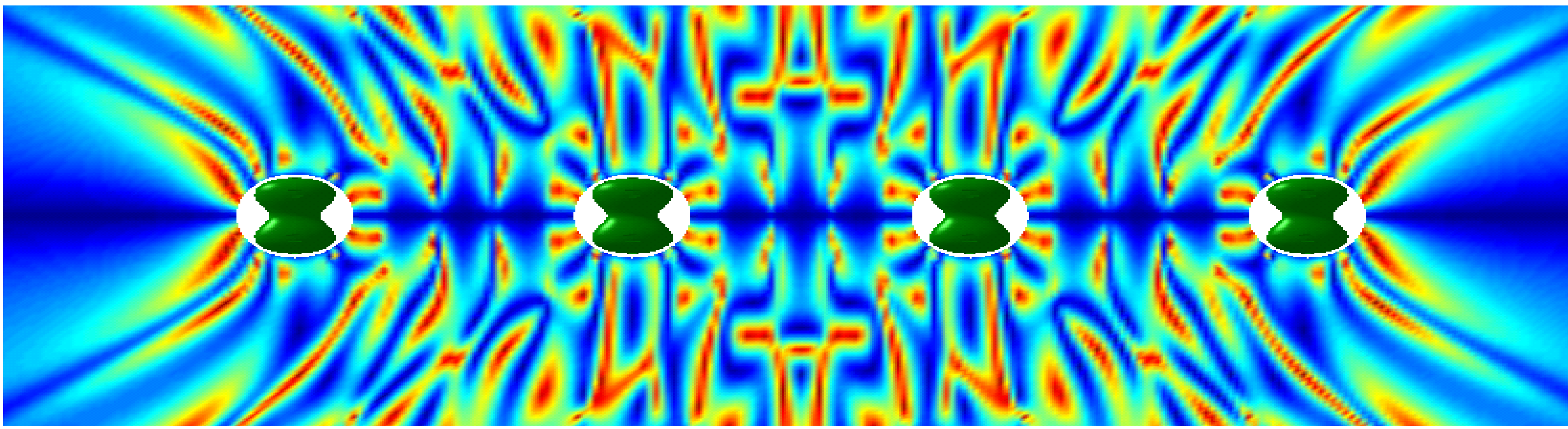
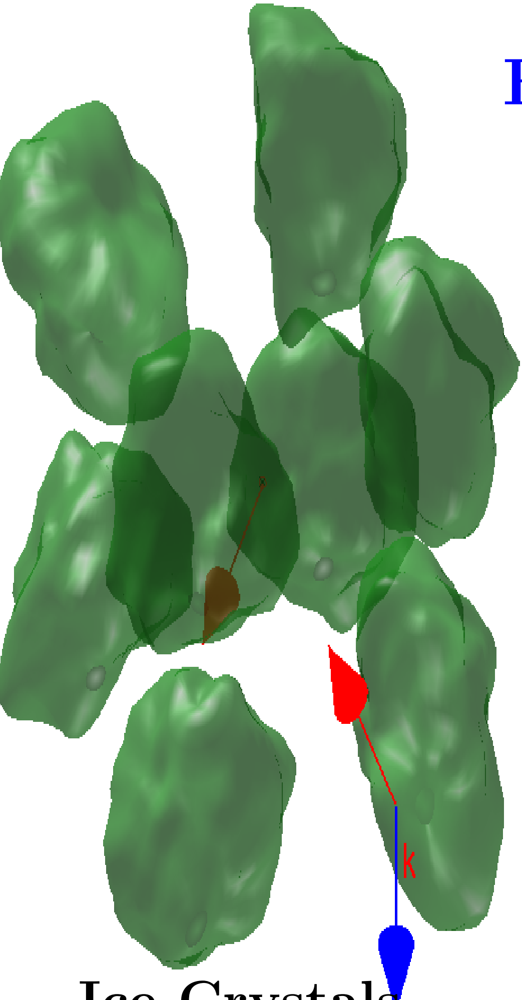
Spectrally accurate algorithms for direct and inverse electromagnetic scattering in three dimensions

M. Ganesh(Colorado School of Mines) & S. Hawkins(UNSW)

R. Kress & M. Pieper (Gottingen Universität)

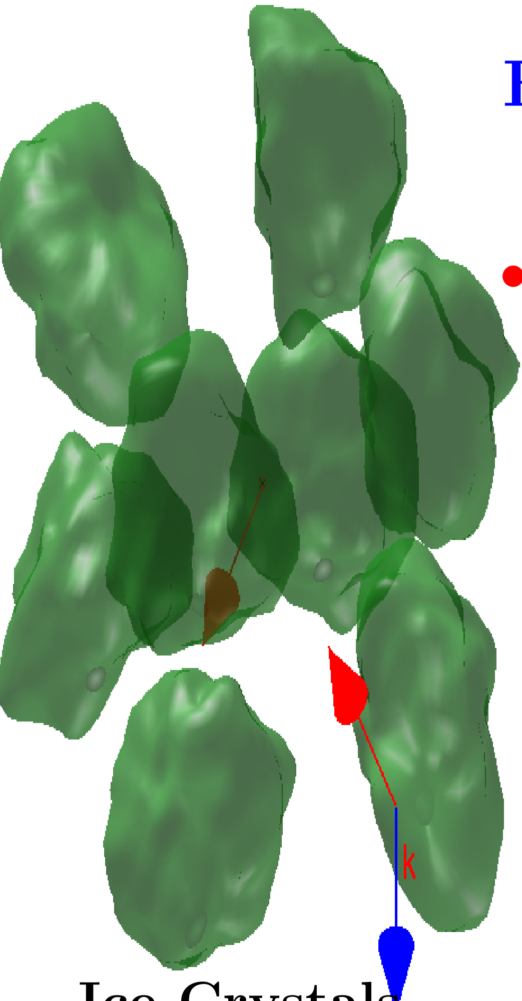


Electromagnetic Scattering in three dimensions



Ice Crystals Modeling Images from CPI

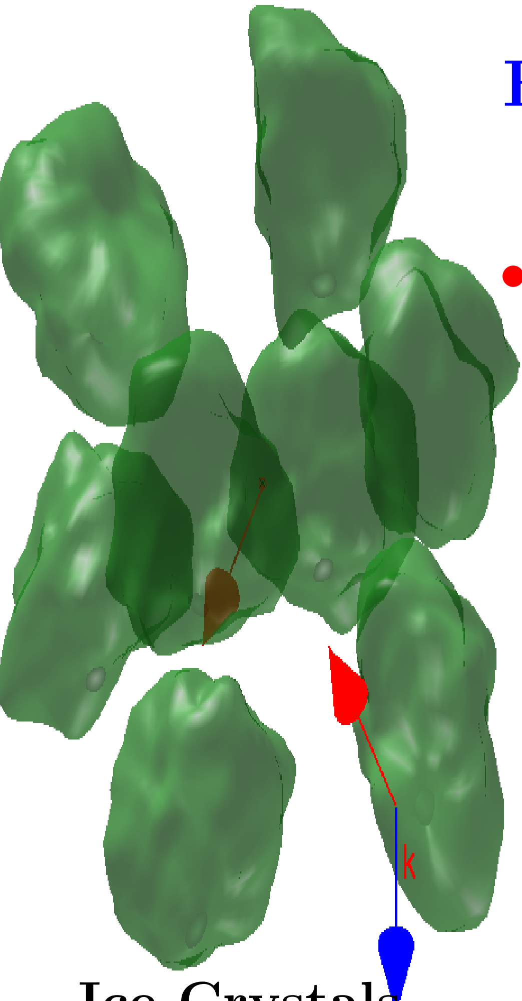
Electromagnetic Scattering in three dimensions



Ice Crystals
Modeling Images
from CPI

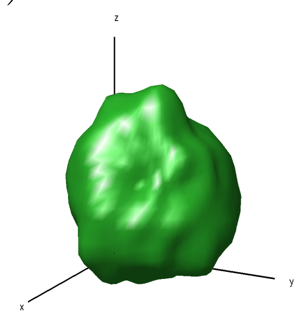
- **Simulate scattering of an incident electromagnetic wave**
  - ⋆ **with frequency ω and wavelength λ (with $\omega\lambda = c$, the speed of light)**
  - ⋆ **impinging on an ensemble $D = \cup_{j=1}^{J} D_i$**
  - ⋆ **of perfectly conducting three dimensional non-/convex stochastic/deterministic obstacles D_j**
  - ⋆ **with electromagnetic size $siz\_obs_j * \lambda \ \ (\geq \lambda)$.**

Electromagnetic Scattering in three dimensions



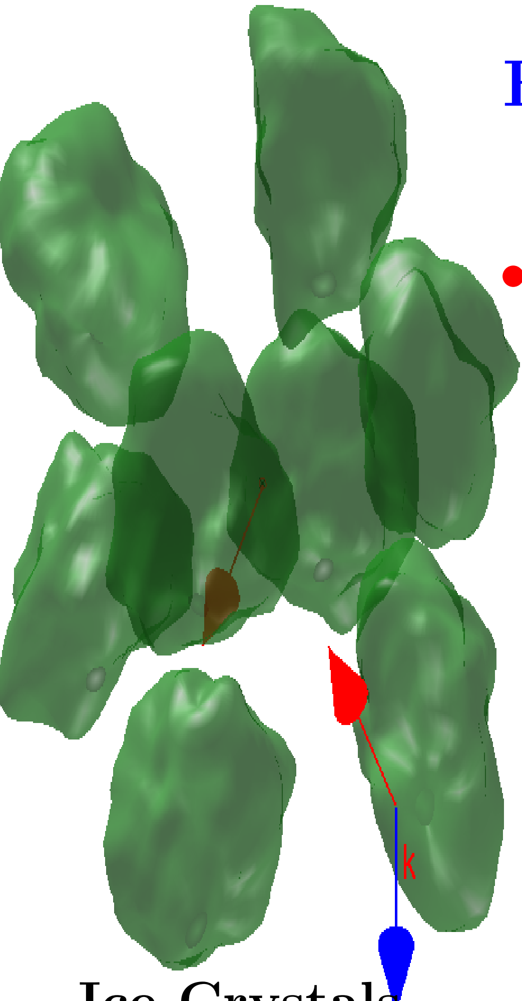
Ice Crystals Modeling Images from CPI

- **Simulate scattering of an incident electromagnetic wave**
  - ⋆ **with frequency ω and wavelength λ (with $\omega\lambda = c$, the speed of light)**
  - ⋆ **impinging on an ensemble $D = \cup_{j=1}^{J} D_i$**
  - ⋆ **of perfectly conducting three dimensional non-/convex stochastic/deterministic obstacles D_j**
  - ⋆ **with electromagnetic size $siz\_obs_j * \lambda \ \ (\geq \lambda)$.**



- **For each surface ∂D_j, $j = 1, \cdots, J$, introduce a local spherical coordinate system.**

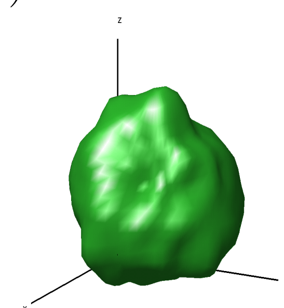
Electromagnetic Scattering in three dimensions



Ice Crystals
Modeling Images
from CPI

- Simulate scattering of an incident electromagnetic wave
  - ⋆ with frequency ω and wavelength λ (with $\omega\lambda = c$, the speed of light)
  - ⋆ impinging on an ensemble $D = \cup_{j=1}^{J} D_i$
  - ⋆ of perfectly conducting three dimensional non-/convex stochastic/deterministic obstacles D_j
  - ⋆ with electromagnetic size $siz\_obs_j * \lambda \quad (\geq \lambda)$.

- For each surface ∂D_j, $j = 1, \cdots, J$, introduce a local spherical coordinate system.



- Assume that each point on $(x, y, z)^T \in \partial D_j$ can be represented locally as

$$(x, y, z)^T = \left(q_1^j(\theta, \phi), q_2^j(\theta, \phi), q_3^j(\theta, \phi) \right)^T, \qquad \theta, \phi \in \mathbb{R},$$

for some nonlinear functionals $q_i^j : \mathbb{R}^2 \to \mathbb{R}$, for $i = 1, 2, 3$ (known analytically or approx. via Fourier coefficients).

Exterior Field, Far Field, Radar Cross Section, Polarization Effects

Exterior Field, Far Field, Radar Cross Section, Polarization Effects

- **Compute time harmonic electric and magnetic fields**

Exterior Field, Far Field, Radar Cross Section, Polarization Effects

- **Compute time harmonic electric and magnetic fields**
  - $\boldsymbol{\mathcal{E}}(\boldsymbol{x}, t) = \frac{1}{\sqrt{\epsilon_0}} \mathrm{Re}\left\{ \boldsymbol{E}(\boldsymbol{x}) e^{-i\omega t} \right\}, \qquad \boldsymbol{\mathcal{H}}(\boldsymbol{x}, t) = \frac{1}{\sqrt{\mu_0}} \mathrm{Re}\left\{ \boldsymbol{H}(\boldsymbol{x}) e^{-i\omega t} \right\}$
  - **free space permittivity ϵ_0 and permeability μ_0.**

Exterior Field, Far Field, Radar Cross Section, Polarization Effects

- **Compute time harmonic electric and magnetic fields**

  ⋆ $\boldsymbol{\mathcal{E}}(\boldsymbol{x}, t) = \dfrac{1}{\sqrt{\epsilon_0}} \mathrm{Re}\left\{ \boldsymbol{E}(\boldsymbol{x}) e^{-i\omega t} \right\}, \qquad \boldsymbol{\mathcal{H}}(\boldsymbol{x}, t) = \dfrac{1}{\sqrt{\mu_0}} \mathrm{Re}\left\{ \boldsymbol{H}(\boldsymbol{x}) e^{-i\omega t} \right\}$

  ⋆ **free space permittivity** ϵ_0 **and permeability** μ_0**.**

Exterior Field, Far Field, Radar Cross Section, Polarization Effects

- **Compute time harmonic electric and magnetic fields**

  ⋆ $\boldsymbol{\mathcal{E}}(\boldsymbol{x},t) = \dfrac{1}{\sqrt{\epsilon_0}}\mathrm{Re}\left\{\boldsymbol{E}(\boldsymbol{x})e^{-i\omega t}\right\}, \qquad \boldsymbol{\mathcal{H}}(\boldsymbol{x},t) = \dfrac{1}{\sqrt{\mu_0}}\mathrm{Re}\left\{\boldsymbol{H}(\boldsymbol{x})e^{-i\omega t}\right\}$

  ⋆ **free space permittivity ϵ_0 and permeability μ_0.**

- **Compute far field (required for solving the inverse problem)**

Exterior Field, Far Field, Radar Cross Section, Polarization Effects

- **Compute time harmonic electric and magnetic fields**

  ⋆ $\boldsymbol{\mathcal{E}}(\boldsymbol{x},t) = \dfrac{1}{\sqrt{\epsilon_0}}\mathrm{Re}\left\{\boldsymbol{E}(\boldsymbol{x})e^{-i\omega t}\right\}, \qquad \boldsymbol{\mathcal{H}}(\boldsymbol{x},t) = \dfrac{1}{\sqrt{\mu_0}}\mathrm{Re}\left\{\boldsymbol{H}(\boldsymbol{x})e^{-i\omega t}\right\}$

  ⋆ **free space permittivity ϵ_0 and permeability μ_0.**

- **Compute far field (required for solving the inverse problem)**

  ⋆ $\boldsymbol{E}_\infty(\widehat{\boldsymbol{x}}) = \lim_{r\to\infty} \boldsymbol{E}(r\widehat{\boldsymbol{x}})e^{-ikr}r \qquad \widehat{\boldsymbol{x}} \in \partial B$ **(unit sphere)**, $\qquad k = 2\pi/\lambda.$

Exterior Field, Far Field, Radar Cross Section, Polarization Effects

- **Compute time harmonic electric and magnetic fields**

  ⋆ $\boldsymbol{\mathcal{E}}(\boldsymbol{x},t) = \dfrac{1}{\sqrt{\epsilon_0}}\mathrm{Re}\left\{\boldsymbol{E}(\boldsymbol{x})e^{-i\omega t}\right\}, \qquad \boldsymbol{\mathcal{H}}(\boldsymbol{x},t) = \dfrac{1}{\sqrt{\mu_0}}\mathrm{Re}\left\{\boldsymbol{H}(\boldsymbol{x})e^{-i\omega t}\right\}$

  ⋆ **free space permittivity ϵ_0 and permeability μ_0.**

- **Compute far field (required for solving the inverse problem)**

  ⋆ $\boldsymbol{E}_\infty(\widehat{\boldsymbol{x}}) = \lim\limits_{r\to\infty} \boldsymbol{E}(r\widehat{\boldsymbol{x}})e^{-ikr}r \qquad \widehat{\boldsymbol{x}} \in \partial B$ **(unit sphere)**, $\qquad k = 2\pi/\lambda.$

- **For direct scattering, compute the radar cross section (RCS), describing visibility (stealth property) of the ensemble D,**

Exterior Field, Far Field, Radar Cross Section, Polarization Effects

- **Compute time harmonic electric and magnetic fields**
  - $\boldsymbol{\mathcal{E}}(\boldsymbol{x},t) = \frac{1}{\sqrt{\epsilon_0}}\mathrm{Re}\left\{\boldsymbol{E}(\boldsymbol{x})e^{-i\omega t}\right\}, \qquad \boldsymbol{\mathcal{H}}(\boldsymbol{x},t) = \frac{1}{\sqrt{\mu_0}}\mathrm{Re}\left\{\boldsymbol{H}(\boldsymbol{x})e^{-i\omega t}\right\}$
  - **free space permittivity** ϵ_0 **and permeability** μ_0**.**

- **Compute far field (required for solving the inverse problem)**
  - $\boldsymbol{E}_\infty(\widehat{\boldsymbol{x}}) = \lim_{r\to\infty} \boldsymbol{E}(r\widehat{\boldsymbol{x}})e^{-ikr}r \qquad \widehat{\boldsymbol{x}} \in \partial B$ **(unit sphere)**, $\qquad k = 2\pi/\lambda.$

- **For direct scattering, compute the radar cross section (RCS), describing visibility (stealth property) of the ensemble** D**,**
  - **measured by a receiver situated in the direction**

$$\widehat{\boldsymbol{x}} = \boldsymbol{p}(\theta,\phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)^T, \qquad \widehat{\boldsymbol{x}} \in \partial B,$$

  - **which is defined as**

$$\sigma(\widehat{\boldsymbol{x}}) = 4\pi\left|\boldsymbol{E}_\infty(\widehat{\boldsymbol{x}})\right|^2/k^2.$$

Maxwell's Equations, Radiation and Boundary Conditions

- **Total field (incident + scattered):** $\boldsymbol{E} = \boldsymbol{E}^{\mathrm{s}} + \boldsymbol{E}^{\mathrm{inc}}$, $\boldsymbol{H} = \boldsymbol{H}^{\mathrm{s}} + \boldsymbol{H}^{\mathrm{inc}}$.

Maxwell's Equations, Radiation and Boundary Conditions

- **Total field (incident + scattered):** $\boldsymbol{E} = \boldsymbol{E}^{\rm s} + \boldsymbol{E}^{\rm inc}$, $\boldsymbol{H} = \boldsymbol{H}^{\rm s} + \boldsymbol{H}^{\rm inc}$.

- **The fields satisfy the exterior harmonic Maxwell Equations**

  ⋆ $\text{curl}\, \boldsymbol{E}(\boldsymbol{x}) - ik\boldsymbol{H}(\boldsymbol{x}) = \boldsymbol{0}, \quad \text{curl}\, \boldsymbol{H}(\boldsymbol{x}) + ik\boldsymbol{E}(\boldsymbol{x}) = \boldsymbol{0}, \quad \boldsymbol{x} \in \boldsymbol{R}^3 \backslash \partial D.$

Maxwell's Equations, Radiation and Boundary Conditions

- **Total field (incident + scattered):** $\boldsymbol{E} = \boldsymbol{E}^{\mathrm{s}} + \boldsymbol{E}^{\mathrm{inc}}$, $\boldsymbol{H} = \boldsymbol{H}^{\mathrm{s}} + \boldsymbol{H}^{\mathrm{inc}}$.

- **The fields satisfy the exterior harmonic Maxwell Equations**

  ⋆ $\mathrm{curl}\, \boldsymbol{E}(\boldsymbol{x}) - ik\boldsymbol{H}(\boldsymbol{x}) = \boldsymbol{0}, \quad \mathrm{curl}\, \boldsymbol{H}(\boldsymbol{x}) + ik\boldsymbol{E}(\boldsymbol{x}) = \boldsymbol{0}, \quad \boldsymbol{x} \in \boldsymbol{R}^3 \backslash \partial D.$

- **The scattered fields satisfy the Silver-Müller radiation condition**

  ⋆ $\lim_{|\boldsymbol{x}| \to \infty} [\boldsymbol{H}^{\mathrm{s}}(\boldsymbol{x}) \times \boldsymbol{x} - |\boldsymbol{x}| \boldsymbol{E}^{\mathrm{s}}(\boldsymbol{x})] = \boldsymbol{0}.$

Maxwell's Equations, Radiation and Boundary Conditions

- **Total field (incident + scattered):** $\boldsymbol{E} = \boldsymbol{E}^{\mathrm{s}} + \boldsymbol{E}^{\mathrm{inc}}$, $\boldsymbol{H} = \boldsymbol{H}^{\mathrm{s}} + \boldsymbol{H}^{\mathrm{inc}}$.

- **The fields satisfy the exterior harmonic Maxwell Equations**

  ⋆ $\mathrm{curl}\,\boldsymbol{E}(\boldsymbol{x}) - ik\boldsymbol{H}(\boldsymbol{x}) = \boldsymbol{0}, \quad \mathrm{curl}\,\boldsymbol{H}(\boldsymbol{x}) + ik\boldsymbol{E}(\boldsymbol{x}) = \boldsymbol{0}, \quad \boldsymbol{x} \in \boldsymbol{R}^3 \backslash \partial D.$

- **The scattered fields satisfy the Silver-Müller radiation condition**

  ⋆ $\lim_{|\boldsymbol{x}| \to \infty} [\boldsymbol{H}^{\mathrm{s}}(\boldsymbol{x}) \times \boldsymbol{x} - |\boldsymbol{x}|\boldsymbol{E}^{\mathrm{s}}(\boldsymbol{x})] = \boldsymbol{0}.$

Maxwell's Equations, Radiation and Boundary Conditions

- **Total field (incident + scattered):** $\boldsymbol{E} = \boldsymbol{E}^{\mathrm{s}} + \boldsymbol{E}^{\mathrm{inc}}$, $\boldsymbol{H} = \boldsymbol{H}^{\mathrm{s}} + \boldsymbol{H}^{\mathrm{inc}}$.

- **The fields satisfy the exterior harmonic Maxwell Equations**

  ⋆ $\mathrm{curl}\, \boldsymbol{E}(\boldsymbol{x}) - ik\boldsymbol{H}(\boldsymbol{x}) = \boldsymbol{0}, \quad \mathrm{curl}\, \boldsymbol{H}(\boldsymbol{x}) + ik\boldsymbol{E}(\boldsymbol{x}) = \boldsymbol{0}, \quad \boldsymbol{x} \in \boldsymbol{R}^3 \backslash \partial D.$

- **The scattered fields satisfy the Silver-Müller radiation condition**

  ⋆ $\lim_{|\boldsymbol{x}| \to \infty} [\boldsymbol{H}^{\mathrm{s}}(\boldsymbol{x}) \times \boldsymbol{x} - |\boldsymbol{x}| \boldsymbol{E}^{\mathrm{s}}(\boldsymbol{x})] = \boldsymbol{0}.$

- **$\boldsymbol{E}^s$ satisfies the perfect conductor boundary condition**

  ⋆ $\boldsymbol{n}(\boldsymbol{x}) \times \boldsymbol{E}^s(\boldsymbol{x}) = -\boldsymbol{n}(\boldsymbol{x}) \times \boldsymbol{E}^{\mathrm{inc}}(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial D.$

  ⋆ **with known incident (plane) electromagnetic wave $[\boldsymbol{E}^{\mathrm{i}}, \boldsymbol{H}^{\mathrm{i}}]$.**

Maxwell's Equations, Radiation and Boundary Conditions

- **Total field (incident + scattered):** $\boldsymbol{E} = \boldsymbol{E}^{\mathrm{s}} + \boldsymbol{E}^{\mathrm{inc}}$, $\boldsymbol{H} = \boldsymbol{H}^{\mathrm{s}} + \boldsymbol{H}^{\mathrm{inc}}$.

- **The fields satisfy the exterior harmonic Maxwell Equations**

  ⋆ $\operatorname{curl} \boldsymbol{E}(\boldsymbol{x}) - ik\boldsymbol{H}(\boldsymbol{x}) = \boldsymbol{0}, \quad \operatorname{curl} \boldsymbol{H}(\boldsymbol{x}) + ik\boldsymbol{E}(\boldsymbol{x}) = \boldsymbol{0}, \quad \boldsymbol{x} \in \boldsymbol{R}^3 \setminus \partial D.$

- **The scattered fields satisfy the Silver-Müller radiation condition**

  ⋆ $\lim_{|\boldsymbol{x}| \to \infty} \left[\boldsymbol{H}^{\mathrm{s}}(\boldsymbol{x}) \times \boldsymbol{x} - |\boldsymbol{x}| \boldsymbol{E}^{\mathrm{s}}(\boldsymbol{x}) \right] = \boldsymbol{0}.$

- $\boldsymbol{E}^s$ **satisfies the perfect conductor boundary condition**

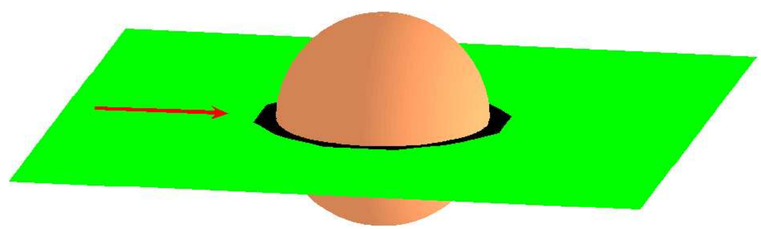
  ⋆ $\boldsymbol{n}(\boldsymbol{x}) \times \boldsymbol{E}^s(\boldsymbol{x}) = -\boldsymbol{n}(\boldsymbol{x}) \times \boldsymbol{E}^{\mathrm{inc}}(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial D.$

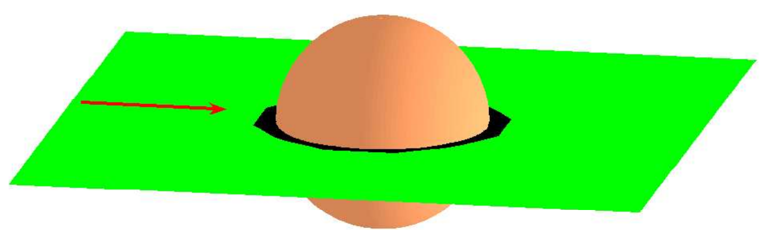
  ⋆ **with known incident (plane) electromagnetic wave** $[\boldsymbol{E}^{\mathrm{i}}, \boldsymbol{H}^{\mathrm{i}}]$.

Monostatic and Bistatic Radar Cross Section

- **The boundary condition is induced by $\left[\boldsymbol{E}^{\mathrm{inc}}, \boldsymbol{H}^{\mathrm{inc}}\right]$, with**
  - ⋆ **direction $\widehat{\boldsymbol{d}}_0$ and polarization $\widehat{\boldsymbol{p}}_0$ (perpendicular to $\widehat{\boldsymbol{d}}_0$),**
  - ⋆ **which originates from the transmitter and is defined as**

$$\boldsymbol{E}^{\mathrm{inc}}(\widehat{\boldsymbol{x}}) = ik\widehat{\boldsymbol{p}}_0 e^{ik\boldsymbol{x}\cdot\widehat{\boldsymbol{d}}_0}, \qquad \boldsymbol{H}^{\mathrm{inc}}(\widehat{\boldsymbol{x}}) = ik(\widehat{\boldsymbol{d}}_0 \times \widehat{\boldsymbol{p}}_0) e^{ik\boldsymbol{x}\cdot\widehat{\boldsymbol{d}}_0}.$$



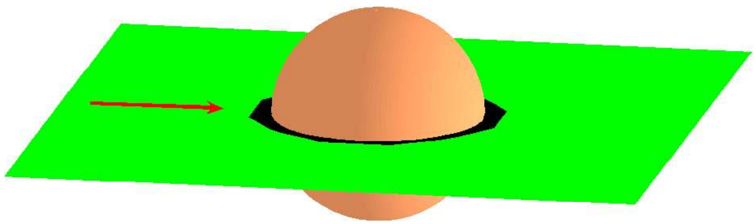


Monostatic and Bistatic Radar Cross Section

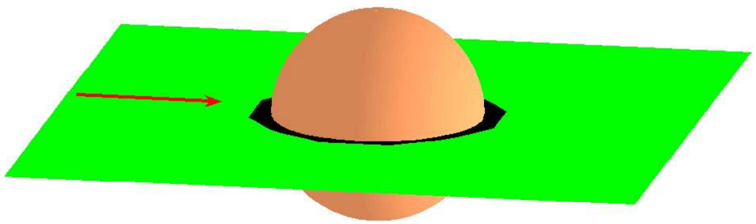
- The boundary condition is induced by $\left[\boldsymbol{E}^{\text{inc}}, \boldsymbol{H}^{\text{inc}}\right]$, with
  - ⋆ direction $\widehat{\boldsymbol{d}}_0$ and polarization $\widehat{\boldsymbol{p}}_0$ (perpendicular to $\widehat{\boldsymbol{d}}_0$),
  - ⋆ which originates from the transmitter and is defined as

$$\boldsymbol{E}^{\text{inc}}(\widehat{\boldsymbol{x}}) = ik\widehat{\boldsymbol{p}}_0 e^{ik\boldsymbol{x}\cdot\widehat{\boldsymbol{d}}_0}, \qquad \boldsymbol{H}^{\text{inc}}(\widehat{\boldsymbol{x}}) = ik(\widehat{\boldsymbol{d}}_0 \times \widehat{\boldsymbol{p}}_0)e^{ik\boldsymbol{x}\cdot\widehat{\boldsymbol{d}}_0}.$$

- Bistatic RCS
  - ⋆ source is fixed
  - ⋆ (one boundary condition)



- Monostatic RCS
  - ⋆ source and receiver Co-located
  - ⋆ (thousands of boundary conditions)

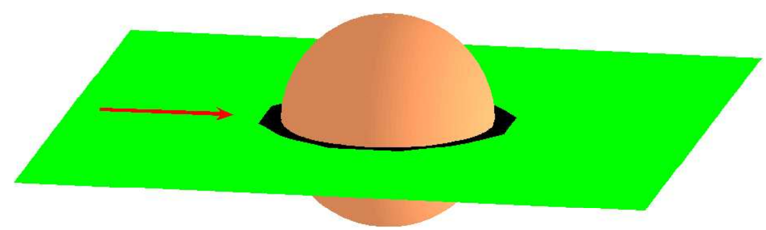


Monostatic and Bistatic Radar Cross Section

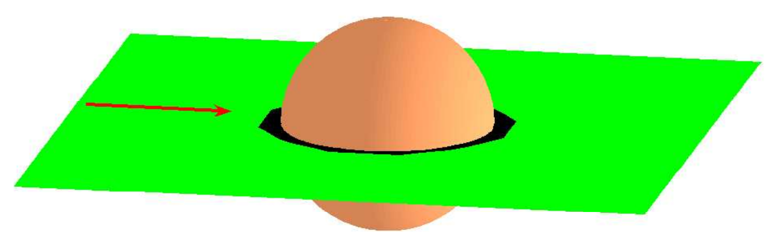
- The boundary condition is induced by $[\boldsymbol{E}^{\mathrm{inc}}, \boldsymbol{H}^{\mathrm{inc}}]$, with
  - ⋆ direction $\widehat{\boldsymbol{d}}_0$ and polarization $\widehat{\boldsymbol{p}}_0$ (perpendicular to $\widehat{\boldsymbol{d}}_0$),
  - ⋆ which originates from the transmitter and is defined as

$$\boldsymbol{E}^{\mathrm{inc}}(\widehat{\boldsymbol{x}}) = ik\widehat{\boldsymbol{p}}_0 e^{ik\boldsymbol{x}\cdot\widehat{\boldsymbol{d}}_0}, \qquad \boldsymbol{H}^{\mathrm{inc}}(\widehat{\boldsymbol{x}}) = ik(\widehat{\boldsymbol{d}}_0 \times \widehat{\boldsymbol{p}}_0) e^{ik\boldsymbol{x}\cdot\widehat{\boldsymbol{d}}_0}.$$

- Bistatic RCS
  - ⋆ source is fixed
  - ⋆ (one boundary condition)



- Monostatic RCS
  - ⋆ source and receiver Co-located
  - ⋆ (thousands of boundary conditions)



Surface Integral Methods

- **Scattered field given by integral over scatterer surface with unknown density (surface current).**

Surface Integral Methods

- **Scattered field given by integral over scatterer surface with unknown density (surface current).**
- **Advantages**
  - ⋆ **solution automatically solves radiation condition**
  - ⋆ **sufficient to compute the density on the surface**
  - ⋆ **truncation of exterior domain is not required**
  - ⋆ **Far field (and hence RCS) are represented by surface integrals.**

Surface Integral Methods

- **Scattered field given by integral over scatterer surface with unknown density (surface current).**

- **Advantages**
  - ⋆ **solution automatically solves radiation condition**
  - ⋆ **sufficient to compute the density on the surface**
  - ⋆ **truncation of exterior domain is not required**
  - ⋆ **Far field (and hence RCS) are represented by surface integrals.**

- **Disadvantage (due to global integral operators as opposed to local differential operators based FEM/FD methods)**
  - ⋆ **lead to linear system with large dense complex matrix**

Surface Integral Algorithms

- **Boundary Element (Method of Moments) solvers**
  - ⋆ **need certain number of points per wavelength**
  - ⋆ **rule of thumb: 10 points per wavelength per dimension**
  - ⋆ **large number of unknowns**
  - ⋆ **industrial standard solver: FISC (Fast Illinois Solver Code)**

Surface Integral Algorithms

- **Boundary Element (Method of Moments) solvers**
  - ⋆ **need certain number of points per wavelength**
  - ⋆ **rule of thumb: 10 points per wavelength per dimension**
  - ⋆ **large number of unknowns**
  - ⋆ **industrial standard solver: FISC (Fast Illinois Solver Code)**
- **Scattering by a sphere of diameter** $48 \times$ **wavelength**
  - ⋆ **FISC — 2, 408, 448 unknowns (to achieve one digit accuracy)**

Surface Integral Algorithms

- **Boundary Element (Method of Moments) solvers**
  - ⋆ **need certain number of points per wavelength**
  - ⋆ **rule of thumb: 10 points per wavelength per dimension**
  - ⋆ **large number of unknowns**
  - ⋆ **industrial standard solver: FISC (Fast Illinois Solver Code)**
- **Scattering by a sphere of diameter** $48 \times$ **wavelength**
  - ⋆ **FISC — 2, 408, 448 unknowns (to achieve one digit accuracy)**
- **How to deal with so many unknowns?**
  - ⋆ **Fast Multipole Method (FMM) and iterative solvers**

Surface Integral Algorithms

- **Boundary Element (Method of Moments) solvers**
  - ⋆ **need certain number of points per wavelength**
  - ⋆ **rule of thumb: 10 points per wavelength per dimension**
  - ⋆ **large number of unknowns**
  - ⋆ **industrial standard solver: FISC (Fast Illinois Solver Code)**
- **Scattering by a sphere of diameter** $48 \times$ **wavelength**
  - ⋆ **FISC — 2, 408, 448 unknowns (to achieve one digit accuracy)**
- **How to deal with so many unknowns?**
  - ⋆ **Fast Multipole Method (FMM) and iterative solvers**
- **Substantial improvement: high-order methods (Bruno et al.)**
  - ⋆ **rule of thumb: 6 points per wavelength per dimension**

Surface Integral Algorithms

- **Boundary Element (Method of Moments) solvers**
  - ⋆ **need certain number of points per wavelength**
  - ⋆ **rule of thumb: 10 points per wavelength per dimension**
  - ⋆ **large number of unknowns**
  - ⋆ **industrial standard solver: FISC (Fast Illinois Solver Code)**
- **Scattering by a sphere of diameter $48 \times$ wavelength**
  - ⋆ **FISC — 2, 408, 448 unknowns (to achieve one digit accuracy)**
- **How to deal with so many unknowns?**
  - ⋆ **Fast Multipole Method (FMM) and iterative solvers**
- **Substantial improvement: high-order methods (Bruno et al.)**
  - ⋆ **rule of thumb: 6 points per wavelength per dimension**
- ***Acoustic* scattering by a bean of diameter $30 \times$ wavelength**
  - ⋆ **Bruno & Kunyanski — 617, 910 unknowns (four digit accuracy)**

Surface Integral Algorithms

- **Boundary Element (Method of Moments) solvers**
  - ⋆ **need certain number of points per wavelength**
  - ⋆ **rule of thumb: 10 points per wavelength per dimension**
  - ⋆ **large number of unknowns**
  - ⋆ **industrial standard solver: FISC (Fast Illinois Solver Code)**
- **Scattering by a sphere of diameter $48 \times$ wavelength**
  - ⋆ **FISC — 2, 408, 448 unknowns (to achieve one digit accuracy)**
- **How to deal with so many unknowns?**
  - ⋆ **Fast Multipole Method (FMM) and iterative solvers**
- **Substantial improvement: high-order methods (Bruno et al.)**
  - ⋆ **rule of thumb: 6 points per wavelength per dimension**
- ***Acoustic* scattering by a bean of diameter $30 \times$ wavelength**
  - ⋆ **Bruno & Kunyanski — 617, 910 unknowns (four digit accuracy)**
- **How to deal with so many unknowns?**
  - ⋆ **FFT and iterative solvers (only 18 iterates; 4.5h per iterate).**

Alternative Approach : fewer unknowns (2 to 5%)

- **Ansatz using spectral basis functions that are tangential on a given surface**

Alternative Approach : fewer unknowns (2 to 5%)

- **Ansatz using spectral basis functions that are tangential on a given surface**

- **Advantages**
  - ⋆ **spectrally accurate high order accuracy**
  - ⋆ **substantially fewer unknowns than industrial standard algorithms**
  - ⋆ **scattering matrix can be set up, leading to parallel LU solver on a cluster with only about 2GB RAM per core (- recall monostatic RCS)**
  - ⋆ **very fast (with FFT) and rapid convergence**

Alternative Approach : fewer unknowns (2 to 5%)

- **Ansatz using spectral basis functions that are tangential on a given surface**

- **Advantages**
  - ⋆ **spectrally accurate high order accuracy**
  - ⋆ **substantially fewer unknowns than industrial standard algorithms**
  - ⋆ **scattering matrix can be set up, leading to parallel LU solver on a cluster with only about 2GB RAM per core (- recall monostatic RCS)**
  - ⋆ **very fast (with FFT) and rapid convergence**

Alternative Approach : fewer unknowns (2 to 5%)

- **Ansatz using spectral basis functions that are tangential on a given surface**

- **Advantages**
  - ⋆ **spectrally accurate high order accuracy**
  - ⋆ **substantially fewer unknowns than industrial standard algorithms**
  - ⋆ **scattering matrix can be set up, leading to parallel LU solver on a cluster with only about 2GB RAM per core (- recall monostatic RCS)**
  - ⋆ **very fast (with FFT) and rapid convergence**

- **Sphere of diameter $48 \times$ wavelength**
  - ⋆ **FISC (Chew et al.) — 2, 408, 448 unknowns (one digit accuracy)**
  - ⋆ **Ganesh & Hawkins — 48, 670 unknowns (one digit accuracy)**
  - ⋆ **Ganesh & Hawkins — 55, 110 unknowns (five digit accuracy)**

Alternative Approach : fewer unknowns (2 to 5%)

- **Ansatz using spectral basis functions that are tangential on a given surface**

- **Advantages**
  - ⋆ **spectrally accurate high order accuracy**
  - ⋆ **substantially fewer unknowns than industrial standard algorithms**
  - ⋆ **scattering matrix can be set up, leading to parallel LU solver on a cluster with only about 2GB RAM per core (- recall monostatic RCS)**
  - ⋆ **very fast (with FFT) and rapid convergence**

- **Sphere of diameter $48 \times$ wavelength**
  - ⋆ **FISC (Chew et al.) — 2, 408, 448 unknowns (one digit accuracy)**
  - ⋆ **Ganesh & Hawkins — 48, 670 unknowns (one digit accuracy)**
  - ⋆ **Ganesh & Hawkins — 55, 110 unknowns (five digit accuracy)**

- **Bean of diameter $30 \times$ wavelength**
  - ⋆ **Bruno & Kunyanski — 619,910 unknowns (acoustic, four digit accuracy**
  - ⋆ **Ganesh & Hawkins — 48, 670 unknowns (four digit accuracy)**
  - ⋆ **Ganesh & Hawkins — 51, 840 unknowns (five digit accuracy)**
  - ⋆ **Time to solve with 2402 incident waves: 10.5 hours using $16\times$ DcOp!**

Boundary Decomposition: disjoint scatterers to each connected scatterer

- **Boundary decomposition technique reduces complexity to**
  - ⋆ **computing scattered field from each connected scatterer** $\partial D_j, \quad j = 1, \cdots, J$
  - ⋆ **as if this is the only scatterer in the configuration with**
  - ⋆ **new boundary conditions involving**
  - ⋆ **original incident field and contribution from other** $J - 1$ **scatterers**

Boundary Decomposition: disjoint scatterers to each connected scatterer

- **Boundary decomposition technique reduces complexity to**
  - ⋆ **computing scattered field from each connected scatterer** $\partial D_j, \quad j = 1, \cdots, J$
  - ⋆ **as if this is the only scatterer in the configuration with**
  - ⋆ **new boundary conditions involving**
  - ⋆ **original incident field and contribution from other** $J-1$ **scatterers**

- **Represent exterior electric and magnetic fields as**

$$\boldsymbol{E}(\boldsymbol{x}) = \sum_{j=1}^{J} \boldsymbol{E}^j(\boldsymbol{x}), \quad \boldsymbol{H}(\boldsymbol{x}) = \sum_{j=1}^{J} \boldsymbol{H}^j(\boldsymbol{x}), \qquad \boldsymbol{E}^j(\boldsymbol{x}) = \textbf{curl} \int_{\partial D_j} \Phi(\boldsymbol{x}, \boldsymbol{y}) \boldsymbol{w}(\boldsymbol{y}) \, ds(\boldsymbol{y}$$

Boundary Decomposition: disjoint scatterers to each connected scatterer

- **Boundary decomposition technique reduces complexity to**
  - ⋆ **computing scattered field from each connected scatterer** $\partial D_j, \quad j = 1, \cdots, J$
  - ⋆ **as if this is the only scatterer in the configuration with**
  - ⋆ **new boundary conditions involving**
  - ⋆ **original incident field and contribution from other** $J-1$ **scatterers**

- **Represent exterior electric and magnetic fields as**

$$\boldsymbol{E}(\boldsymbol{x}) = \sum_{j=1}^{J} \boldsymbol{E}^j(\boldsymbol{x}), \quad \boldsymbol{H}(\boldsymbol{x}) = \sum_{j=1}^{J} \boldsymbol{H}^j(\boldsymbol{x}), \qquad \boldsymbol{E}^j(\boldsymbol{x}) = \textbf{curl} \int_{\partial D_j} \Phi(\boldsymbol{x}, \boldsymbol{y}) \boldsymbol{w}(\boldsymbol{y}) \, ds(\boldsymbol{y}$$

- **The unknown density** w **on** ∂D_i **(and** H^i**) can be computed by the fact tha**
  - ⋆ **for each** $i = 1, \cdots J$**,** $[\boldsymbol{E}^i, \boldsymbol{H}^i]$ **represents the unique radiating solution of the time harmonic Maxwell equations exterior to** $\overline{D}_i$**,**
  - ⋆ **subject to the boundary condition**

$$\boldsymbol{n}(\boldsymbol{x}) \times \boldsymbol{E}^i(\boldsymbol{x}) = \boldsymbol{n}(\boldsymbol{x}) \times \boldsymbol{E}^{\text{inc}}(\boldsymbol{x}) - \sum_{\substack{j=1 \\ j \neq i}}^{J} \boldsymbol{n} \times \boldsymbol{E}^j(\boldsymbol{x}) =: \boldsymbol{f}(\boldsymbol{x}) \qquad \boldsymbol{x} \in \partial D_i.$$

Surface Integral Representations: $\partial D^i \to \partial D$, $\boldsymbol{E}^i \to E$

- **The surface current $\boldsymbol{w}$ is tangential on ∂D**

- **Surface Integral Equation**
  - ⋆ $\boldsymbol{w}(\boldsymbol{x}) + (\mathcal{M}\boldsymbol{w})(\boldsymbol{x}) = 2\boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial D$

- **Magnetic dipole operator**
  - ⋆ $(\mathcal{M}\boldsymbol{a})(\boldsymbol{x}) = 2\int_{\partial D} \boldsymbol{n}(\boldsymbol{x}) \times \mathrm{curl}_{\boldsymbol{x}} \{\Phi(\boldsymbol{x},\boldsymbol{y})\boldsymbol{a}(\boldsymbol{y})\} \; ds(\boldsymbol{y}), \quad \boldsymbol{x} \in \partial D$
  - ⋆ **(Other representations & formulations ...: Kress & Piper (2007) use $\mathcal{M}'$**

Surface Integral Representations: $\partial D^i \to \partial D$, $\boldsymbol{E}^i \to E$

- **The surface current $\boldsymbol{w}$ is tangential on ∂D**

- **Surface Integral Equation**
  - $\boldsymbol{w}(\boldsymbol{x}) + (\mathcal{M}\boldsymbol{w})(\boldsymbol{x}) = 2\boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial D$

- **Magnetic dipole operator**
  - $$(\mathcal{M}\boldsymbol{a})(\boldsymbol{x}) = 2\int_{\partial D} \boldsymbol{n}(\boldsymbol{x}) \times \mathrm{curl}_{\boldsymbol{x}} \{\Phi(\boldsymbol{x},\boldsymbol{y})\boldsymbol{a}(\boldsymbol{y})\}\; ds(\boldsymbol{y}), \quad \boldsymbol{x} \in \partial D$$
  - **(Other representations & formulations ...: Kress & Piper (2007) use $\mathcal{M}'$**

- **Use bijective parametrization in spherical coordinates: $\boldsymbol{q} : \partial B \to \partial D$**

- **Transplanted Surface Integral Equation**
  - $\boldsymbol{W}(\widehat{\boldsymbol{x}}) + (\mathcal{M}\boldsymbol{W})(\widehat{\boldsymbol{x}}) = 2\boldsymbol{F}(\widehat{\boldsymbol{x}}), \quad \widehat{\boldsymbol{x}} \in \partial B \qquad \boldsymbol{W} = \boldsymbol{w} \circ \boldsymbol{q}$

- **Definition: $\boldsymbol{W}$ is $\boldsymbol{q}$-tangential if $\boldsymbol{w}$ is tangential on ∂D**

Galerkin Scheme

- **Find $\boldsymbol{W}_n$ in W_n (ansatz space) such that**
  - ⋆ $\langle \boldsymbol{W}_n + \mathcal{M}\boldsymbol{W}_n, \boldsymbol{V}\rangle = \langle 2\boldsymbol{F}, \boldsymbol{V}\rangle, \quad \forall\, \boldsymbol{V} \in W_n$

Galerkin Scheme

- **Find $\boldsymbol{W}_n$ in W_n (ansatz space) such that**
  - ⋆ $\langle \boldsymbol{W}_n + \mathcal{M}\boldsymbol{W}_n, \boldsymbol{V} \rangle = \langle 2\boldsymbol{F}, \boldsymbol{V} \rangle, \quad \forall\, \boldsymbol{V} \in W_n$

Galerkin Scheme

- **Find $\boldsymbol{W}_n$ in W_n (ansatz space) such that**
  - $\star$ $\langle \boldsymbol{W}_n + \mathcal{M}\boldsymbol{W}_n, \boldsymbol{V} \rangle = \langle 2\boldsymbol{F}, \boldsymbol{V} \rangle, \quad \forall\, \boldsymbol{V} \in W_n$

- **Four Main Ingredients**
  - $\star$ **Spectrally accurate evaluation of $\mathcal{M}\boldsymbol{V}$ for $\boldsymbol{V} \in W_n$**
  - $\star$ **Spectrally accurate approximation of $\boldsymbol{W}$ in W_n**
  - $\star$ **Fully discrete Galerkin scheme**
  - $\star$ **Fast assembly of the Galerkin matrix**

Galerkin Scheme

- **Find $\boldsymbol{W}_n$ in W_n (ansatz space) such that**
  - ⋆ $\langle \boldsymbol{W}_n + \mathcal{M}\boldsymbol{W}_n, \boldsymbol{V} \rangle = \langle 2\boldsymbol{F}, \boldsymbol{V} \rangle, \quad \forall\, \boldsymbol{V} \in W_n$

- **Four Main Ingredients**
  - ⋆ **Spectrally accurate evaluation of $\mathcal{M}\boldsymbol{V}$ for $\boldsymbol{V} \in W_n$**
  - ⋆ **Spectrally accurate approximation of $\boldsymbol{W}$ in W_n**
  - ⋆ **Fully discrete Galerkin scheme**
  - ⋆ **Fast assembly of the Galerkin matrix**

Spectral boundary integral algorithms in 3-d: potential and acoustic

Spectral boundary integral algorithms in 3-d: potential and acoustic

- **Boundary integral spectral algorithm for 3-D potential problems was investigated by Atkinson (1984), Ganesh, Graham, ...**

Spectral boundary integral algorithms in 3-d: potential and acoustic

- **Boundary integral spectral algorithm for 3-D potential problems was investigated by Atkinson (1984), Ganesh, Graham, ...**

- **Boundary integral spectral algorithm for 3-D acoustic scattering was studied by Ganesh, Graham, Kress, Sloan, Wienert ,.......**

Spectral boundary integral algorithms in 3-d: potential and acoustic

- **Boundary integral spectral algorithm for 3-D potential problems was investigated by Atkinson (1984), Ganesh, Graham, ...**

- **Boundary integral spectral algorithm for 3-D acoustic scattering was studied by Ganesh, Graham, Kress, Sloan, Wienert ,.......**

- **Fully discrete Galerkin spectral algorithm analysis and implementation for acoustic scattering was completed in 2004.**

Spectral boundary integral algorithms in 3-d: potential and acoustic

- **Boundary integral spectral algorithm for 3-D potential problems was investigated by Atkinson (1984), Ganesh, Graham, ...**

- **Boundary integral spectral algorithm for 3-D acoustic scattering was studied by Ganesh, Graham, Kress, Sloan, Wienert ,.......**

- **Fully discrete Galerkin spectral algorithm analysis and implementation for acoustic scattering was completed in 2004.**

- **Ansatz space and essential tool for all approximations in spectrally accurate acoustic algorithm:**
  Space spanned by scalar spherical harmonics of degree up to n

$$Y_{l,j}, \quad 0 \le l \le n, \quad |j| \le l$$

Spectral boundary integral algorithms in 3-d: potential and acoustic

- **Boundary integral spectral algorithm for 3-D potential problems was investigated by Atkinson (1984), Ganesh, Graham, ...**

- **Boundary integral spectral algorithm for 3-D acoustic scattering was studied by Ganesh, Graham, Kress, Sloan, Wienert ,.......**

- **Fully discrete Galerkin spectral algorithm analysis and implementation for acoustic scattering was completed in 2004.**

- **Ansatz space and essential tool for all approximations in spectrally accurate acoustic algorithm:**
  Space spanned by scalar spherical harmonics of degree up to n

$$Y_{l,j}, \quad 0 \leq l \leq n, \quad |j| \leq l$$

- **Dimension of ansatz space (and hence # of unknowns)** $= (n+1)^2$

Spectral boundary integral algorithms in 3-d: potential and acoustic

- **Boundary integral spectral algorithm for 3-D potential problems was investigated by Atkinson (1984), Ganesh, Graham, ...**

- **Boundary integral spectral algorithm for 3-D acoustic scattering was studied by Ganesh, Graham, Kress, Sloan, Wienert ,.......**

- **Fully discrete Galerkin spectral algorithm analysis and implementation for acoustic scattering was completed in 2004.**

- **Ansatz space and essential tool for all approximations in spectrally accurate acoustic algorithm:**
  Space spanned by scalar spherical harmonics of degree up to n

$$Y_{l,j}, \quad 0 \leq l \leq n, \quad |j| \leq l$$

- **Dimension of ansatz space (and hence # of unknowns)** $= (n+1)^2$

Standard approach for EM following acoustics : stagnated convergence

- **Sample error result for a plane-wave acoustic scattering from a sphere of diameter 1λ (Ganesh & Graham, J. Comp. Phys., 2004):**

$n = 5,\ \mathbf{err} = 2.9e - 03;$

Standard approach for EM following acoustics : stagnated convergence

- **Sample error result for a plane-wave acoustic scattering from a sphere of diameter 1λ (Ganesh & Graham, J. Comp. Phys., 2004):**

  $n = 5$, $\mathbf{err} = 2.9e - 03$; $n = 10$, $\mathbf{err} = 2.2e - 10$;

Standard approach for EM following acoustics : stagnated convergence

- **Sample error result for a plane-wave acoustic scattering from a sphere of diameter 1λ (Ganesh & Graham, J. Comp. Phys., 2004):**

$$n = 5, \ \mathbf{err} = 2.9e-03; \quad n = 10, \ \mathbf{err} = 2.2e-10; \quad n = 15, \ \mathbf{err} = 6.2e-14$$

Standard approach for EM following acoustics : stagnated convergence

- **Sample error result for a plane-wave acoustic scattering from a sphere of diameter 1λ (Ganesh & Graham, J. Comp. Phys., 2004):**

$$n = 5,\ \mathbf{err} = 2.9e - 03; \quad n = 10,\ \mathbf{err} = 2.2e - 10; \quad n = 15,\ \mathbf{err} = 6.2e - 14$$

- **EM scattering - compute tangential electric field:**
use tangential spectral basis on sphere - W_n spanned by tangential harmonics of degree up to n (dim $= 2(n+1)^2 - 2$)

  ⋆ Grad $Y_{l,j}$ **and** $\boldsymbol{n}_{\partial B} \times$ Grad $Y_{l,j}$, $\qquad 1 \le l \le n, \quad |j| \le l$

Standard approach for EM following acoustics : stagnated convergence

- **Sample error result for a plane-wave acoustic scattering from a sphere of diameter 1λ (Ganesh & Graham, J. Comp. Phys., 2004):**

$$n = 5,\ \textbf{err} = 2.9e-03; \quad n = 10,\ \textbf{err} = 2.2e-10; \quad n = 15,\ \textbf{err} = 6.2e-14$$

- **EM scattering - compute tangential electric field: use tangential spectral basis on sphere - W_n spanned by tangential harmonics of degree up to n (dim $= 2(n+1)^2 - 2$)**
  - $\star$ Grad $Y_{l,j}$ **and** $\boldsymbol{n}_{\partial B} \times$ Grad $Y_{l,j}$, $\quad 1 \le l \le n, \quad |j| \le l$

- **This standard approach (leading to stagnated/quad convergence) was investigated by**
  - $\star$ **Ganesh & Hawkins (Proc. of CTAC conference, 2006)**
  - $\star$ **Kress & Pieper (PhD thesis, Gottingen, 2007).**

Standard approach for EM following acoustics : stagnated convergence

- **Sample error result for a plane-wave acoustic scattering from a sphere of diameter 1λ (Ganesh & Graham, J. Comp. Phys., 2004):**

$$n = 5,\ \mathbf{err} = 2.9e-03; \quad n = 10,\ \mathbf{err} = 2.2e-10; \quad n = 15,\ \mathbf{err} = 6.2e-14$$

- **EM scattering - compute tangential electric field:**
use tangential spectral basis on sphere - W_n spanned by tangential harmonics of degree up to n (dim $= 2(n+1)^2 - 2$)
  - ⋆ $\mathrm{Grad}\, Y_{l,j}$ **and** $\boldsymbol{n}_{\partial B} \times \mathrm{Grad}\, Y_{l,j}, \qquad 1 \le l \le n, \quad |j| \le l$

- **This standard approach (leading to stagnated/quad convergence) was investigated by**
  - ⋆ **Ganesh & Hawkins (Proc. of CTAC conference, 2006)**
  - ⋆ **Kress & Pieper (PhD thesis, Gottingen, 2007).**

- **Sample error result for a plane-wave EM scattering from a sphere of diameter less than 1λ (Pieper thesis, Univ. Gottingen, 2007):**

$$n = 5,\ \mathbf{err} = 7.9e-03;$$

Standard approach for EM following acoustics : stagnated convergence

- **Sample error result for a plane-wave acoustic scattering from a sphere of diameter 1λ (Ganesh & Graham, J. Comp. Phys., 2004):**

$$n = 5,\ \mathbf{err} = 2.9e-03; \quad n = 10,\ \mathbf{err} = 2.2e-10; \quad n = 15,\ \mathbf{err} = 6.2e-14$$

- **EM scattering - compute tangential electric field:**
use tangential spectral basis on sphere - W_n spanned by tangential harmonics of degree up to n (dim $= 2(n+1)^2 - 2$)

  ⋆ $\text{Grad}\, Y_{l,j}$ **and** $\boldsymbol{n}_{\partial B} \times \text{Grad}\, Y_{l,j}, \qquad 1 \le l \le n, \quad |j| \le l$

- **This standard approach (leading to stagnated/quad convergence) was investigated by**

  ⋆ **Ganesh & Hawkins (Proc. of CTAC conference, 2006)**

  ⋆ **Kress & Pieper (PhD thesis, Gottingen, 2007).**

- **Sample error result for a plane-wave EM scattering from a sphere of diameter less than 1λ (Pieper thesis, Univ. Gottingen, 2007):**

$$n = 5,\ \mathbf{err} = 7.9e-03; \quad n = 11,\ \mathbf{err} = 5.9e-04;$$

Standard approach for EM following acoustics : stagnated convergence

- **Sample error result for a plane-wave acoustic scattering from a sphere of diameter 1λ (Ganesh & Graham, J. Comp. Phys., 2004):**

$$n = 5,\ \mathbf{err} = 2.9e-03; \quad n = 10,\ \mathbf{err} = 2.2e-10; \quad n = 15,\ \mathbf{err} = 6.2e-14$$

- **EM scattering - compute tangential electric field:**
use tangential spectral basis on sphere - W_n spanned by tangential harmonics of degree up to n (dim $= 2(n+1)^2 - 2$)

  ⋆ $\text{Grad}\, Y_{l,j}$ **and** $\boldsymbol{n}_{\partial B} \times \text{Grad}\, Y_{l,j}, \qquad 1 \le l \le n, \quad |j| \le l$

- **This standard approach (leading to stagnated/quad convergence) was investigated by**

  ⋆ **Ganesh & Hawkins (Proc. of CTAC conference, 2006)**

  ⋆ **Kress & Pieper (PhD thesis, Gottingen, 2007).**

- **Sample error result for a plane-wave EM scattering from a sphere of diameter less than 1λ (Pieper thesis, Univ. Gottingen, 2007):**

$$n = 5,\ \mathbf{err} = 7.9e-03; \quad n = 11,\ \mathbf{err} = 5.9e-04; \quad n = 15,\ \mathbf{err} = 3.0e-04$$

Standard approach for EM following acoustics : stagnated convergence

- **Sample error result for a plane-wave acoustic scattering from a sphere of diameter 1λ (Ganesh & Graham, J. Comp. Phys., 2004):**

$$n = 5,\ \mathbf{err} = 2.9e-03;\quad n = 10,\ \mathbf{err} = 2.2e-10;\quad n = 15,\ \mathbf{err} = 6.2e-14$$

- **EM scattering - compute tangential electric field: use tangential spectral basis on sphere - W_n spanned by tangential harmonics of degree up to n (dim $= 2(n+1)^2 - 2$)**
  - ⋆ $\mathrm{Grad}\, Y_{l,j}$ **and** $\boldsymbol{n}_{\partial B} \times \mathrm{Grad}\, Y_{l,j}, \qquad 1 \le l \le n, \quad |j| \le l$

- **This standard approach (leading to stagnated/quad convergence) was investigated by**
  - ⋆ **Ganesh & Hawkins (Proc. of CTAC conference, 2006)**
  - ⋆ **Kress & Pieper (PhD thesis, Gottingen, 2007).**

- **Sample error result for a plane-wave EM scattering from a sphere of diameter less than 1λ (Pieper thesis, Univ. Gottingen, 2007):**

$$n = 5,\ \mathbf{err} = 7.9e-03;\quad n = 11,\ \mathbf{err} = 5.9e-04;\quad n = 15,\ \mathbf{err} = 3.0e-04$$

Non-standard approach for EM : spectral convergence

- **Modify the Stratton-Chu formula that is tolerant to non-tangential basis functions but outcome is a tangential electric field**

Non-standard approach for EM : spectral convergence

- **Modify the Stratton-Chu formula that is tolerant to non-tangential basis functions but outcome is a tangential electric field**
- **Ansatz space is spanned by both tangential and normal components**

Non-standard approach for EM : spectral convergence

- **Modify the Stratton-Chu formula that is tolerant to non-tangential basis functions but outcome is a tangential electric field**
- **Ansatz space is spanned by both tangential and normal components**
- **Spectral convergence (Ganesh & Hawkins, Numer. Alg., 2007):**

Non-standard approach for EM : spectral convergence

- **Modify the Stratton-Chu formula that is tolerant to non-tangential basis functions but outcome is a tangential electric field**
- **Ansatz space is spanned by both tangential and normal components**
- **Spectral convergence (Ganesh & Hawkins, Numer. Alg., 2007):**
- **If $3(n+1)^2$ is the total number unknowns in an approximate surface curren (and hence the dimension of the resulting linear system), then**

Non-standard approach for EM : spectral convergence

- **Modify the Stratton-Chu formula that is tolerant to non-tangential basis functions but outcome is a tangential electric field**
- **Ansatz space is spanned by both tangential and normal components**
- **Spectral convergence (Ganesh & Hawkins, Numer. Alg., 2007):**
- **If $3(n+1)^2$ is the total number unknowns in an approximate surface curren (and hence the dimension of the resulting linear system), then**

$$\max\{\|\boldsymbol{w}-\boldsymbol{w}_n\|_{\infty,\partial D}, \|\boldsymbol{E}_\infty-\boldsymbol{E}_{n,\infty}\|_{\infty,\partial B}, \|\sigma_\infty-\sigma_{n,\infty}\|_{\infty,\partial B}\} \le c_q\,\frac{1}{n^q}\|\boldsymbol{w}\|_{q+1,\infty,\partial D}$$

Non-standard approach for EM : spectral convergence

- **Modify the Stratton-Chu formula that is tolerant to non-tangential basis functions but outcome is a tangential electric field**
- **Ansatz space is spanned by both tangential and normal components**
- **Spectral convergence (Ganesh & Hawkins, Numer. Alg., 2007):**
- **If $3(n+1)^2$ is the total number unknowns in an approximate surface curren (and hence the dimension of the resulting linear system), then**

$$\max\{\|\boldsymbol{w}-\boldsymbol{w}_n\|_{\infty,\partial D}, \|\boldsymbol{E}_\infty-\boldsymbol{E}_{n,\infty}\|_{\infty,\partial B}, \|\sigma_\infty-\sigma_{n,\infty}\|_{\infty,\partial B}\} \leq c_q \frac{1}{n^q}\|\boldsymbol{w}\|_{q+1,\infty,\partial D}$$

- **In addition to full theoretical analysis, spectral convergence was demonstrated by simulating electromagnetic scattering by several benchmark targets with siz_obs from 1 to 24 with $n \leq 150$**

Non-standard approach for EM : spectral convergence

- **Modify the Stratton-Chu formula that is tolerant to non-tangential basis functions but outcome is a tangential electric field**
- **Ansatz space is spanned by both tangential and normal components**
- **Spectral convergence (Ganesh & Hawkins, Numer. Alg., 2007):**
- **If $3(n+1)^2$ is the total number unknowns in an approximate surface curren (and hence the dimension of the resulting linear system), then**

$$\max\{\|\boldsymbol{w}-\boldsymbol{w}_n\|_{\infty,\partial D}, \|\boldsymbol{E}_\infty-\boldsymbol{E}_{n,\infty}\|_{\infty,\partial B}, \|\sigma_\infty-\sigma_{n,\infty}\|_{\infty,\partial B}\} \le c_q \frac{1}{n^q}\|\boldsymbol{w}\|_{q+1,\infty,\partial D}$$

- **In addition to full theoretical analysis, spectral convergence was demonstrated by simulating electromagnetic scattering by several benchmark targets with** siz_obs **from 1 to 24 with** $n \le 150$
- **However, dimension of the above ansatz space is about** $(n+1)^2$ **more than that required for purely tangential functions based space**

Non-standard approach for EM : spectral convergence

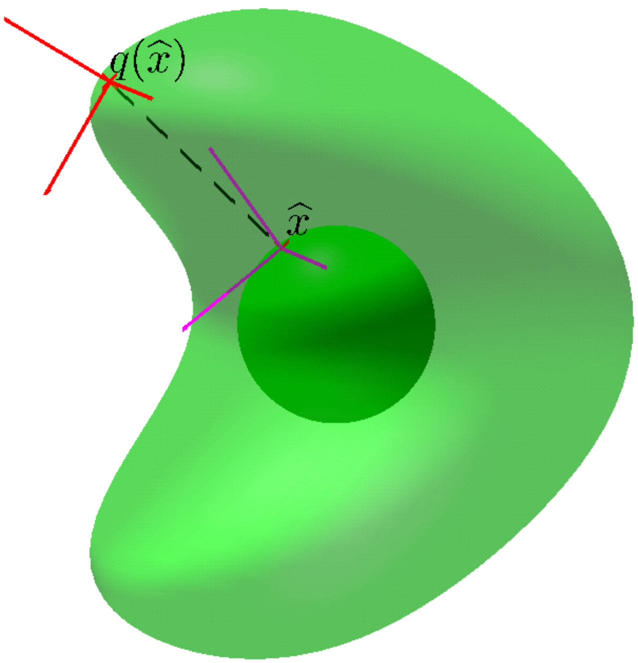
- **Modify the Stratton-Chu formula that is tolerant to non-tangential basis functions but outcome is a tangential electric field**
- **Ansatz space is spanned by both tangential and normal components**
- **Spectral convergence (Ganesh & Hawkins, Numer. Alg., 2007):**
- **If $3(n+1)^2$ is the total number unknowns in an approximate surface curren (and hence the dimension of the resulting linear system), then**

$$\max\{\|\boldsymbol{w}-\boldsymbol{w}_n\|_{\infty,\partial D}, \|\boldsymbol{E}_\infty-\boldsymbol{E}_{n,\infty}\|_{\infty,\partial B}, \|\sigma_\infty-\sigma_{n,\infty}\|_{\infty,\partial B}\} \leq c_q \frac{1}{n^q}\|\boldsymbol{w}\|_{q+1,\infty,\partial D}$$

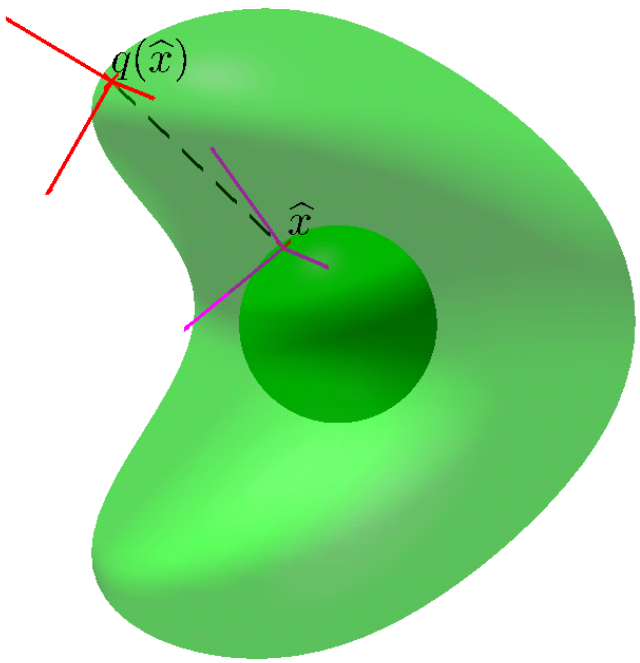
- **In addition to full theoretical analysis, spectral convergence was demonstrated by simulating electromagnetic scattering by several benchmark targets with siz_obs from 1 to 24 with $n \leq 150$**
- **However, dimension of the above ansatz space is about $(n+1)^2$ more than that required for purely tangential functions based space**
- **In particular for $n = 150$, number of unknowns is increased by more than $22,500$ (i.e., complex dense matrix of size added by $22,500 \times 22,500$)**

Spectrally Accurate Tangential Approximation

- **Tangential spectral basis on sphere**
  - $\star$ $\boldsymbol{Z}_{l,j}^{(1)} = \text{Grad } Y_{l,j}$ **and** $\boldsymbol{Z}_{l,j}^{(2)} = \boldsymbol{n}_{\partial B} \times \text{Grad } Y_{l,j}$



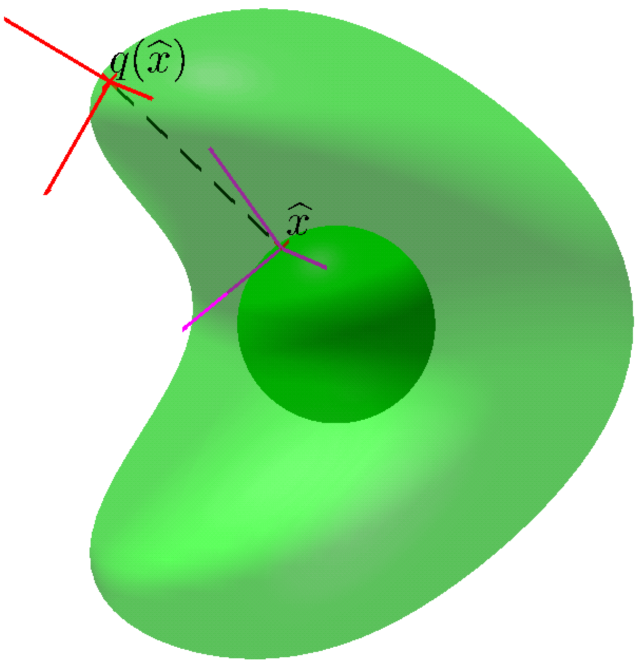
Spectrally Accurate Tangential Approximation



- **Tangential spectral basis on sphere**
  - ⋆ $\boldsymbol{Z}_{l,j}^{(1)} = \text{Grad}\, Y_{l,j}$ **and** $\boldsymbol{Z}_{l,j}^{(2)} = \boldsymbol{n}_{\partial B} \times \text{Grad}\, Y_{l,j}$

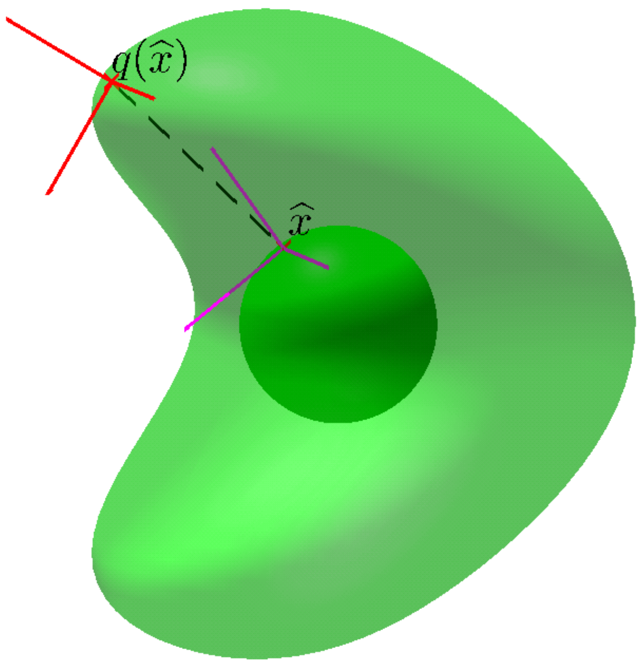
- **Tangential transformation $\mathcal{F}$**
  - ⋆ **if $\boldsymbol{V}$ is tangential on ∂B then $\mathcal{F}\boldsymbol{V}$ is $\boldsymbol{q}$-tangential**
  - ⋆ **idea: rotate $\boldsymbol{n}_{\partial B}$ to $\boldsymbol{n}_{\partial D}$**

Spectrally Accurate Tangential Approximation



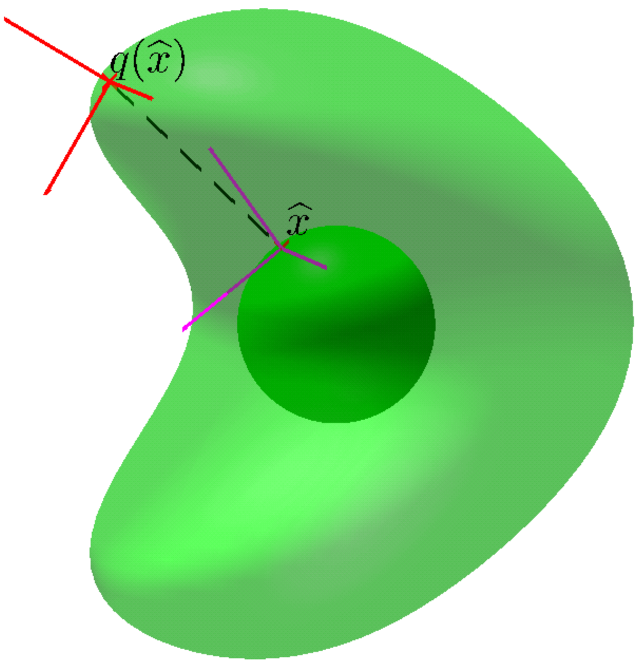
- **Tangential spectral basis on sphere**
  - $\star$ $\boldsymbol{Z}_{l,j}^{(1)} = \mathrm{Grad}\ Y_{l,j}$ **and** $\boldsymbol{Z}_{l,j}^{(2)} = \boldsymbol{n}_{\partial B} \times \mathrm{Grad}\ Y_{l,j}$
- **Tangential transformation $\mathcal{F}$**
  - $\star$ **if $\boldsymbol{V}$ is tangential on ∂B then $\mathcal{F}\boldsymbol{V}$ is $\boldsymbol{q}$-tangential**
  - $\star$ **idea: rotate $\boldsymbol{n}_{\partial B}$ to $\boldsymbol{n}_{\partial D}$**
- **Combination of $\mathcal{F}$ with tangential spherical harmonics leads to spectral approximation**

Spectrally Accurate Tangential Approximation



- **Tangential spectral basis on sphere**
  - ⋆ $\boldsymbol{Z}_{l,j}^{(1)} = \text{Grad } Y_{l,j}$ **and** $\boldsymbol{Z}_{l,j}^{(2)} = \boldsymbol{n}_{\partial B} \times \text{Grad } Y_{l,j}$
- **Tangential transformation $\mathcal{F}$**
  - ⋆ **if $\boldsymbol{V}$ is tangential on ∂B then $\mathcal{F}\boldsymbol{V}$ is $\boldsymbol{q}$-tangential**
  - ⋆ **idea: rotate $\boldsymbol{n}_{\partial B}$ to $\boldsymbol{n}_{\partial D}$**
- **Combination of $\mathcal{F}$ with tangential spherical harmonics leads to spectral approximation**
- **Ansatz space (dimension $2(n+1)^2 - 2$) spanned by $\mathcal{F}\boldsymbol{Z}_{l,j}^{(1)}, \ \mathcal{F}\boldsymbol{Z}_{l,j}^{(2)} \qquad 1 \le l \le n, \quad |j| \le l$**
- **Details in Ganesh & Hawkins (2007, pre-print)**

Spectrally Accurate Tangential Approximation



- **Tangential spectral basis on sphere**
  - ⋆ $\boldsymbol{Z}_{l,j}^{(1)} = \text{Grad } Y_{l,j}$ **and** $\boldsymbol{Z}_{l,j}^{(2)} = \boldsymbol{n}_{\partial B} \times \text{Grad } Y_{l,j}$
- **Tangential transformation $\mathcal{F}$**
  - ⋆ **if $\boldsymbol{V}$ is tangential on ∂B then $\mathcal{F}\boldsymbol{V}$ is $\boldsymbol{q}$-tangential**
  - ⋆ **idea: rotate $\boldsymbol{n}_{\partial B}$ to $\boldsymbol{n}_{\partial D}$**
- **Combination of $\mathcal{F}$ with tangential spherical harmonics leads to spectral approximation**
- **Ansatz space (dimension $2(n+1)^2 - 2$) spanned by $\mathcal{F}\boldsymbol{Z}_{l,j}^{(1)},\ \mathcal{F}\boldsymbol{Z}_{l,j}^{(2)} \qquad 1 \le l \le n, \quad |j| \le l$**
- **Details in Ganesh & Hawkins (2007, pre-print)**

Performance of FISC and GH (present) algorithms

Electromagnetic Scattering by a sphere of diameter 48λ

Algorithm	Unknowns	RMS Err. (MD)	RMS Err. (ED)	RMS Err. (PW)

Performance of FISC and GH (present) algorithms

Electromagnetic Scattering by a sphere of diameter 48λ

Algorithm	Unknowns	RMS Err. (MD)	RMS Err. (ED)	RMS Err. (PW)
GH	48,670	2.9e-11	1.9e-11	9.9e-02

Performance of FISC and GH (present) algorithms

Electromagnetic Scattering by a sphere of diameter 48λ

Algorithm	Unknowns	RMS Err. (MD)	RMS Err. (ED)	RMS Err. (PW)
GH	48,670	2.9e-11	1.9e-11	9.9e-02
FISC	2,408,448	–	–	3.3e-01

Performance of FISC and GH (present) algorithms

Electromagnetic Scattering by a sphere of diameter 48λ

Algorithm	Unknowns	RMS Err. (MD)	RMS Err. (ED)	RMS Err. (PW)
GH	48,670	2.9e-11	1.9e-11	9.9e-02
FISC	2,408,448	–	–	3.3e-01
GH	51,840	–	–	3.7e-03

Performance of FISC and GH (present) algorithms

Electromagnetic Scattering by a sphere of diameter 48λ

Algorithm	Unknowns	RMS Err. (MD)	RMS Err. (ED)	RMS Err. (PW)
GH	48,670	2.9e-11	1.9e-11	9.9e-02
FISC	2,408,448	–	–	3.3e-01
GH	51,840	–	–	3.7e-03
GH	55,110	–	–	5.9e-05

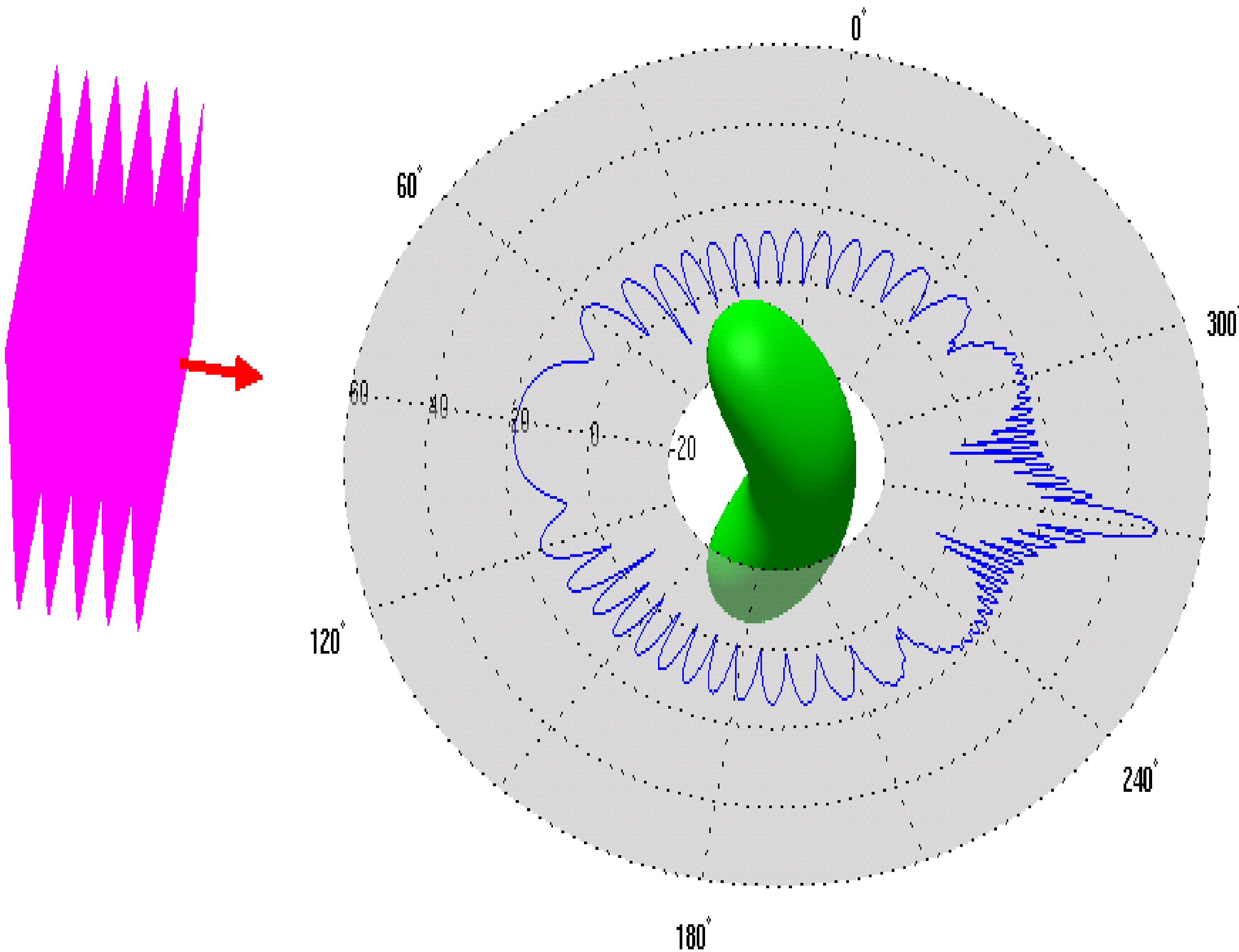
Performance of FISC and GH (present) algorithms

Electromagnetic Scattering by a sphere of diameter 48λ

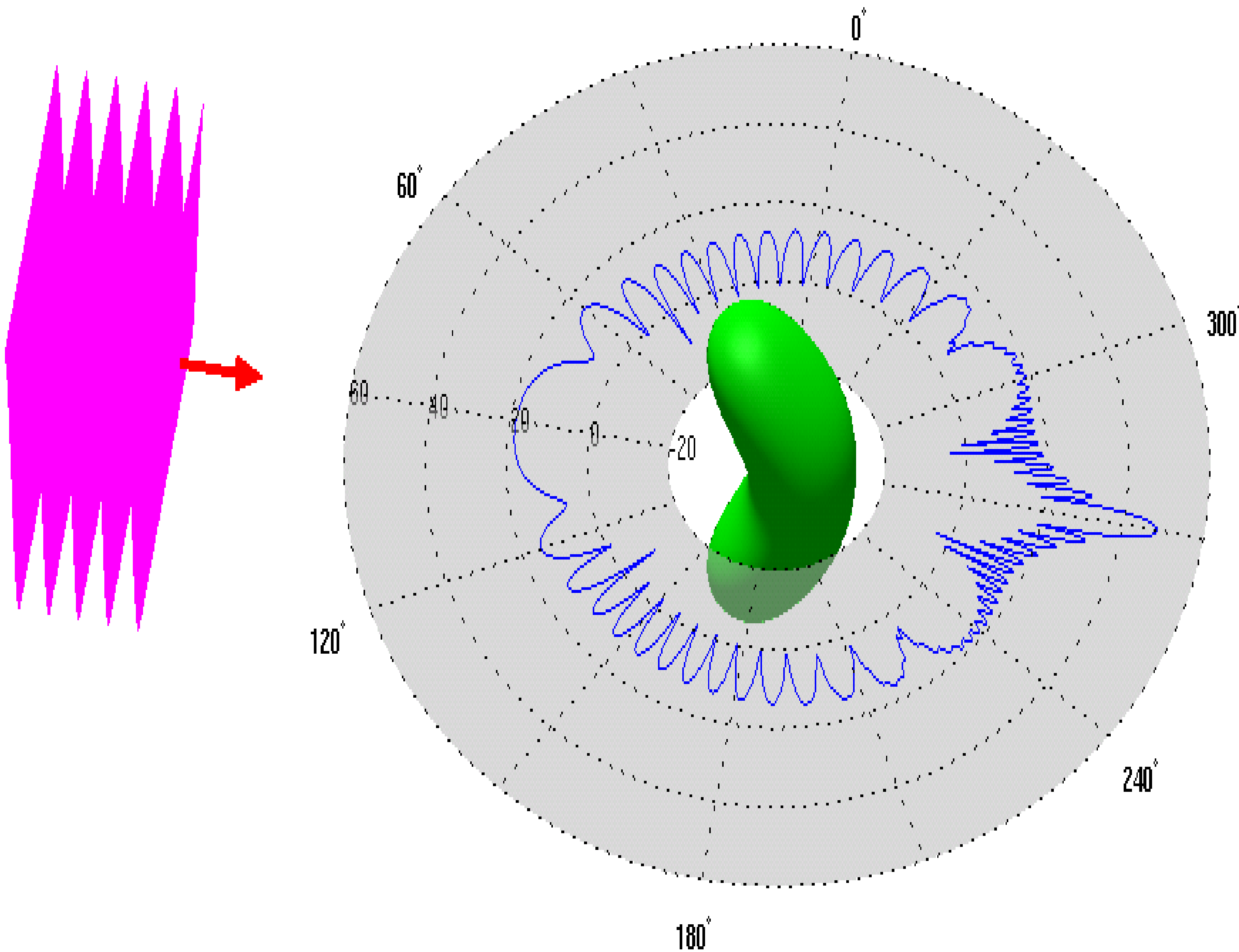
Algorithm	Unknowns	RMS Err. (MD)	RMS Err. (ED)	RMS Err. (PW)
GH	48,670	2.9e-11	1.9e-11	9.9e-02
FISC	2,408,448	–	–	3.3e-01
GH	51,840	–	–	3.7e-03
GH	55,110	–	–	5.9e-05

n	**bean**(30λ)	**bean**(30λ)	**fount**(30λ)	**fount**(30λ)
	$k = 47.1239$	$k = 47.1239$	$k = 78.5398$	$k = 78.5398$
	MD	**ED**	**MD**	**ED**
	Rel. Err.	**Rel. Err.**	**Rel. Err.**	**Rel. Err.**
150	**6.73e-04**	**9.09e-04**	**2.41e-06**	**1.18e-06**
155	**5.04e-05**	**7.54e-05**	**3.70e-07**	**2.08e-07**
160	**6.03e-06**	**7.87e-06**	**2.98e-07**	**1.45e-07**

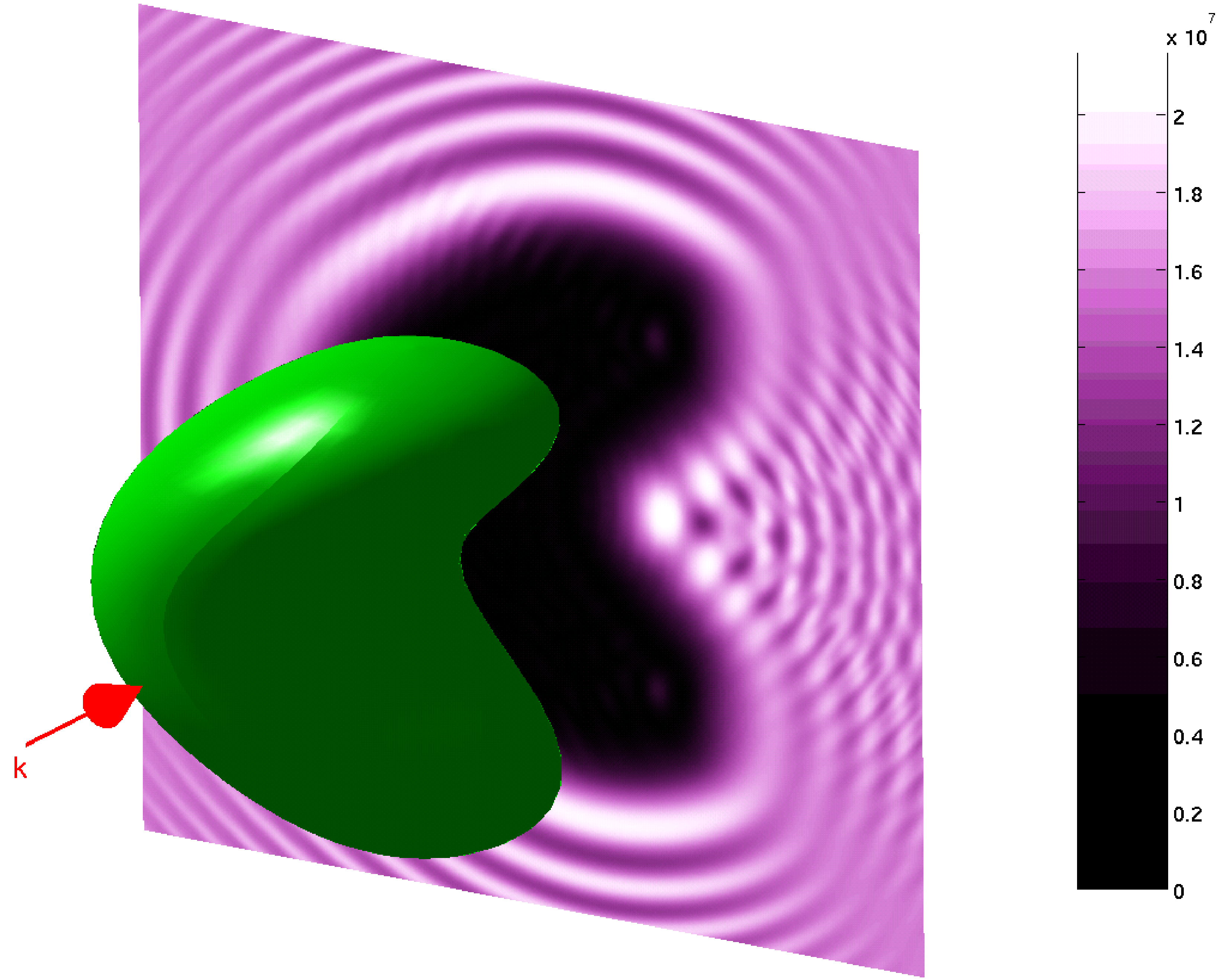
n	**ice**(15λ)	**ice**(15λ)	**dust**(10λ)	**dust**(10λ)
	$k = 47.1239$	$k = 47.1239$	$k = 31.4159$	$k = 31.4159$
	MD	**ED**	**MD**	**ED**
	Rel. Err.	**Rel. Err.**	**Rel. Err.**	**Rel. Err.**
150	**3.10e-07**	**2.58e-07**	**5.00e-05**	**2.02e-05**
155	**8.29e-08**	**6.72e-08**	**1.55e-05**	**6.20e-06**
160	**3.99e-08**	**3.35e-08**	**4.84e-06**	**1.93e-06**



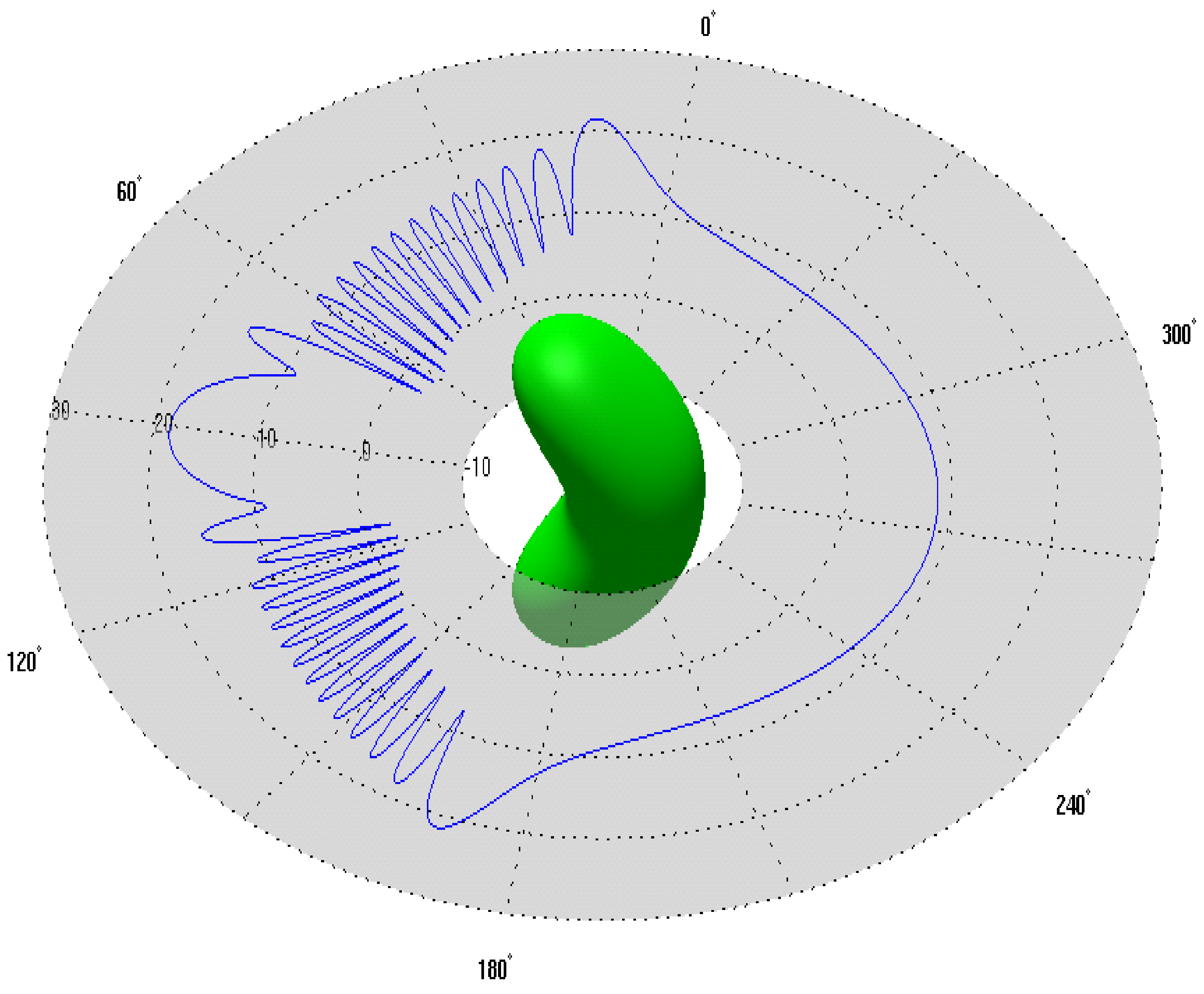
Bistatic RCS of bean with diameter 30λ for a single incident wave with HH polarization,



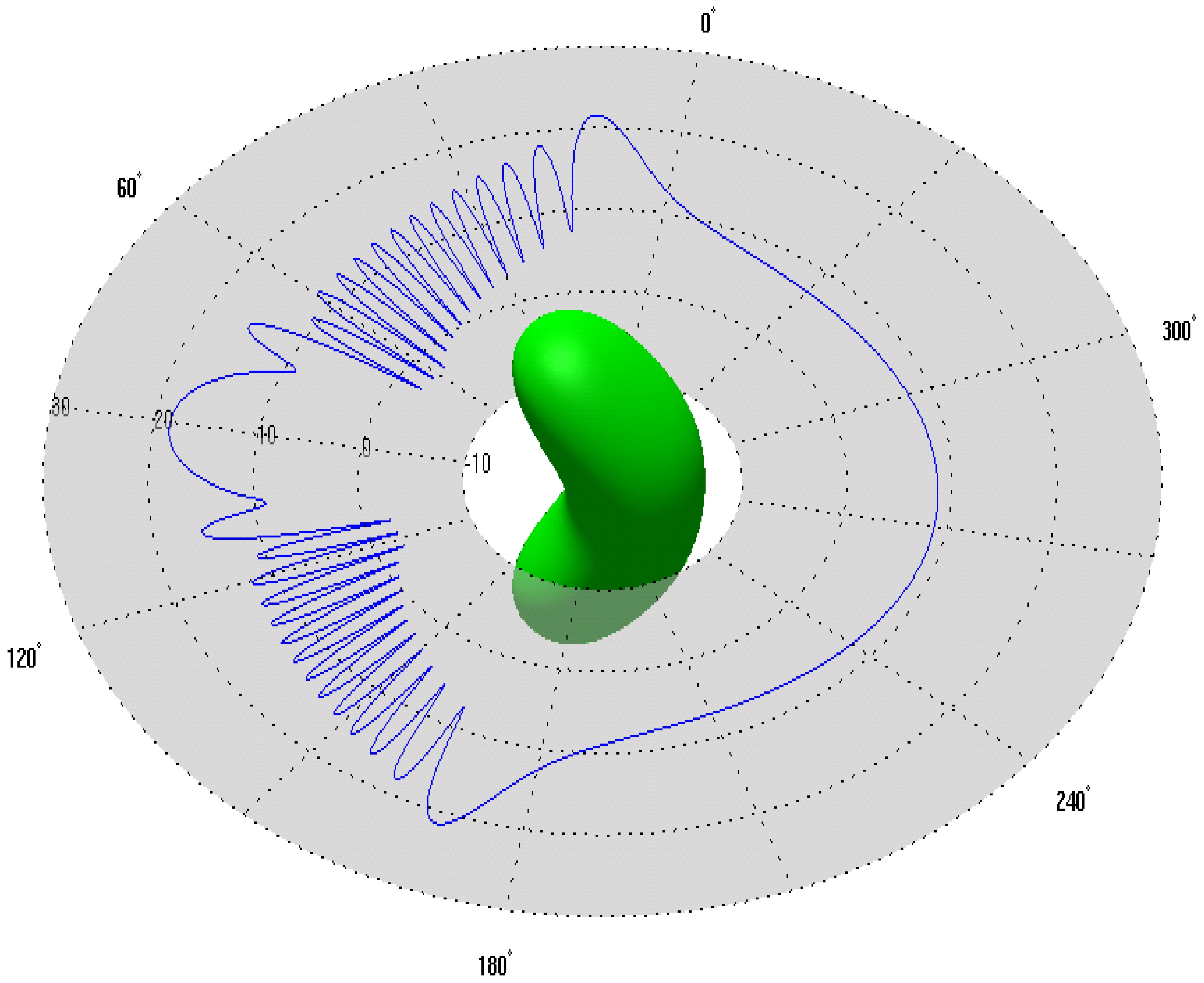
Bistatic RCS of bean with diameter 30λ for a single incident wave with HH polarization, simulated using 48,670 unknowns (with 10^{-6} accuracy)



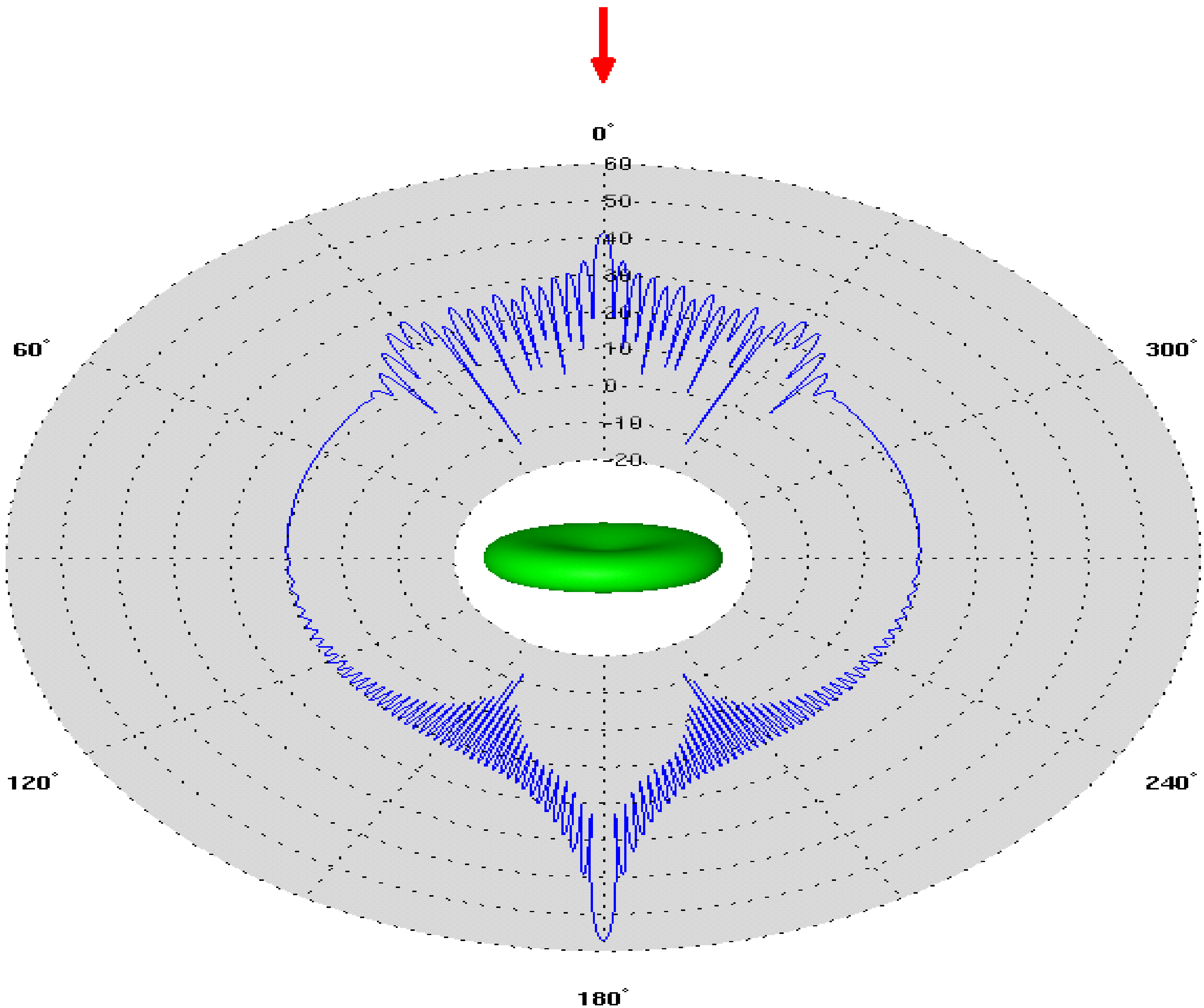
Visualization of the intensity $|\boldsymbol{\mathcal{E}}(\cdot,t)|$ of the total exterior field behind the obstacle bean(30λ) and detection of radiation free (shadow) region (simulated using 48,670 unknowns).



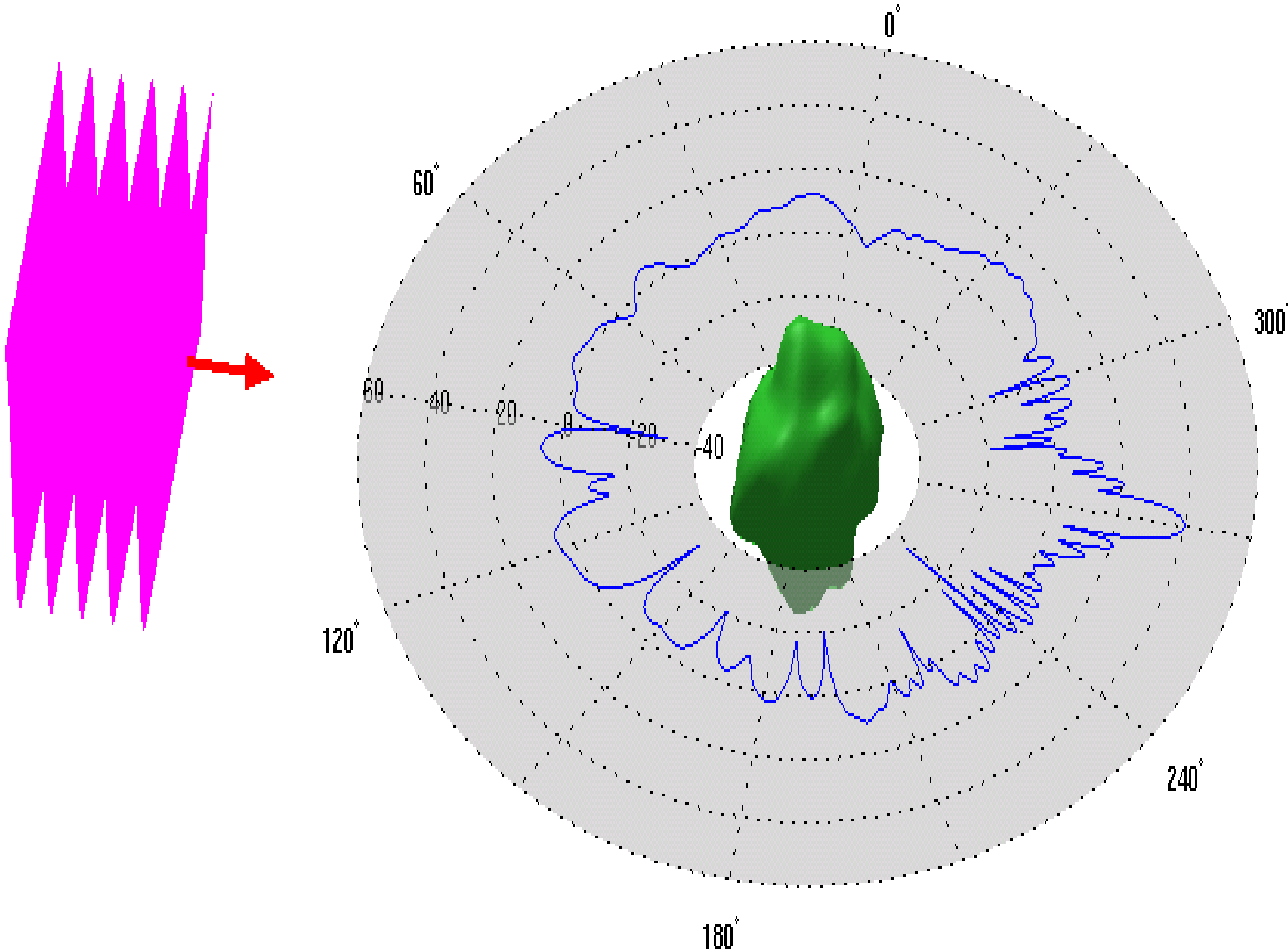
Monostatic RCS of bean with diameter 30λ for a single incident wave with HH polarization,



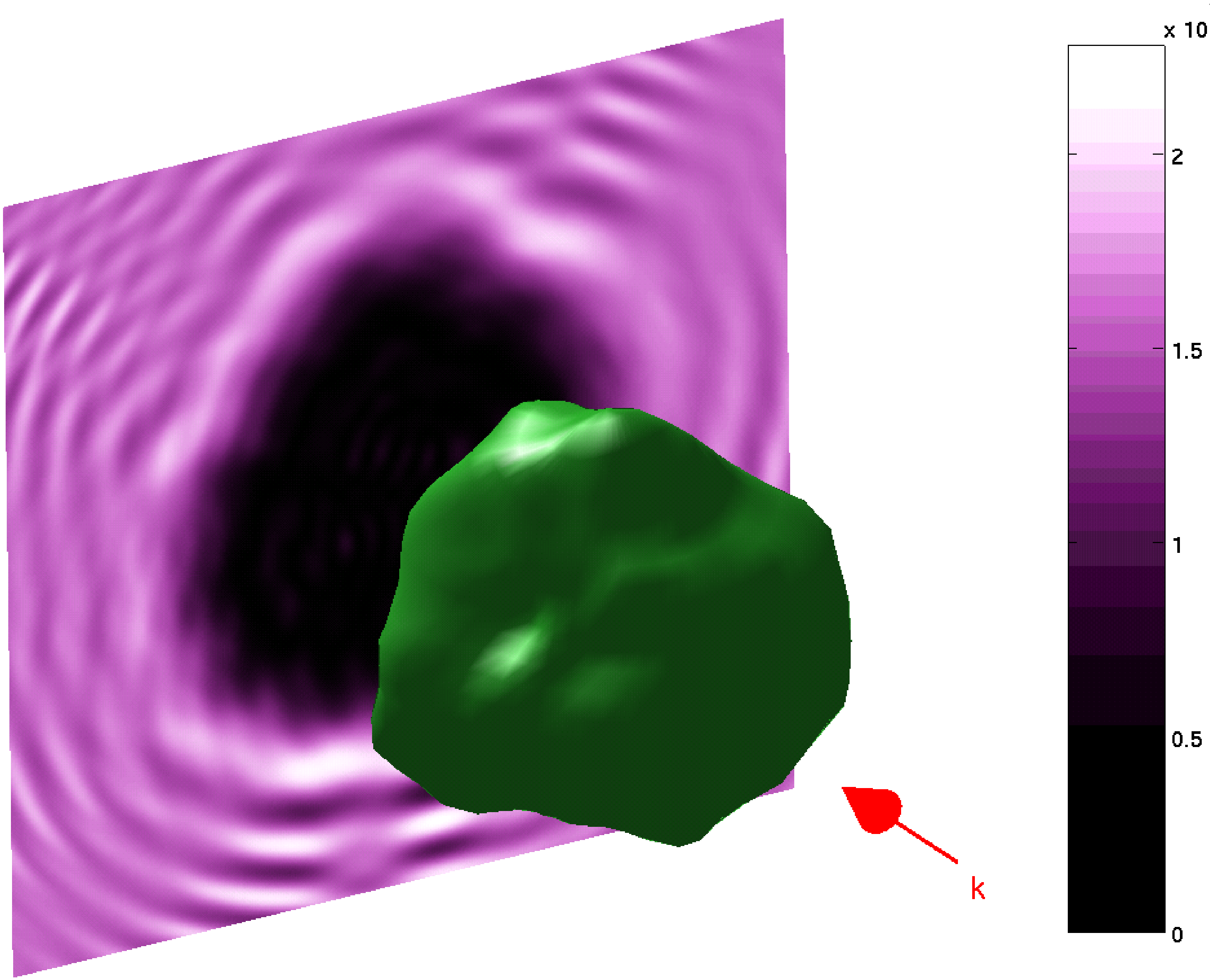
Monostatic RCS of bean with diameter 30λ for a single incident wave with HH polarization,
simulated using 2,402 incident waves with 48,670 unknowns per incident wave (with 10^{-6} acc



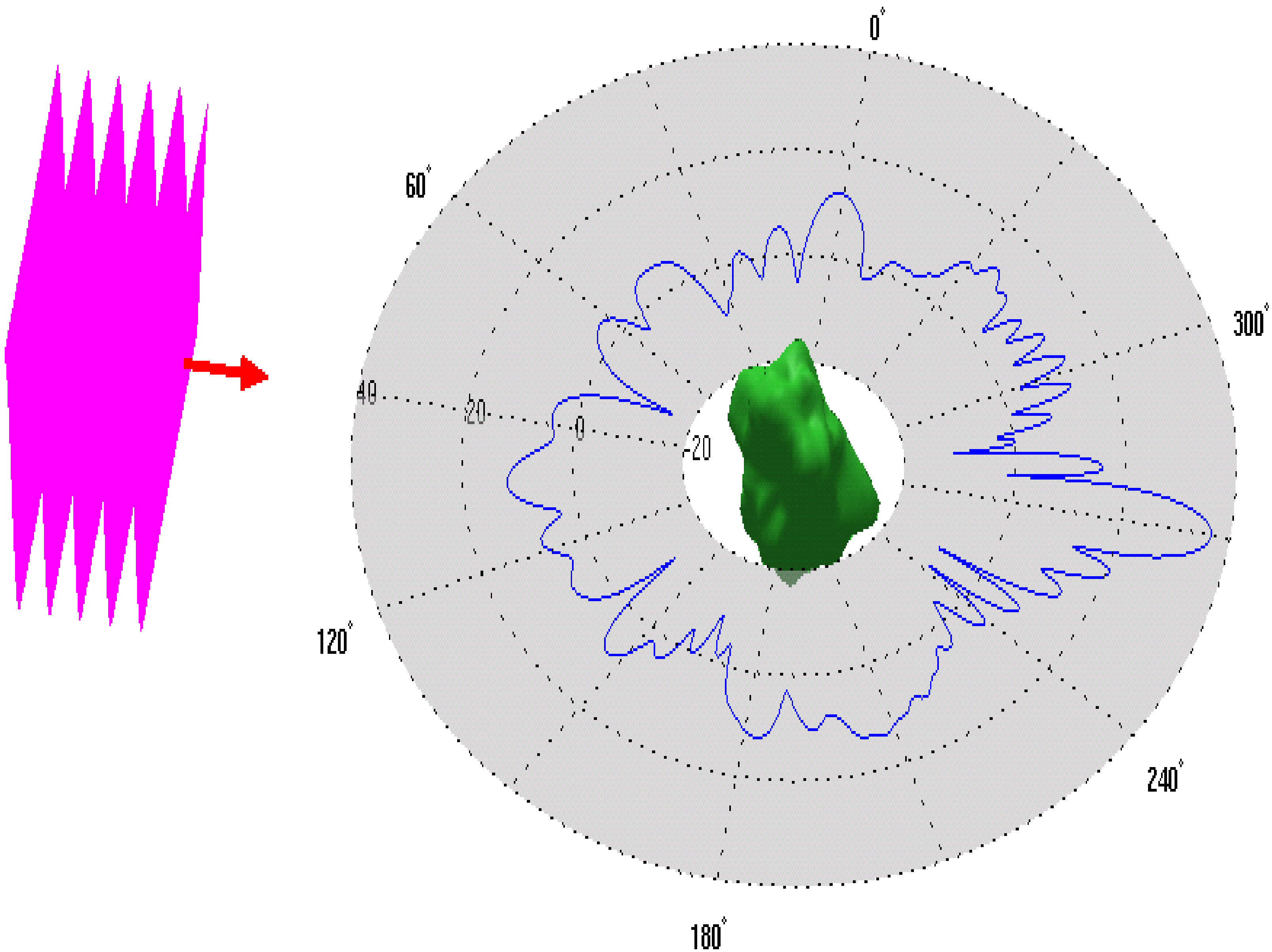
Bistatic RCS of erythrocyte with diameter 40λ for a single incident wave with HH polarization, simulated using 48,670 unknowns (with 10^{-7} accuracy)



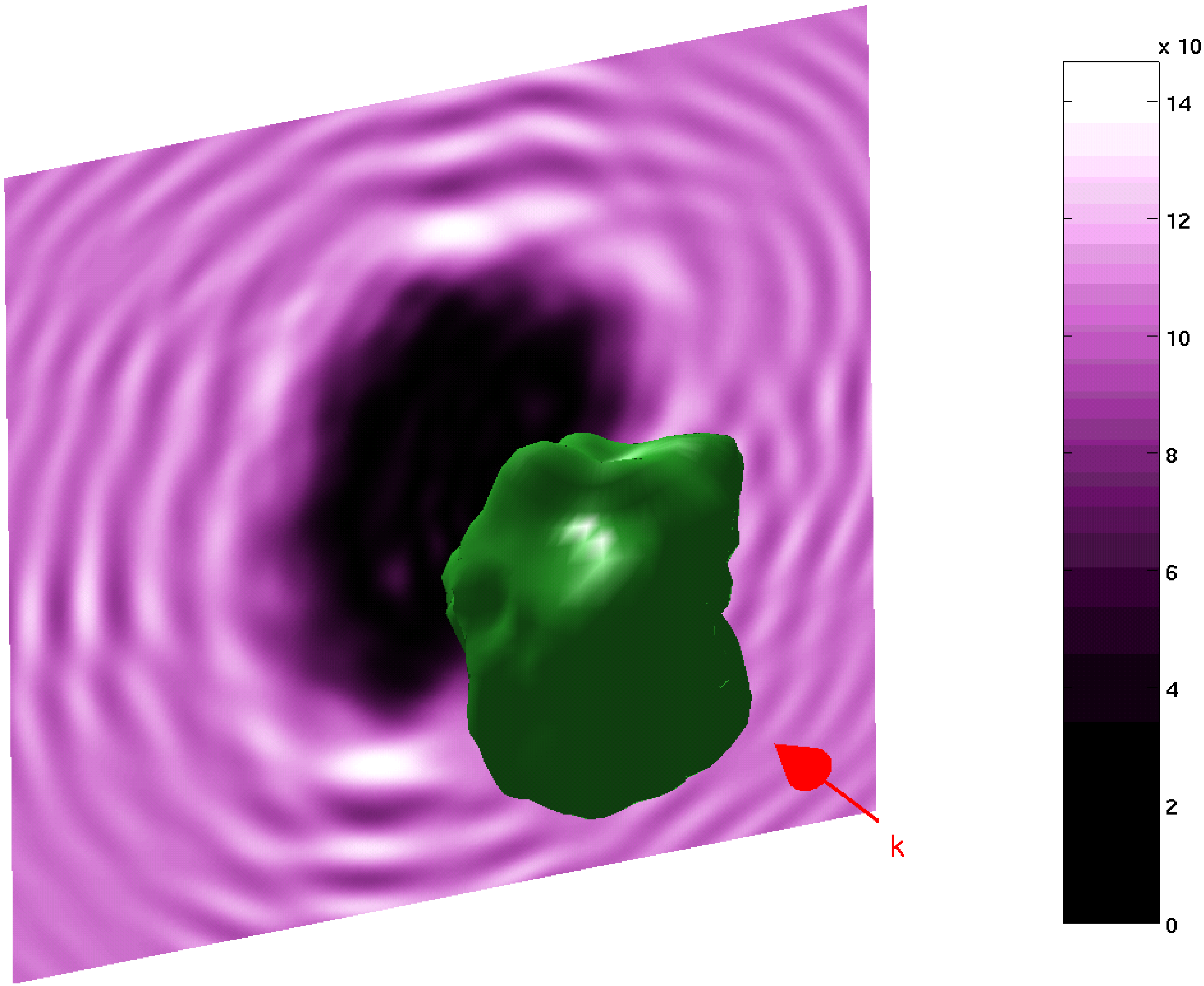
Bistatic RCS of random ice crystal with diameter 15λ for a single incident wave with HH polarization, simulated using 48,670 unknowns (with 10^{-8} accuracy)



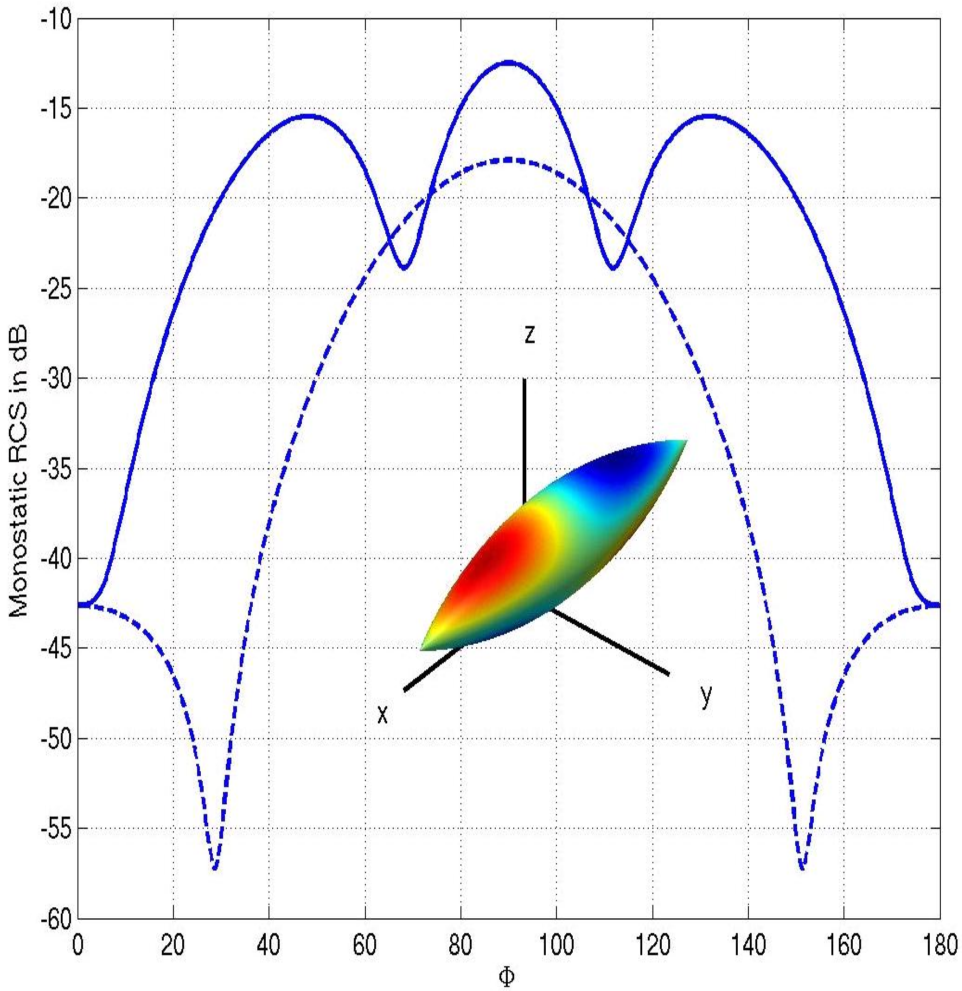
Visualization of the intensity $|\boldsymbol{\mathcal{E}}(\cdot,t)|$ of the total exterior field behind the obstacle ice(15λ) and detection of radiation free (shadow) region (simulated using 48,670 unknowns).



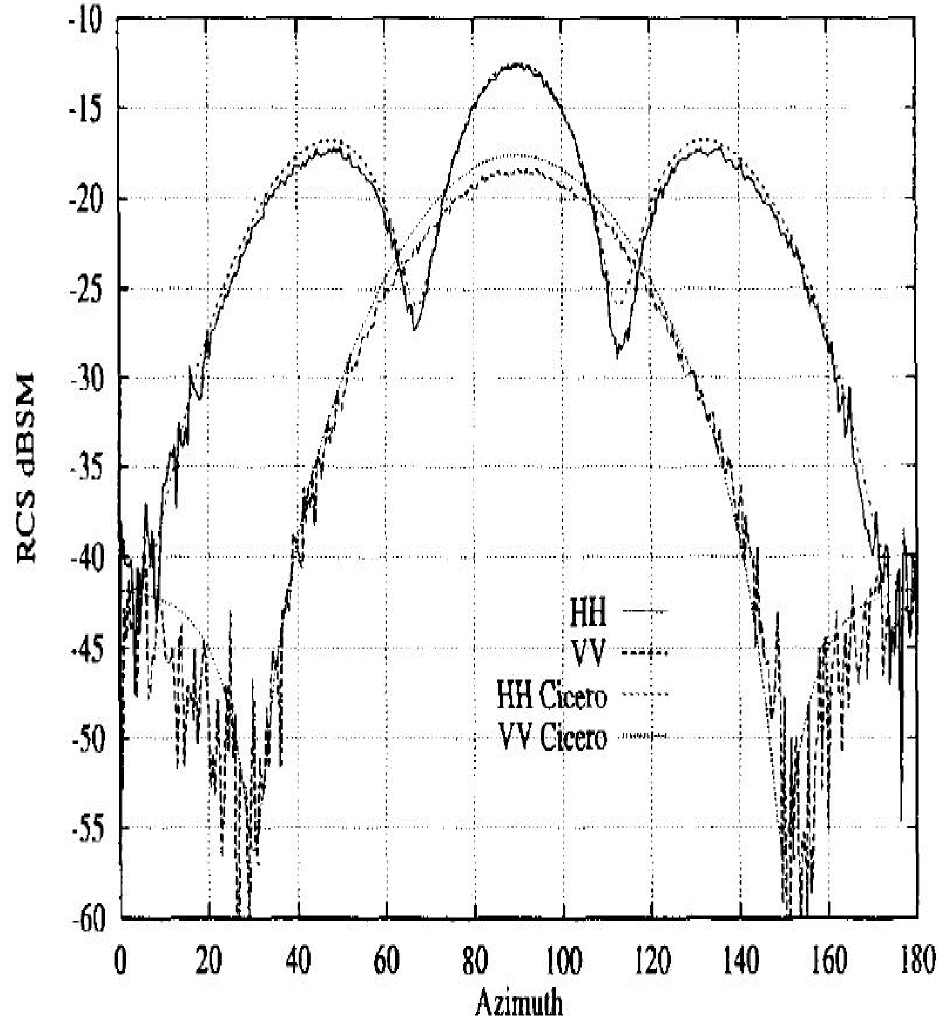
Bistatic RCS of random dust particle with diameter 10λ for a single incident wave with HH polarization, simulated using 48,670 unknowns (with 10^{-7} accuracy)



Visualization of the intensity $|\mathcal{E}(\cdot,t)|$ of the total exterior field behind the obstacle dust(10λ) and detection of radiation free (shadow) region (simulated using 48,670 unknowns).

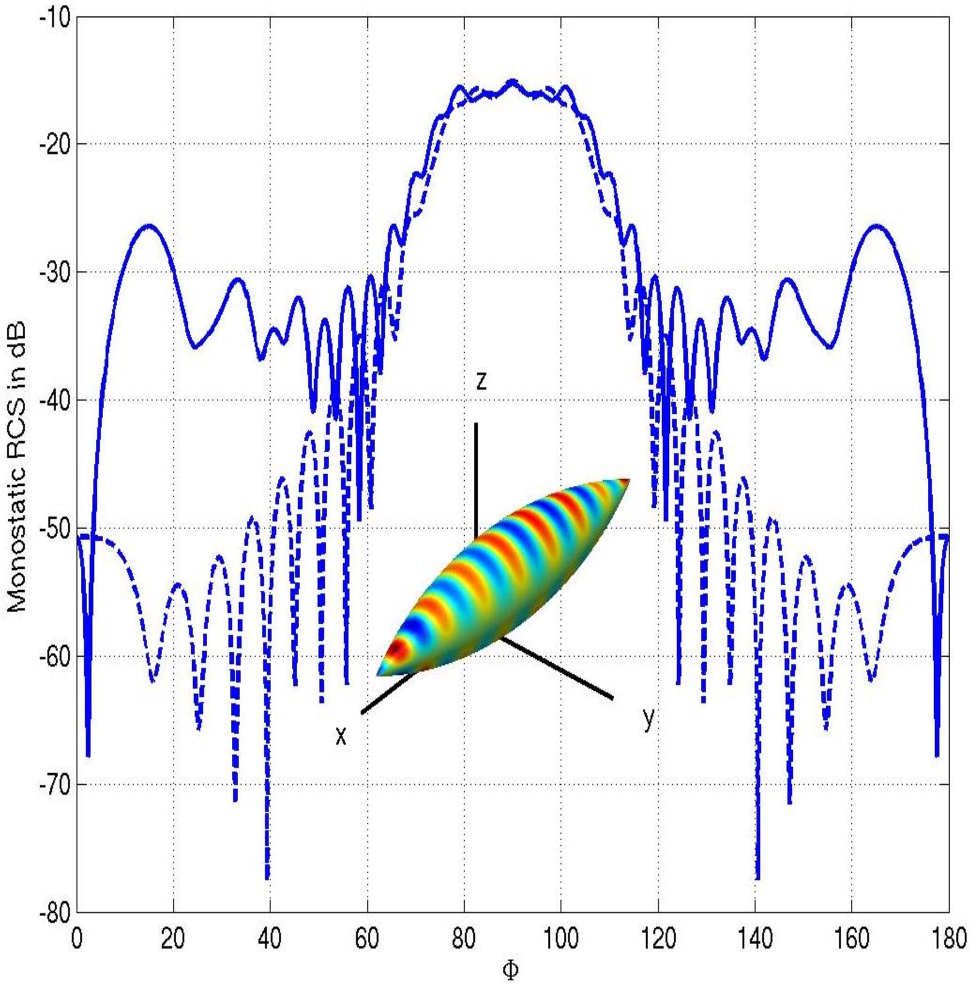


Numerical using GH (present)

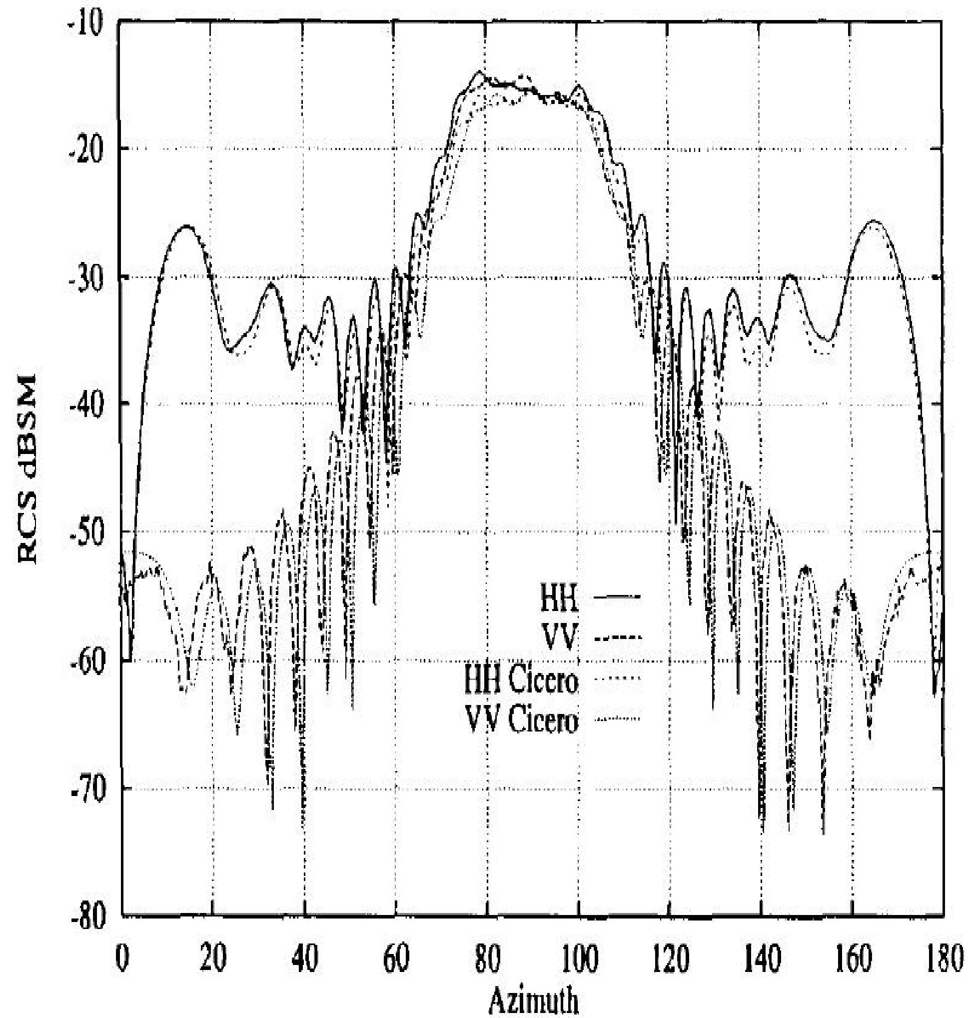


Experimental and Cicero

- **Comparison of monostatic RCS of Ogive at 1.18GHz with H-H and V-V polarization at observed angles $\phi \in [0, 180]$.**



Numerical using GH (present)



Experimental and Cicero

- **Comparison of monostatic RCS of Ogive at 9.0GHz with H-H and V-V polarization at observed angles $\phi \in [0, 180]$.**

Inverse Electromagnetic scattering

- **Give far-field measurements $\boldsymbol{E}_\infty^{\mathrm{mea}}$ for a few incident fields $\boldsymbol{E}^{\mathrm{inc}}$, find the shape (surface) of the electromagnetic scatterer that produced the measurements.**

Inverse Electromagnetic scattering

- **Give far-field measurements $\boldsymbol{E}_\infty^{\mathrm{mea}}$ for a few incident fields $\boldsymbol{E}^{\mathrm{inc}}$, find the shape (surface) of the electromagnetic scatterer that produced the measurements.**
- **For a fixed incident field, if we consider a nonlinear map F that maps a given surface to the generated far-field, the problem is to find the surface ∂D^* such that**

$$F(\partial D^*) = \boldsymbol{E}^{\mathrm{mea}}$$

Inverse Electromagnetic scattering

- **Give far-field measurements $\boldsymbol{E}_\infty^{\text{mea}}$ for a few incident fields $\boldsymbol{E}^{\text{inc}}$, find the shape (surface) of the electromagnetic scatterer that produced the measurements.**
- **For a fixed incident field, if we consider a nonlinear map F that maps a given surface to the generated far-field, the problem is to find the surface ∂D^* such that**

$$F(\partial D^*) = \boldsymbol{E}^{\text{mea}}$$

- **Methods for reconstruction of the surface fall into two broad class:**
  - ⋆ **Methods that require solutions to several direct problems (iterative methods)**
  - ⋆ **Methods that do not require solutions to direct problems (decomposition methods), except possibly to generate synthetic data.**

Inverse Electromagnetic scattering

- **Give far-field measurements $\boldsymbol{E}_\infty^{\text{mea}}$ for a few incident fields $\boldsymbol{E}^{\text{inc}}$, find the shape (surface) of the electromagnetic scatterer that produced the measurements.**
- **For a fixed incident field, if we consider a nonlinear map F that maps a given surface to the generated far-field, the problem is to find the surface ∂D^* such that**

$$F(\partial D^*) = \boldsymbol{E}^{\text{mea}}$$

- **Methods for reconstruction of the surface fall into two broad class:**
  - ⋆ **Methods that require solutions to several direct problems (iterative methods)**
  - ⋆ **Methods that do not require solutions to direct problems (decomposition methods), except possibly to generate synthetic data.**
- **Regularized iterative Newton methods are known to produce accurate reconstruction. Variants are based on Kress & Rundall (1985): Ivanyshyn and Kress (2006) for acoustic and Piper (2007) for EM.**

Inverse Electromagnetic scattering

- **Give far-field measurements $\boldsymbol{E}_{\infty}^{\mathrm{mea}}$ for a few incident fields $\boldsymbol{E}^{\mathrm{inc}}$, find the shape (surface) of the electromagnetic scatterer that produced the measurements.**

- **For a fixed incident field, if we consider a nonlinear map F that maps a given surface to the generated far-field, the problem is to find the surface ∂D^* such that**

$$F(\partial D^*) = \boldsymbol{E}^{\mathrm{mea}}$$

- **Methods for reconstruction of the surface fall into two broad class:**
  - ⋆ **Methods that require solutions to several direct problems (iterative methods)**
  - ⋆ **Methods that do not require solutions to direct problems (decomposition methods), except possibly to generate synthetic data.**

- **Regularized iterative Newton methods are known to produce accurate reconstruction. Variants are based on Kress & Rundall (1985): Ivanyshyn and Kress (2006) for acoustic and Piper (2007) for EM.**

- **However such methods in 3-D are practical provided that very fast methods to solve hundreds/thousands of direct problems (with various boundary conditions) are available. (We have one now for EM!).**

Inverse Electromagnetic scattering

- Give far-field measurements $\boldsymbol{E}_\infty^{\text{mea}}$ for a few incident fields $\boldsymbol{E}^{\text{inc}}$, find the shape (surface) of the electromagnetic scatterer that produced the measurements.

- For a fixed incident field, if we consider a nonlinear map F that maps a given surface to the generated far-field, the problem is to find the surface ∂D^* such that

$$F(\partial D^*) = \boldsymbol{E}^{\text{mea}}$$

- Methods for reconstruction of the surface fall into two broad class:
  - ⋆ Methods that require solutions to several direct problems (iterative methods)
  - ⋆ Methods that do not require solutions to direct problems (decomposition methods), except possibly to generate synthetic data.

- Regularized iterative Newton methods are known to produce accurate reconstruction. Variants are based on Kress & Rundall (1985): Ivanyshyn and Kress (2006) for acoustic and Piper (2007) for EM.

- However such methods in 3-D are practical provided that very fast methods to solve hundreds/thousands of direct problems (with various boundary conditions) are available. (We have one now for EM!).

Inverse Electromagnetic scattering : star-shaped

- **As a first step, it is convenient to restrict to star shaped obstacles ∂D and assume that**

$$\boldsymbol{x} = r(\widehat{\boldsymbol{x}})\widehat{\boldsymbol{x}}, \qquad \boldsymbol{x} \in \partial D, \ \widehat{\boldsymbol{x}} = \boldsymbol{x}/|\boldsymbol{x}|.$$

Inverse Electromagnetic scattering : star-shaped

- **As a first step, it is convenient to restrict to star shaped obstacles ∂D and assume that**

$$\boldsymbol{x} = r(\widehat{\boldsymbol{x}})\widehat{\boldsymbol{x}}, \qquad \boldsymbol{x} \in \partial D,\ \widehat{\boldsymbol{x}} = \boldsymbol{x}/|\boldsymbol{x}|.$$

- **We seek an approximation to $\partial D^\star$ of the above form with**

$$r(\widehat{\boldsymbol{x}}) = \sum_{l=1}^{N_q} \sum_{|j|\le l} \mathrm{r}_{lj} \boldsymbol{Y}_{l,j}.$$

Inverse Electromagnetic scattering : star-shaped

- **As a first step, it is convenient to restrict to star shaped obstacles ∂D and assume that**

$$\boldsymbol{x} = r(\widehat{\boldsymbol{x}})\widehat{\boldsymbol{x}}, \qquad \boldsymbol{x} \in \partial D,\ \widehat{\boldsymbol{x}} = \boldsymbol{x}/|\boldsymbol{x}|.$$

- **We seek an approximation to $\partial D^\star$ of the above form with**

$$r(\widehat{\boldsymbol{x}}) = \sum_{l=1}^{N_q} \sum_{|j| \le l} \mathrm{r}_{lj} \boldsymbol{Y}_{l,j}.$$

- **Thus our approximation to $\partial D^\star$ is described by $\mathrm{r} = (\mathrm{r}_{lj})_{l=1,\dots,N_q,\ |j|\le l} \in \mathbb{C}^{(N_q+1)^2}$.**

Inverse Electromagnetic scattering : star-shaped

- **As a first step, it is convenient to restrict to star shaped obstacles ∂D and assume that**

$$\boldsymbol{x} = r(\widehat{\boldsymbol{x}})\widehat{\boldsymbol{x}}, \qquad \boldsymbol{x} \in \partial D, \ \widehat{\boldsymbol{x}} = \boldsymbol{x}/|\boldsymbol{x}|.$$

- **We seek an approximation to $\partial D^\star$ of the above form with**

$$r(\widehat{\boldsymbol{x}}) = \sum_{l=1}^{N_q} \sum_{|j|\le l} \mathrm{r}_{lj} \boldsymbol{Y}_{l,j}.$$

- **Thus our approximation to $\partial D^\star$ is described by $\mathrm{r} = (\mathrm{r}_{lj})_{l=1,\ldots,N_q,\ |j|\le l} \in \mathbb{C}^{(N_q+1)^2}$.**

- **The coefficients r_{lj} are also required to satisfy a symmetry condition to ensure that r is real.**

Inverse Electromagnetic scattering : star-shaped

- As a first step, it is convenient to restrict to star shaped obstacles ∂D and assume that

$$\boldsymbol{x} = r(\widehat{\boldsymbol{x}})\widehat{\boldsymbol{x}}, \qquad \boldsymbol{x} \in \partial D,\ \widehat{\boldsymbol{x}} = \boldsymbol{x}/|\boldsymbol{x}|.$$

- We seek an approximation to $\partial D^\star$ of the above form with

$$r(\widehat{\boldsymbol{x}}) = \sum_{l=1}^{N_q} \sum_{|j|\le l} \mathrm{r}_{lj} \boldsymbol{Y}_{l,j}.$$

- Thus our approximation to $\partial D^\star$ is described by $\mathrm{r} = (\mathrm{r}_{lj})_{l=1,\ldots,N_q,\ |j|\le l} \in \mathbb{C}^{(N_q+1)^2}$.

- The coefficients r_{lj} are also required to satisfy a symmetry condition to ensure that r is real.

- The inverse problem can now be formulated in terms of the mapping

$$\boldsymbol{F} : \mathbb{C}^{(N_q+1)^2} \longrightarrow \underline{C}(\partial B), \qquad \mathrm{r} \mapsto \boldsymbol{E}_{\infty,n}$$

  where

  ⋆ $\boldsymbol{E}_{\infty,n}$ is the spectrally accurate approximation to $\boldsymbol{E}_\infty$ described earlier

  ⋆ ∂D is described by r and the above equations.

- It is convenient to write $\boldsymbol{F}(\widehat{\boldsymbol{x}};\mathrm{r})$ for $\boldsymbol{F}(\mathrm{r})$ evaluated at $\widehat{\boldsymbol{x}} \in \partial B$.

Inverse Electromagnetic scattering : algebraic and linearization

- **In practice we discretize using collocation at points $\widehat{\boldsymbol{x}}_1, \ldots, \widehat{\boldsymbol{x}}_M \in \partial B$.**

Inverse Electromagnetic scattering : algebraic and linearization

- **In practice we discretize using collocation at points $\widehat{\boldsymbol{x}}_1, \ldots, \widehat{\boldsymbol{x}}_M \in \partial B$.**
- **Thus solve the discrete system**

$$\mathrm{F}(\mathrm{r}) = \mathrm{E}_\infty^{\mathrm{ref}} \tag{2}$$

where

- ⋆ $\mathrm{F}(\mathrm{r}) = \left(\boldsymbol{F}(\widehat{\boldsymbol{x}}_i; \mathrm{r})\right)_{i=1,\ldots,M}$
- ⋆ $\mathrm{E}_\infty^{\mathrm{mea}} = \left(\boldsymbol{E}_\infty^{\mathrm{mea}}(\widehat{\boldsymbol{x}}_i)\right)_{i=1,\ldots,M}$.

Inverse Electromagnetic scattering : algebraic and linearization

- **In practice we discretize using collocation at points $\widehat{\boldsymbol{x}}_1, \ldots, \widehat{\boldsymbol{x}}_M \in \partial B$.**
- **Thus solve the discrete system**

$$\mathrm{F}(\mathrm{r}) = \mathrm{E}_{\infty}^{\mathrm{ref}} \tag{2}$$

where

  - ⋆ $\mathrm{F}(\mathrm{r}) = \left(\boldsymbol{F}(\widehat{\boldsymbol{x}}_i; \mathrm{r})\right)_{i=1,\ldots,M}$
  - ⋆ $\mathrm{E}_{\infty}^{\mathrm{mea}} = \left(\boldsymbol{E}_{\infty}^{\mathrm{mea}}(\widehat{\boldsymbol{x}}_i)\right)_{i=1,\ldots,M}$.

- **Solve the nonlinear equation using Newton's method with initial guess r_0 and**

$$\mathrm{r}_{k+1} = \mathrm{r}_k + \mathrm{q}_k, \quad \mathrm{G}'_{\mathrm{r}}(\mathrm{r}_k)\mathrm{q}_k = \mathrm{G}(\mathrm{r}_k), \quad k = 0, 1, 2, \ldots,$$

where

  - ⋆ $\mathrm{G}(\mathrm{r}) = \mathrm{E}_{\infty}^{\mathrm{mea}} - \mathrm{F}(\mathrm{r})$
  - ⋆ G'_{r} **is the Frechet derivative of G with respect to r.**

Inverse Electromagnetic scattering : algebraic and linearization

- **In practice we discretize using collocation at points $\widehat{\boldsymbol{x}}_1, \ldots, \widehat{\boldsymbol{x}}_M \in \partial B$.**
- **Thus solve the discrete system**

$$\mathrm{F}(\mathrm{r}) = \mathrm{E}_\infty^{\mathrm{ref}} \tag{2}$$

where

- ⋆ $\mathrm{F}(\mathrm{r}) = \left(\boldsymbol{F}(\widehat{\boldsymbol{x}}_i; \mathrm{r})\right)_{i=1,\ldots,M}$
- ⋆ $\mathrm{E}_\infty^{\mathrm{mea}} = \left(\boldsymbol{E}_\infty^{\mathrm{mea}}(\widehat{\boldsymbol{x}}_i)\right)_{i=1,\ldots,M}$.

- **Solve the nonlinear equation using Newton's method with initial guess r_0 and**

$$\mathrm{r}_{k+1} = \mathrm{r}_k + \mathrm{q}_k, \quad \mathrm{G}'_{\mathrm{r}}(\mathrm{r}_k)\mathrm{q}_k = \mathrm{G}(\mathrm{r}_k), \quad k = 0, 1, 2, \ldots,$$

where

- ⋆ $\mathrm{G}(\mathrm{r}) = \mathrm{E}_\infty^{\mathrm{mea}} - \mathrm{F}(\mathrm{r})$
- ⋆ G'_{r} **is the Frechet derivative of G with respect to r.**
- ⋆ **The Frechet derivative is computable by solving direct scattering problems with with several different boundary conditions.**

Inverse Electromagnetic scattering : algebraic and linearization

- **In practice we discretize using collocation at points $\widehat{\boldsymbol{x}}_1, \ldots, \widehat{\boldsymbol{x}}_M \in \partial B$.**
- **Thus solve the discrete system**

$$\mathrm{F}(\mathrm{r}) = \mathrm{E}_\infty^{\mathrm{ref}} \tag{2}$$

where

⋆ $\mathrm{F}(\mathrm{r}) = \left(\boldsymbol{F}(\widehat{\boldsymbol{x}}_i; \mathrm{r})\right)_{i=1,\ldots,M}$

⋆ $\mathrm{E}_\infty^{\mathrm{mea}} = \left(\boldsymbol{E}_\infty^{\mathrm{mea}}(\widehat{\boldsymbol{x}}_i)\right)_{i=1,\ldots,M}$.

- **Solve the nonlinear equation using Newton's method with initial guess r_0 and**

$$\mathrm{r}_{k+1} = \mathrm{r}_k + \mathrm{q}_k, \quad \mathrm{G}'_{\mathrm{r}}(\mathrm{r}_k)\mathrm{q}_k = \mathrm{G}(\mathrm{r}_k), \quad k = 0, 1, 2, \ldots,$$

where

⋆ $\mathrm{G}(\mathrm{r}) = \mathrm{E}_\infty^{\mathrm{mea}} - \mathrm{F}(\mathrm{r})$

⋆ G'_{r} **is the Frechet derivative of G with respect to r.**

⋆ **The Frechet derivative is computable by solving direct scattering problems with with several different boundary conditions.**

⋆ **Another approach is to compute the Frechet derivative is by use finite difference approximation, which also require solutions of several direct problems.**

Inverse Electromagnetic scattering : least-squares & regularization

- ⋆ **The systems of equations that must be solved for q_k is in general overdetermined and must be solved in a least squares sense.**

Inverse Electromagnetic scattering : least-squares & regularization

- ⋆ **The systems of equations that must be solved for q_k is in general overdetermined and must be solved in a least squares sense.**

- ⋆ **The system of equations is badly conditioned because of the ill-posedness of the inverse problems.**

Inverse Electromagnetic scattering : least-squares & regularization

- ⋆ **The systems of equations that must be solved for q_k is in general overdetermined and must be solved in a least squares sense.**

- ⋆ **The system of equations is badly conditioned because of the ill-posedness of the inverse problems.**

- ⋆ **Thus in practice we compute q_k using Tikhonov regularization, by solving**

$$\left(\alpha_k \mathrm{I} + \overline{\mathrm{G}'_{\mathrm{r}}(\mathrm{r}_k)}^T \mathrm{G}'_{\mathrm{r}}(\mathrm{r}_k)\right) \mathrm{q}_k = \overline{\mathrm{G}'_{\mathrm{r}}(\mathrm{r}_k)}^T \mathrm{G}(\mathrm{r}_k)$$

for some appropriately chosen regularization parameter $\alpha_k > 0$.

Inverse Electromagnetic scattering : least-squares & regularization

- ⋆ **The systems of equations that must be solved for q_k is in general overdetermined and must be solved in a least squares sense.**

- ⋆ **The system of equations is badly conditioned because of the ill-posedness of the inverse problems.**

- ⋆ **Thus in practice we compute q_k using Tikhonov regularization, by solving**

$$\left(\alpha_k \mathrm{I} + \overline{\mathrm{G}'_{\mathrm{r}}(\mathrm{r}_k)}^T \mathrm{G}'_{\mathrm{r}}(\mathrm{r}_k)\right) \mathrm{q}_k = \overline{\mathrm{G}'_{\mathrm{r}}(\mathrm{r}_k)}^T \mathrm{G}(\mathrm{r}_k)$$

  for some appropriately chosen regularization parameter $\alpha_k > 0$.

- ⋆ **An easily computed measure of the convergence of the Newton iteration is $\|\mathrm{G}(\mathrm{r}_k)\|_\infty$.**

Inverse Electromagnetic scattering : least-squares & regularization

- ⋆ **The systems of equations that must be solved for q_k is in general overdetermined and must be solved in a least squares sense.**

- ⋆ **The system of equations is badly conditioned because of the ill-posedness of the inverse problems.**

- ⋆ **Thus in practice we compute q_k using Tikhonov regularization, by solving**

$$\left(\alpha_k \mathrm{I} + \overline{\mathrm{G}'_{\mathrm{r}}(\mathrm{r}_k)}^T \mathrm{G}'_{\mathrm{r}}(\mathrm{r}_k)\right) \mathrm{q}_k = \overline{\mathrm{G}'_{\mathrm{r}}(\mathrm{r}_k)}^T \mathrm{G}(\mathrm{r}_k)$$

  for some appropriately chosen regularization parameter $\alpha_k > 0$.

- ⋆ **An easily computed measure of the convergence of the Newton iteration is $\|\mathrm{G}(\mathrm{r}_k)\|_\infty$.**

- ⋆ **There are several approach to choose regularization parameters.**

Inverse Electromagnetic scattering : least-squares & regularization

- ⋆ **The systems of equations that must be solved for q_k is in general overdetermined and must be solved in a least squares sense.**

- ⋆ **The system of equations is badly conditioned because of the ill-posedness of the inverse problems.**

- ⋆ **Thus in practice we compute q_k using Tikhonov regularization, by solving**
$$\left(\alpha_k \mathrm{I} + \overline{\mathrm{G}'_\mathrm{r}(\mathrm{r}_k)}^T \mathrm{G}'_\mathrm{r}(\mathrm{r}_k)\right) \mathrm{q}_k = \overline{\mathrm{G}'_\mathrm{r}(\mathrm{r}_k)}^T \mathrm{G}(\mathrm{r}_k)$$
for some appropriately chosen regularization parameter $\alpha_k > 0$.

- ⋆ **An easily computed measure of the convergence of the Newton iteration is $\|\mathrm{G}(\mathrm{r}_k)\|_\infty$.**

- ⋆ **There are several approach to choose regularization parameters.**

- ⋆ **In our experiments the regularization parameter at step $k+1$ is chosen according to the residual from the previous step:**
$$\alpha_k = \min\{10^{-1}, \|\mathrm{G}(\mathrm{r}_k)\|_\infty\}.$$

Inverse Electromagnetic scattering : least-squares & regularization

- ⋆ **The systems of equations that must be solved for** q_k **is in general overdetermined and must be solved in a least squares sense.**

- ⋆ **The system of equations is badly conditioned because of the ill-posedness of the inverse problems.**

- ⋆ **Thus in practice we compute** q_k **using Tikhonov regularization, by solving**
$$\left(\alpha_k \mathrm{I} + \overline{\mathrm{G}_\mathrm{r}'(\mathrm{r}_k)}^T \mathrm{G}_\mathrm{r}'(\mathrm{r}_k)\right) \mathrm{q}_k = \overline{\mathrm{G}_\mathrm{r}'(\mathrm{r}_k)}^T \mathrm{G}(\mathrm{r}_k)$$
for some appropriately chosen regularization parameter $\alpha_k > 0$**.**

- ⋆ **An easily computed measure of the convergence of the Newton iteration is** $\|\mathrm{G}(\mathrm{r}_k)\|_\infty$**.**

- ⋆ **There are several approach to choose regularization parameters.**

- ⋆ **In our experiments the regularization parameter at step** $k+1$ **is chosen according to the residual from the previous step:**
$$\alpha_k = \min\{10^{-1}, \|\mathrm{G}(\mathrm{r}_k)\|_\infty\}.$$

Inverse EM scattering : general surface, current work & simulation

⋆ **More general surface representation:**

$$\boldsymbol{x} = [q_1(\widehat{\boldsymbol{x}}), q_2(\widehat{\boldsymbol{x}}), q_3(\widehat{\boldsymbol{x}})] \qquad \boldsymbol{x} \in \partial D, \quad q_i(\widehat{\boldsymbol{x}}) = \sum_{l=1}^{N_q} \sum_{|j|\le l} \mathrm{r}_{lj}^{i} \boldsymbol{Y}_{l,j}.$$

Inverse EM scattering : general surface, current work & simulation

⋆ **More general surface representation:**

$$\boldsymbol{x} = [q_1(\widehat{\boldsymbol{x}}), q_2(\widehat{\boldsymbol{x}}), q_3(\widehat{\boldsymbol{x}})] \qquad \boldsymbol{x} \in \partial D, \quad q_i(\widehat{\boldsymbol{x}}) = \sum_{l=1}^{N_q} \sum_{|j|\le l} \mathrm{r}_{lj}^{i} \boldsymbol{Y}_{l,j}.$$

⋆ **The inverse problem can now be formulated in terms of the mapping**

$$\boldsymbol{F} : \mathbb{C}^{3(N_q+1)^2} \longrightarrow \underline{C}(\partial B), \quad \mathrm{r} \mapsto \boldsymbol{E}_{\infty,n}$$

Inverse EM scattering : general surface, current work & simulation

⋆ **More general surface representation:**

$$\boldsymbol{x} = [q_1(\widehat{\boldsymbol{x}}), q_2(\widehat{\boldsymbol{x}}), q_3(\widehat{\boldsymbol{x}})] \qquad \boldsymbol{x} \in \partial D, \quad q_i(\widehat{\boldsymbol{x}}) = \sum_{l=1}^{N_q} \sum_{|j| \le l} \mathrm{r}_{lj}^{i} \boldsymbol{Y}_{l,j}.$$

⋆ **The inverse problem can now be formulated in terms of the mapping**

$$\boldsymbol{F} : \mathbb{C}^{3(N_q+1)^2} \to \underline{C}(\partial B), \quad \mathrm{r} \mapsto \boldsymbol{E}_{\infty,n}$$

⋆ **Our direct EM solver allows such representations and solutions for resonance region case takes only a few minutes.**

⋆ **Some preliminary simulation results using both the representations (Ganesh & Hawkins, 2007/2008):**

Inverse EM scattering : reconstruction of bean using star-shaped rep

Inverse EM scattering : reconstruction of bean using general rep.

Inverse EM scattering : reconstruction of ellipse using star-shaped

Inverse EM scattering : reconstruction of ellipse using general rep

Inverse EM scattering : reconstruction of fount using star-shaped