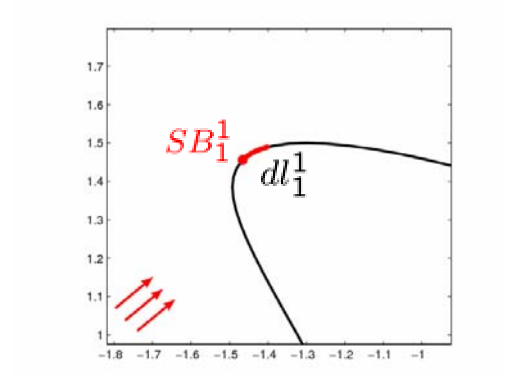
[A. Anand, Y. Boubendir, F. Ecevit & Reitich, ’06]

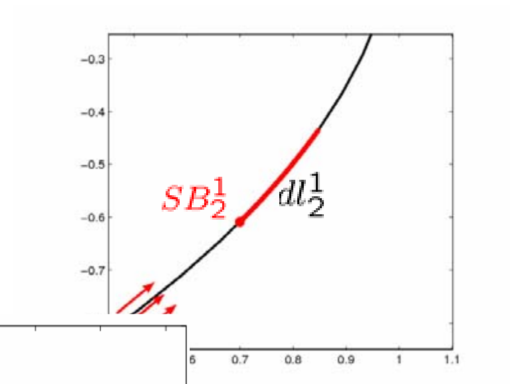
A Convergent High-Frequency Approach

"Putting it all together": Example

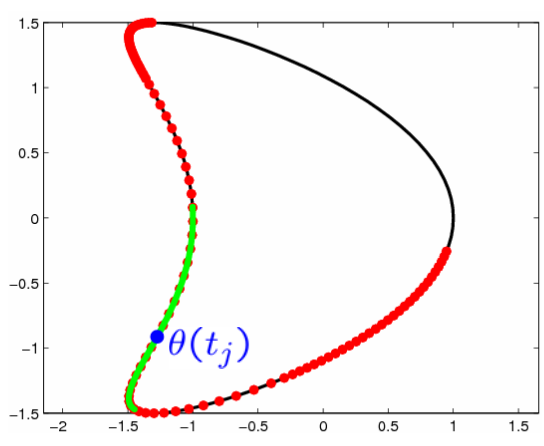
- *Discretization*

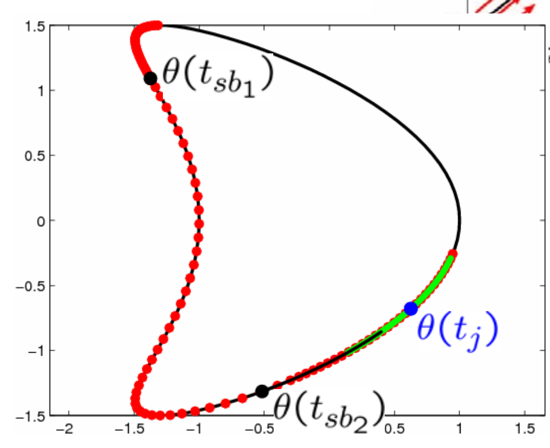






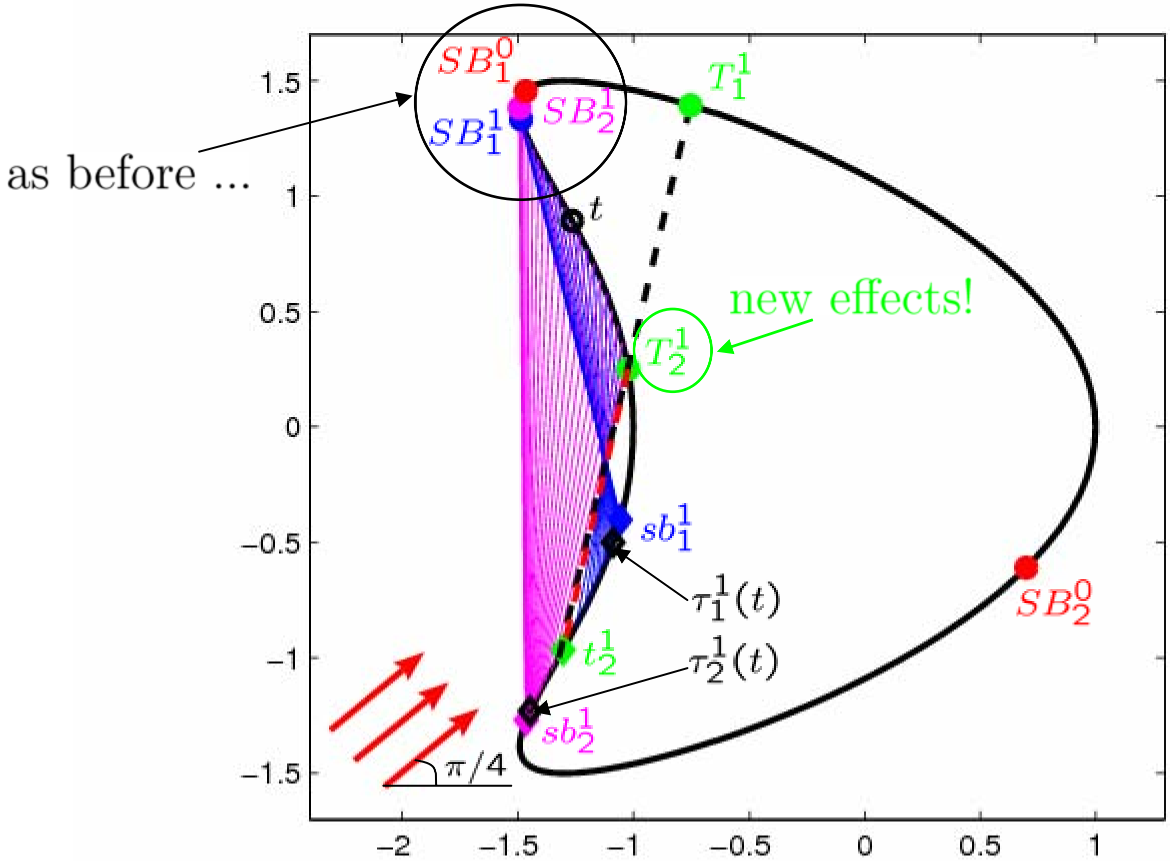
- *Localized Integration*





A Convergent High-Frequency Approach
Example

- *Iteration 1:* Geometrical Optics

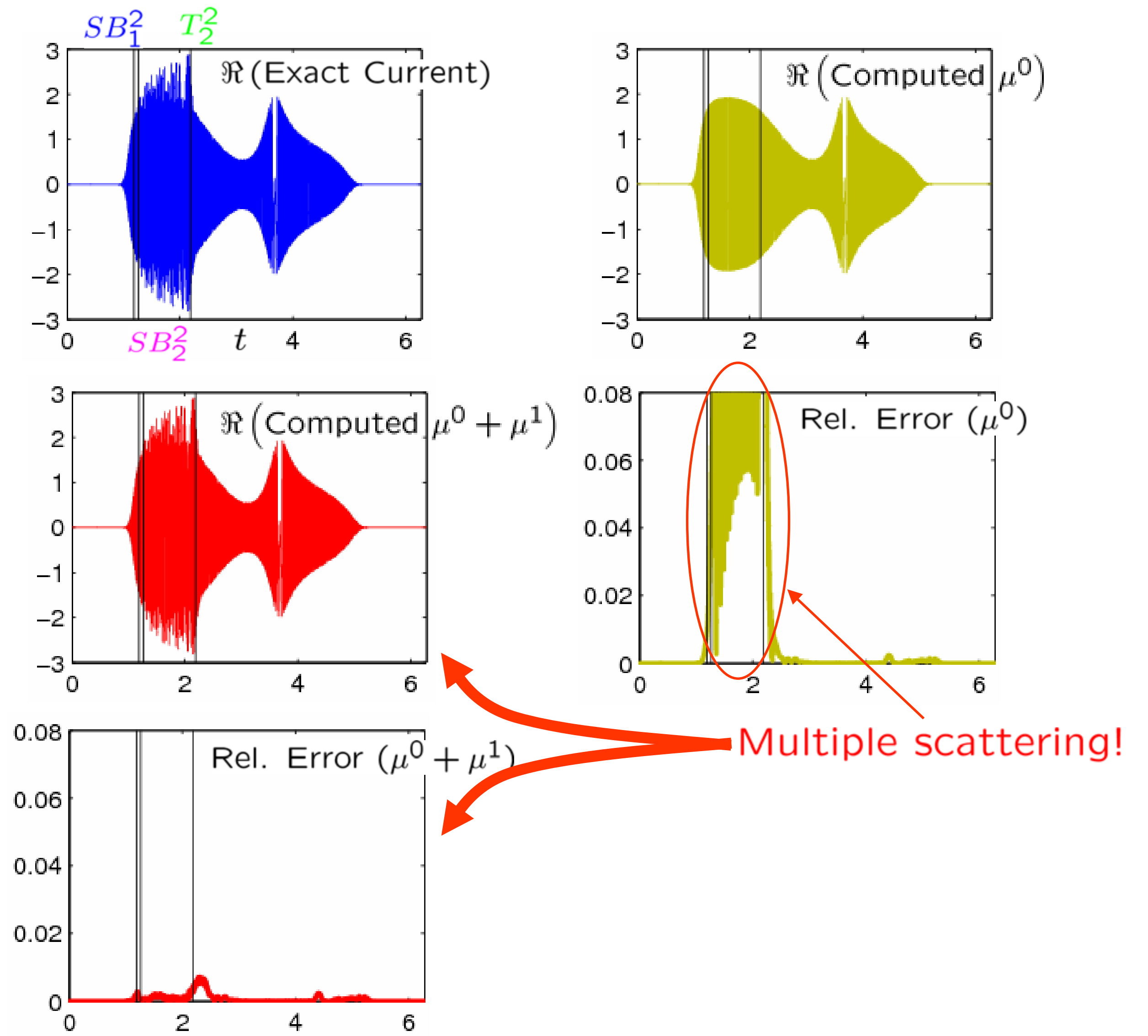


A Convergent High-Frequency Approach

Example

- *Iteration 1:*

  Comparison with exact solution



A Convergent High-Frequency Approach

Rough Surfaces (Gratings) Asymptotic Integration

$$\frac{1}{2}\mu(\mathbf{x}) + \underbrace{\int_S \frac{\partial G_k(\mathbf{x},\mathbf{x}')}{\partial \mathbf{n}(\mathbf{x})}}_{} \mu(\mathbf{x}')\, d\sigma(\mathbf{x}') = ik\vec{\mathbf{d}}\cdot\mathbf{n}(\mathbf{x})e^{ik\vec{d}\cdot\mathbf{x}}, \;\; \mathbf{x}\in S$$

$$G_k^{per} = \frac{i}{4}\sum_{n=-\infty}^{\infty} e^{-ikd_1 Ln} H_0^1(k|\mathbf{x}-(\mathbf{x}'-nL)|)$$

Phase extraction

$$\mathcal{I}(\mathbf{x},k) \equiv \int_S \left[\frac{\partial \Phi_k(\mathbf{x},\mathbf{x}')}{\partial \mathbf{n}(\mathbf{x})} e^{i\vec{\mathbf{k}}\cdot(\mathbf{x}'-\mathbf{x})}\right]\mu_{slow}(\mathbf{x}')\, d\sigma(\mathbf{x}') :$$

- *Singular points* $(\mathbf{x}'=\mathbf{x})$
- ~~*Stationary points*~~ $\cancel{(\nabla_S\varphi = 0)}$

$\Rightarrow$ *Asymptotics:* $\mathcal{I}(\mathbf{x},k) \approx \mathcal{I}^0(\mathbf{x}) + O\left(\frac{1}{k}\right)$

Want convergence! $\Rightarrow$ $\mathcal{I}(\mathbf{x},k) = \sum_{n=0}^{\infty}\frac{\mathcal{I}^n(\mathbf{x})}{k^n}$

$$\Rightarrow \mu_{slow}(\mathbf{x}) = \mu_{slow}(\mathbf{x},k) = k\mu_{slow}^{-1}(\mathbf{x}) + \sum_{n=0}^{\infty}\frac{\mu_{slow}^n(\mathbf{x})}{k^n}$$

Kirchhoff approximation

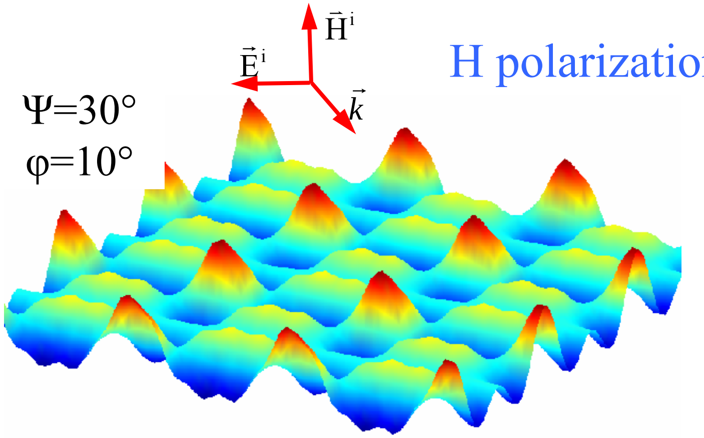
Recursively computable

A Convergent High-Frequency Approach

Asymptotic Integration

Oblique incidence

$\lambda = 0.021, \quad h = 0.015$



Efficiency	Scattered energy	Order 0	Order 1	Order 3	Order 5	Order 10
(0,0)	0.2168524774056718e-00	3.67e-04	5.21e-06	1.27e-07	3.13e-09	1.51e-11
(1,0)	0.2163867201690774e-01	2.89e-04	8.52e-06	1.71e-07	9.98e-10	1.11e-11
(1,1)	0.3014199854789860e-01	1.59e-03	1.99e-05	7.13e-08	3.72e-09	4.35e-12
(1,2)	0.2763092444554548e-01	3.44e-04	1.22e-05	2.78e-07	8.41e-09	9.16e-12
(2,2)	0.4699008279945933e-03	3.81e-03	2.35e-04	3.31e-06	1.75e-08	5.59e-11
(2,4)	0.2901776163762193e-03	9.75e-05	1.40e-04	2.50e-06	1.06e-08	7.54e-11
ϵ	2.05e-15	1.23e-06	4.62e-09	3.33e-12	9.44e-15	6.96e-15

[*Reitich & Turc*, '05]

A Convergent High-Frequency Approach

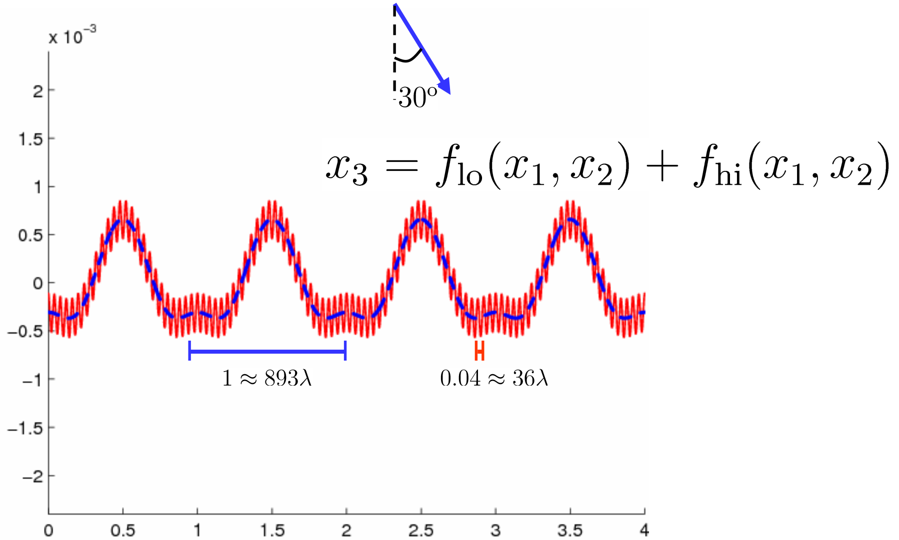
Asymptotic Integration

Multi-scale surface

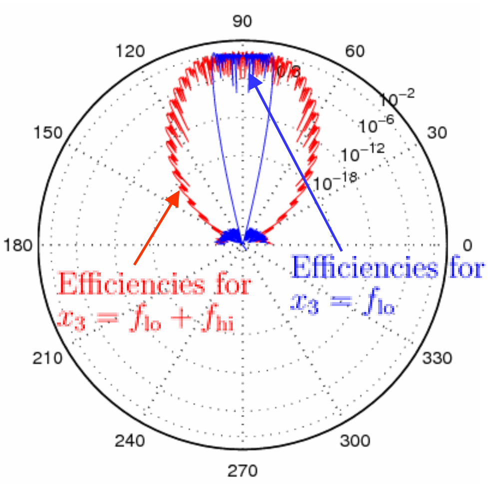
Oblique incidence

$$\lambda = 0.00112, \quad h = 0.001 \approx 0.9\lambda$$

$$h_1 = 0.0002 \approx 0.2\lambda$$



Efficiency	Scattered energy	Order 0	Order 3	Order 7	Order 11	Order 18
0	0.1999444889329398e-02	4.48e+01	1.32e-01	3.73e-02	5.53e-05	1.40e-11
1	0.3482870139291976e-02	4.45e+01	1.46e-01	3.71e-02	5.49e-05	1.95e-12
2	0.9312287660659891e-03	4.43e+01	1.60e-01	3.69e-02	5.46e-05	1.40e-11
3	0.4426354000069988e-04	4.40e+01	1.75e-01	3.67e-02	5.42e-05	2.01e-10
4	0.5340069488986556e-03	4.38e+01	1.89e-01	3.65e-02	5.38e-05	5.20e-11
5	0.1410308180715168e-02	4.35e+01	2.03e-01	3.63e-02	5.34e-05	3.24e-11



[*Reitich & Turc*, ‘06]

High-Order Nystrom Method: Diffraction Gratings (HF)

Window functions (POU) [Bruno & Monro '07]

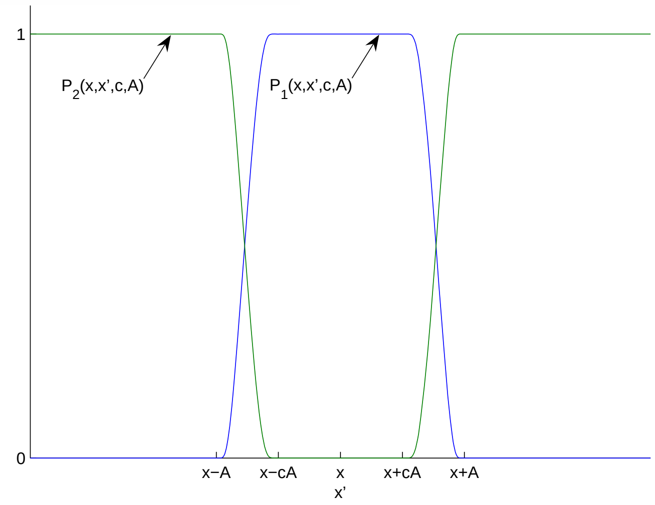
$$\frac{1}{2}\mu(\mathbf{x}) + \underbrace{\int_S \frac{\partial G_k(\mathbf{x},\mathbf{x}')}{\partial \mathbf{n}(\mathbf{x})}\mu(\mathbf{x}')\,d\sigma(\mathbf{x}')}_{} = ik\vec{\mathbf{d}}\cdot\mathbf{n}(\mathbf{x})e^{ik\vec{d}\cdot\mathbf{x}}, \;\; \mathbf{x}\in S$$

$$\approx \int_{x-A}^{x+A} P_1(x,x',c,A)\frac{i}{4}\frac{\partial H_0^1(k|\mathbf{x}-\mathbf{x}'|)}{\partial \mathbf{n}(\mathbf{x})}\mu(x')\sqrt{1+[f'(x')]^2}\,dx'$$

$$Ex: I = \int_0^\infty \frac{e^{ik_n x}}{\sqrt{x}}dx, \;\; k_n = k - kd_1 - \frac{2\pi n}{L} \neq 0$$

$$I_{per} = \int_0^A \frac{e^{ik_n x}}{\sqrt{x}}dx \qquad I_{pou} = \int_0^A P_1(0,x,c,A))\frac{e^{ik_n x}}{\sqrt{x}}dx$$

A	$\lvert I - I_{per}\rvert$	$\lvert I - I_{pou}(c=0.1)\rvert$
10	$5.0e-2$	$8.5e-5$
20	$3.6e-2$	$9.7e-7$
30	$2.9e-2$	$4.9e-8$
40	$2.5e-2$	$4.5e-9$
80	$1.8e-2$	$4.5e-11$
100	$1.6e-2$	$7.7e-14$



High-Order Nystrom Method: Diffraction Gratings (HF)

Window functions (POU)

$$\frac{1}{2}\mu(\mathbf{x}) + \int_S \frac{\partial G_k(\mathbf{x},\mathbf{x}')}{\partial \mathbf{n}(\mathbf{x})}\mu(\mathbf{x}')\,d\sigma(\mathbf{x}') = ik\vec{\mathbf{d}}\cdot\mathbf{n}(\mathbf{x})e^{ik\vec{d}\cdot\mathbf{x}},\ \mathbf{x}\in S$$

$Multiple\ scatt:\ \mu(\mathbf{x}) = e^{ikd_1 x}\mu_1(k,\mathbf{x})$ $\quad$ $Single\ scatt:\ \mu(\mathbf{x}) = e^{ik\vec{d}\cdot\mathbf{x}}\mu_2(k,\mathbf{x})$

$$\frac{1}{2}\mu_i(x) + \int_{-\infty}^{\infty} P_1(x,x',c,A)g(x,x')\frac{h(ku(x,x'))}{e^{iku(x,x')}}e^{ik\phi_i(x,x')}\mu_i(x')\,dx'$$

$$\cancel{+I_{i,2}(x,k,c,A)} = q_i(x),\ i=1,2$$

$$Th:\ |kL(1\pm d_1)+2\pi n| \geq \gamma > 0 \Rightarrow I_{i,2}(x,k,c,A) = \mathcal{O}\left(\left(\frac{A}{L}\right)^{-p}\right),\ \forall p\geq 1$$

High-order Nystrom method:

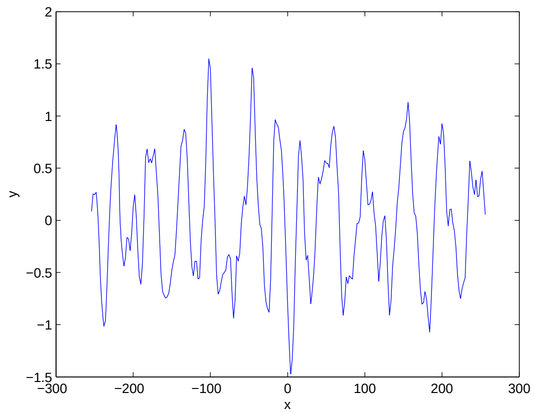
- *use additional floating Pou of size A_{sp} to treat the singularity of the kernel;*
- *use a coarse grid for the 'target' points and a refined grid for 'integration' points; resolve the oscillations in μ_1 via FFT interpolation*

→ *Single scattering: O(1) unknowns; Multiple scattering: highly efficient*

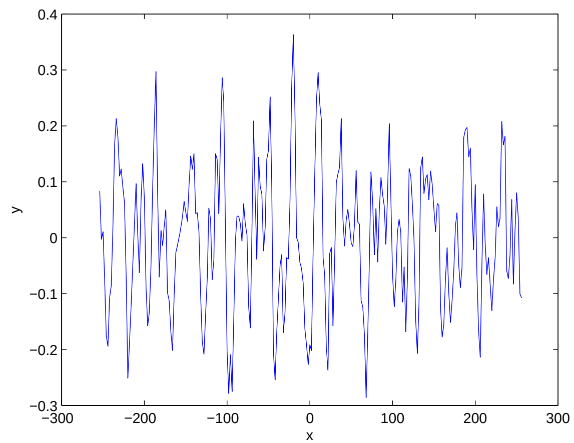
High-Order Nystrom Method: Diffraction Gratings (HF)

Ocean surface *(JPL code: directional wavespectrum for wind-driven ocean surfaces; randomly generated Fourier coefficients; used wavelengths similar to those of GPS satellites: period = 2048λ)*

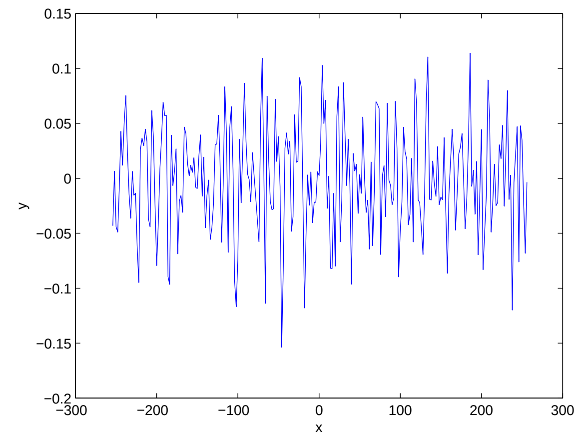
Ocean surface



First derivative



Second derivative



Incidence θ =85 °; Multiple scattering configuration; 3GHz processor

method	n_t	n_i	A_{sp}	A	energy balance	time (sec)	time w/o eff (sec)
Br. & Mon.	768	11520	0.875	2	$6.6 * 10^{-4}$	90	75
Br. & Mon.	1024	15360	0.875	150	$6.5 * 10^{-8}$	5556	—
KA	768	768	—	—	$1.6 * 10^{-1}$	15	0.06
KA	3840	3840	—	—	$1.6 * 10^{-1}$	16	0.07