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Plane Wave Discontinuous Galerkin Methods

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also based on work of:

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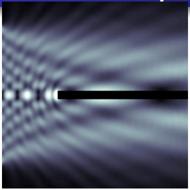
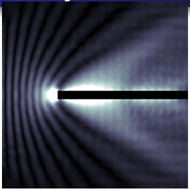
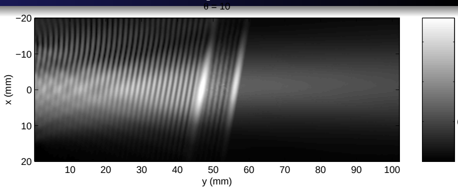
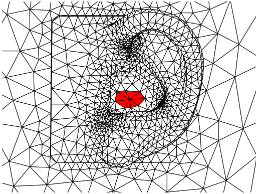
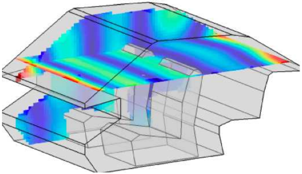
UWVF examples by Tomi Huttunen, Kuopio, Finland

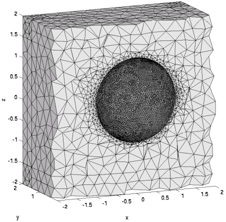


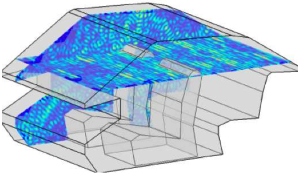












1.1 Model Problems

☞ Propagation of time-harmonic acoustic waves in homogeneous medium

➤ Helmholtz equation

Given: ▷ angular frequency (wave number) $\omega > 0$ ($[\omega] = \mathrm{m}^{-1}$)
▷ bounded computational domain $\Omega \subset \mathbb{R}^2$
▷ source term $f \in L^2(\Omega)$ (supported in parts of Ω)

- 1D model problem:

$$-u'' - \omega^2 u = f \quad \text{in }]-1,1[\,, \quad \pm u'(\pm 1) - i\omega u(\pm 1) = 0\,. \tag{1.1.1}$$

- 2D model problem: $\Omega \subset \mathbb{R}^2$ bounded polygon

$$\begin{aligned} -\Delta u - \omega^2 u &= f \quad \text{in } \Omega\,, \\ \nabla u \cdot \boldsymbol{n} - i\omega u &= g \quad \text{on } \partial\Omega\,. \end{aligned} \tag{1.1.2}$$

"Real world":

$$-\nabla \cdot (\boldsymbol{A}(\boldsymbol{x})\nabla u) - \omega^2 n(\boldsymbol{x}) u = f \quad \text{in } \Omega\,,$$

PML-ABC at cut-off boundary $\partial\Omega$.

1.2 Discontinuous Galerkin Discretization

(1.1.2) ▷ first order system:

$$\begin{aligned} i\omega\,\boldsymbol{\sigma} &= \nabla u && \text{in } \Omega, \\ i\omega\,u - \nabla\cdot\boldsymbol{\sigma} &= \frac{1}{i\omega} f && \text{in } \Omega, \\ i\omega\boldsymbol{\sigma}\cdot\boldsymbol{n} + i\omega\,u &= g && \text{on } \partial\Omega. \end{aligned} \tag{1.2.1}$$

+ partitioning $\mathcal{T}_h = \{K\}$ of Ω into cells (⟶ finite element mesh)

← test & integrate by parts locally

$$\begin{aligned} &\int_K i\omega\,\boldsymbol{\sigma}\cdot\overline{\boldsymbol{\tau}}\,\mathrm{d}V + \int_K u\,\overline{\nabla\cdot\boldsymbol{\tau}}\,\mathrm{d}V - \int_{\partial K} u\,\overline{\boldsymbol{\tau}\cdot\boldsymbol{n}}\,\mathrm{d}S = 0 \quad \forall\boldsymbol{\tau}\in\mathbf{H}(\mathrm{div};K) \\ &\int_K i\omega\,u\,\overline{v}\,\mathrm{d}V + \int_K \boldsymbol{\sigma}\cdot\overline{\nabla v}\,\mathrm{d}V - \int_{\partial K} \boldsymbol{\sigma}\cdot\boldsymbol{n}\,\overline{v}\,\mathrm{d}S = \frac{1}{i\omega}\int_K f\,\overline{v}\,\mathrm{d}V \quad \forall v\in H^1(K)\,. \end{aligned} \tag{1.2.2}$$

+ numerical fluxes $\hat{u}$, $\hat{\boldsymbol{\sigma}}$ on cell interfaces

$$\begin{aligned} &\int_K i\omega\,\boldsymbol{\sigma}_h\cdot\overline{\boldsymbol{\tau}}_h\,\mathrm{d}V + \int_K u_h\,\overline{\nabla\cdot\boldsymbol{\tau}}_h\,\mathrm{d}V - \int_{\partial K} \hat{u}_h\,\overline{\boldsymbol{\tau}_h\cdot\boldsymbol{n}}\,\mathrm{d}S = 0 \quad \forall\boldsymbol{\tau}_h\in\boldsymbol{\Sigma}_h(K) \\ &\int_K i\omega\,u_h\,\overline{v}_h\,\mathrm{d}V + \int_K \boldsymbol{\sigma}_h\cdot\overline{\nabla v}_h\,\mathrm{d}V - \int_{\partial K} \hat{\boldsymbol{\sigma}}_h\cdot\boldsymbol{n}\,\overline{v}_h\,\mathrm{d}S = \frac{1}{i\omega}\int_K f\,\overline{v}_h\,\mathrm{d}V \;\forall v_h\in V_h(K)\,. \end{aligned} \tag{1.2.3}$$

Here: $\Sigma_h(K) \;\rightarrow\;$ trial/test space for cell fluxes
$V_h(K) \;\rightarrow\;$ local trial/test space for u

Note: **no interelement continuity conditions (→ discontinuous Galkerin)**

Traditional choice: polynomial spaces $\Sigma_h(K)$, $V_h(K)$ (fixed or variable degree)

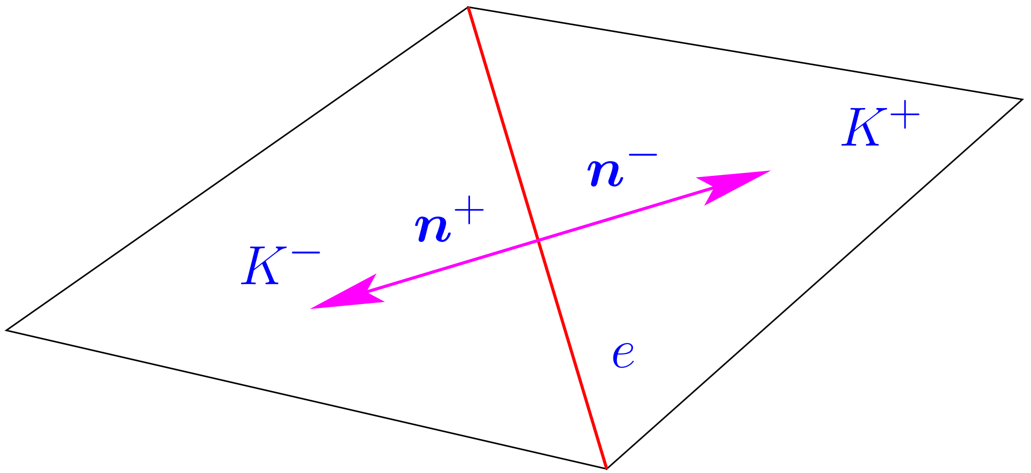
D. Arnold, F. Brezzi, B. Cockburn and L. Marini (2002), ‘Unified analysis of discontinuous Galerkin methods for elliptic problems’, *SIAM J. Numer. Anal.* **39**(5), 1749–1779.

P. Castillo, B. Cockburn, I. Perugia and D. Schötzau (2000), ‘An a priori error analysis of the local discontinuous galerkin method for elliptic problems’, *SIAM J. Numer. Anal.* **38**(5), 1676–1706.

✎ “DG-speak”

avarages:
$$\{\!\{u_h\}\!\} := \tfrac{1}{2}(u_h^+ + u_h^-)\,,$$
$$\{\!\{\boldsymbol{\sigma}_h\}\!\} := \tfrac{1}{2}(\boldsymbol{\sigma}_h^+ + \boldsymbol{\sigma}_h^-)\,,$$

jumps:
$$[\![u_h]\!] := u_h^+\boldsymbol{n}^+ + u_h^-\boldsymbol{n}^-\,,$$
$$[\![\boldsymbol{\sigma}_h]\!] := \boldsymbol{\sigma}_h^+\cdot\boldsymbol{n}^+ + \boldsymbol{\sigma}_h^-\cdot\boldsymbol{n}^-\,.$$



Common nmerical fluxes (with polynomial spaces):

penalty parameter

interior penalty (IP) DG: $\hat{u}_h = \{\!\{u_h\}\!\}$, $\hat{\boldsymbol{\sigma}}_h = \{\!\{\nabla u_h\}\!\} - \alpha(\boldsymbol{x})[\![u_h]\!]$,

local DG (LDG): $\hat{u}_h = \{\!\{u_h\}\!\} - \beta[\![u_h]\!]$, $\hat{\boldsymbol{\sigma}} = \{\!\{\boldsymbol{\sigma}_h\}\!\} - \beta[\![u_h]\!] - \alpha[\![\boldsymbol{\sigma}_h]\!]$.

mixed DG: $\hat{u}_h = \{\!\{u_h\}\!\} + \boldsymbol{\gamma} \cdot [\![u_h]\!] - \beta\,[\![\boldsymbol{\sigma}_h]\!]$, $\hat{\boldsymbol{\sigma}}_h = \{\!\{\boldsymbol{\sigma}_h\}\!\} - \alpha\,[\![u_h]\!] - \boldsymbol{\gamma}\,[\![\boldsymbol{\sigma}_h]\!]$.

Terminology: $\hat{u}_h = \hat{u}_h(\{\!\{u_h\}\!\}, [\![u_h]\!])$ ➤ primal DG

1.3 The Ultra-weak Connection

O. Cessenat and B. Despres (1998), ‘Application of an ultra weak variational formulation of elliptic PDEs to the two-dimensional Helmholtz equation’, *SIAM J. Numer. Anal.* **35**(1), 255–299.

O. Cessenat and B. Despres (2003), ‘Using plane waves as base functions for solving time harmonic equations with the ultra weak variational formulation’, *J. Computational Acoustics* **11**, 227–238.

T. Huttunen, P. Monk and J. Kaipio (2002), ‘Computational aspects of the ultra-weak variational formulation’, *J. Comp. Phys.* **182**(1), 27–46 **& many more**

A. Buffa and P. Monk (2007), ‘Error estimates for the ultra weak variational formulation of the Helmholtz equation’, Preprint

$$
\begin{aligned}
&\int_K i\omega\,\boldsymbol{\sigma}_h\cdot\overline{\boldsymbol{\tau}}_h\,\mathrm{d}V + \int_K u_h\,\overline{\nabla\cdot\boldsymbol{\tau}}_h\,\mathrm{d}V - \int_{\partial K}\hat{u}_h\,\overline{\boldsymbol{\tau}_h\cdot\boldsymbol{n}}\,\mathrm{d}S = 0 \quad \forall\boldsymbol{\tau}_h\in\boldsymbol{\Sigma}_h(K)\\
&\int_K i\omega\,u_h\,\overline{v}_h\,\mathrm{d}V + \int_K \boldsymbol{\sigma}_h\cdot\overline{\nabla v}_h\,\mathrm{d}V - \int_{\partial K}\hat{\boldsymbol{\sigma}}_h\cdot\boldsymbol{n}\,\overline{v}_h\,\mathrm{d}S = \frac{1}{i\omega}\int_K f\,\overline{v}_h\,\mathrm{d}V\ \forall v_h\in V_h(K)\,.
\end{aligned}
\tag{1.2.3}
$$

$\longleftarrow$ another integration by parts

$$\int_K \sigma_h \cdot \overline{\boldsymbol{\tau}}_h \,\mathrm{d}V = \frac{1}{i\omega}\int_K \nabla u_h \cdot \overline{\boldsymbol{\tau}}_h \,\mathrm{d}V - \frac{1}{i\omega}\int_{\partial K} (u_h - \hat{u}_h)\,\overline{\boldsymbol{\tau}_h \cdot \boldsymbol{n}}\,\mathrm{d}S \;. \tag{1.3.1}$$

Assume $\nabla_h V_h \subseteq \boldsymbol{\Sigma}_h$

$$\int_K \nabla u_h \cdot \nabla \overline{v}_h - \omega^2 u_h \overline{v}_h \,\mathrm{d}V - \int_{\partial K} (u_h - \hat{u}_h)\,\overline{\nabla v_h \cdot \boldsymbol{n}}\,\mathrm{d}S - \int_{\partial K} i\omega \hat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{n}\,\overline{v}_h \,\mathrm{d}S = \int_K f\,\overline{v}_h \,\mathrm{d}V \;. \tag{1.3.2}$$

$\longleftarrow$ another integration by parts

$$\int_K \overline{(-\Delta v_h - \omega^2 v_h)}\, u_h \,\mathrm{d}V + \int_{\partial K} \hat{u}_h\,\overline{\nabla v_h \cdot \boldsymbol{n}}\,\mathrm{d}S - \int_{\partial K} i\omega \hat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{n}\,\overline{v}_h \,\mathrm{d}S = \int_K f\,\overline{v}_h \,\mathrm{d}V \;. \tag{1.3.3}$$

Assume $\boxed{-\Delta v_h - \omega^2 v_h = 0} \quad \forall v_h \in V_h(K)\;, \quad \forall K\;.$

$$\text{(1.3.3)} \;\Rightarrow\; \int_{\partial K} \hat{u}_h\,\overline{\nabla v_h \cdot \boldsymbol{n}}\,\mathrm{d}S - \int_{\partial K} i\omega \hat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{n}\,\overline{v}_h \,\mathrm{d}S = \int_K f\,\overline{v}_h \,\mathrm{d}\boldsymbol{x}\;. \tag{1.3.4}$$

Ultra-weak fluxes:

$$\hat{\boldsymbol{\sigma}}_h = \frac{1}{i\omega}\{\!\{\nabla_h u_h\}\!\} - \frac{1}{2}[\![u_h]\!] \,, \qquad \hat{u}_h = \{\!\{u_h\}\!\} - \frac{1}{2i\omega}[\![\nabla_h u_h]\!] \,. \tag{1.3.5}$$

Note: special boundary fluxes required

Note: non-standard fluxes ($\hat{u}_h = \{\!\{u_h\}\!\} - \frac{1}{2i\omega}[\![\nabla_h u_h]\!]$)

Variational problem on skeleton of $\mathcal{T}_h$ ($\mathcal{F}^{\mathcal{I}} \hat{=}$ interior edges):

$$\begin{aligned} u_h \in V_h\colon \quad & \sum_{K\in\mathcal{T}_h} \int_{\partial K} (-\nabla u_h \cdot \boldsymbol{n} + i\omega u_h)\overline{(-\nabla v_h \cdot \boldsymbol{n} + i\omega v_h)}\,\mathrm{d}S \\ & - \int_{\mathcal{F}^{\mathcal{I}}} (-\nabla u_h^- \cdot \boldsymbol{n}^- + i\omega u_h^-)\overline{(\nabla v_h^+ \cdot \boldsymbol{n}^+ + i\omega v_h^+)}\,\mathrm{d}S \\ & - \int_{\mathcal{F}^{\mathcal{I}}} (-\nabla u_h^+ \cdot \boldsymbol{n}^+ + i\omega u_h^+)\overline{(\nabla v_h^- \cdot \boldsymbol{n}^- + i\omega v_h^-)}\,\mathrm{d}S \\ & = -2i\omega\,(f, v_h) + \int_{\mathcal{F}_h^{\mathcal{B}}} g\overline{(\nabla v_h \cdot \boldsymbol{n} + i\omega v_h)}\,\mathrm{d}S \,, \quad \forall v_h \in V_h \,. \end{aligned} \tag{1.3.6}$$

✎ notations: mesh cells K_i, $i = 1, \dots, N$, interfaces $\Sigma_{j,l} := \bar{K}_j \cap \bar{K}_l$

Define: On ∂K_j: $\mathcal{X}_j := i\omega u_{h|K_k} - \nabla u_{h|K_j} \cdot \boldsymbol{n}_j$
$\mathcal{Y}_j := i\omega v_{h|K_k} - \nabla v_{h|K_j} \cdot \boldsymbol{n}_j$
$F_j(\mathcal{Y}_j) := i\omega v_{h|K_j} - \nabla v_{h|K_j} \cdot \boldsymbol{n}_j$

▶ for an interior cell K_j ($f = 0$):

$$\int_{\partial K_j} \mathcal{X}_j \bar{\mathcal{Y}}_j \, \mathrm{d}V - \sum_l \int_{\Sigma_{j,l}} \mathcal{X}_l \bar{F}_j(\mathcal{Y}_j) \, \mathrm{d}V = 0 \; .$$

unknowns: functions on cell boundaries

▶ Variational problem on $\boxed{X := \prod_{K \in \mathcal{T}_h} L^2(\partial K)}$:

Seek $\vec{\mathcal{X}} := (\mathcal{X}_1, \ldots, \mathcal{X}_N) \in X$: $\quad \mathsf{a}(\vec{\mathcal{X}}, \vec{\mathcal{Y}}) = f(\vec{\mathcal{Y}}) \quad \forall \vec{\mathcal{Y}} \in X$,

$$\mathsf{a}(\vec{\mathcal{X}}, \vec{\mathcal{Y}}) := \sum_{j=1}^{N} \int_{\partial K_j} \mathcal{X}_j \bar{\mathcal{Y}}_j \,\mathrm{d}V - \sum_{l} \int_{\Sigma_{j,l}} \mathcal{X}_l \bar{F}_j(\mathcal{Y}_j) \,\mathrm{d}V \ ,$$

$$f(\vec{\mathcal{Y}}) := \sum_{j=1}^{N} g \bar{F}_j(\mathcal{Y}_j) \,\mathrm{d}V \ .$$

Trial/test space for the original ultra-weak formulation:

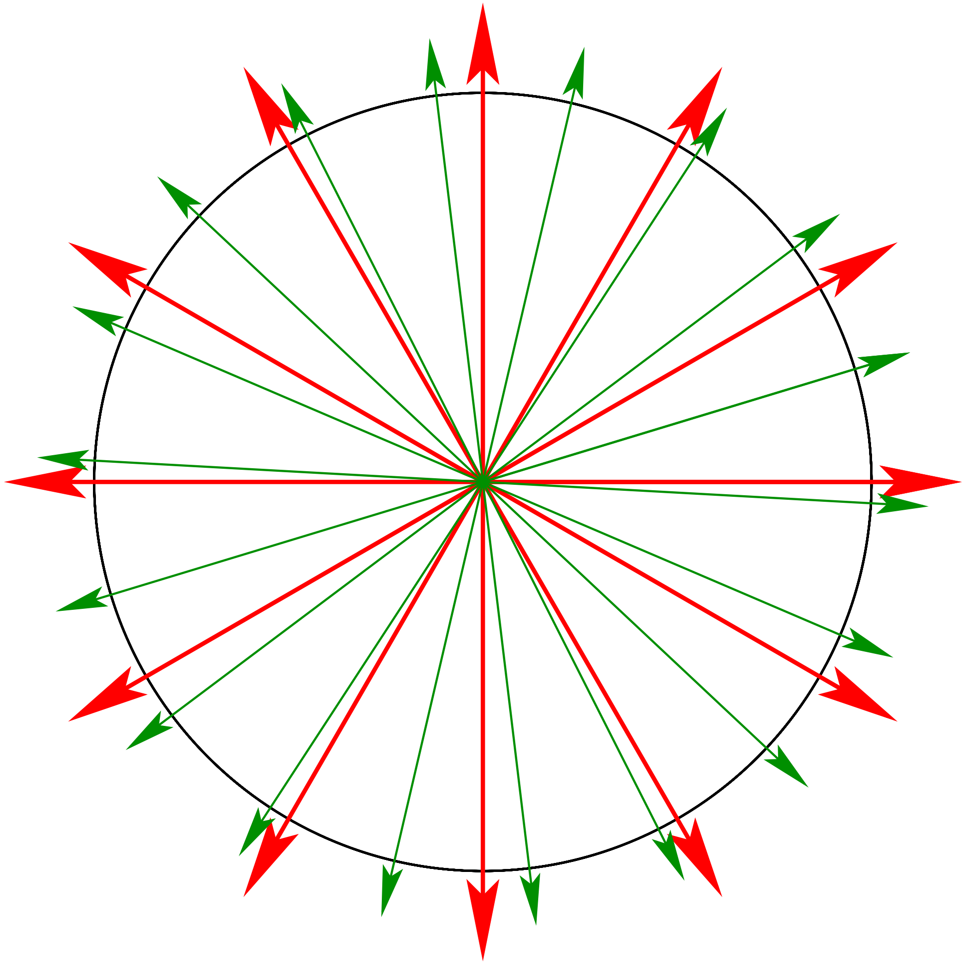
$\mathcal{T}_h \hat{=}$ triangular finite element mesh

plane wave space:

$$V_h(K) := \underbrace{\text{Span}\left\{\boldsymbol{x} \mapsto \exp(i\omega \boldsymbol{d}_j \cdot \boldsymbol{x})\right\}_{j=1}^{p}}_{=:W_p},$$

$$\boldsymbol{d}_j = \begin{pmatrix} \cos(\frac{2\pi}{p}(j-1)) \\ \sin(\frac{2\pi}{p}(j-1)) \end{pmatrix}, \quad j = 1, \ldots, p\,.$$

$p \in \mathbb{N} \hat{=}$ no. of plane wave directions



1.4 Plane waves

1.4.1 Approximation of homogeneous Helmholtz solutions

J. Melenk (1995), On Generalized Finite Element Methods, PhD thesis, University of Maryland, USA.

✎ notation: plane wave space $W_p := \mathrm{Span}\left\{\boldsymbol{x} \mapsto \exp(i\omega \boldsymbol{d}_j \cdot \boldsymbol{x})\right\}_{j=1}^{p}$, $p \in \mathbb{N}$

Theorem 1.4.1 (Proposition 8.4.14 in (Melenk 1995))**.**
D *star-shaped Lipschitz domain, exterior angles* $\geq \lambda\pi$, $u \in H^s(D)$, $s > 1$, $-\Delta u - \omega^2 u = 0$

$$\inf_{w \in W_p} |u - w|_{1,D} \leq C(D,\omega,s) \left(\frac{\log^2 p}{p}\right)^{\lambda(s-1)} \|u\|_{s,D} \; .$$

Theorem 1.4.2 (Sect. 8.6 of (Melenk 1995)).
$u \hat{=}$ *homogeneous Helmholtz solution in bounded domain* D, $D_0 \subset\subset D$ *compact,*

$$\exists C > 0,\ 0 < q < 1: \quad \inf_{w \in W_p} \|u - w\|_\infty \leq C q^p \quad \forall p \in \mathbb{N} .$$

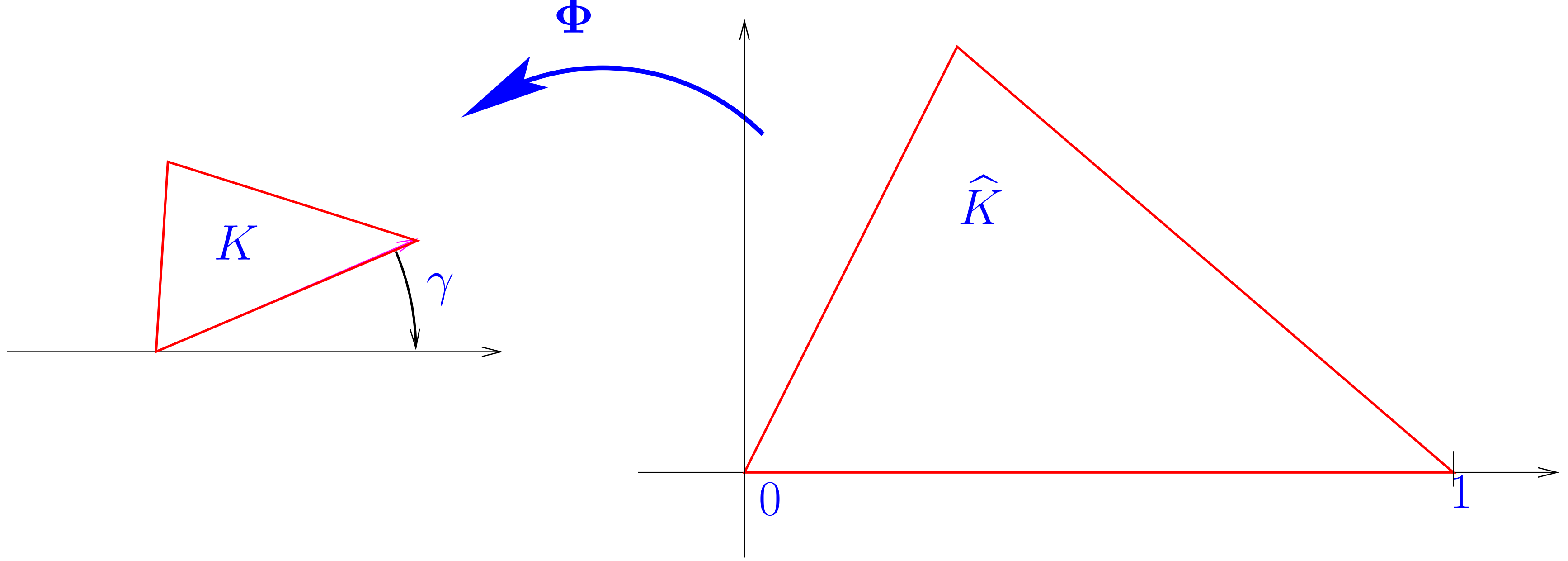
Approximation of Helmholtz solutions	:	plane waves $\sim$ spectral polynomial approximation

1.4.2 Estimates for low frequencies

☞ relevant for h-version of plane-wave DG

☞ focus: triangular meshes

❶ Technical tool: *similarity* transformation techniques (*cf.* finite element theory)



Note: change of frequency $\omega \to h_K\omega$ under similarity pullback

❷ A $\omega \to 0$ stable basis of W_p, p odd

$$\langle b_1, \dots, b_p \rangle = \underbrace{\langle \boldsymbol{x} \mapsto \exp(i\omega \boldsymbol{d}_1 \boldsymbol{x}), \dots, \boldsymbol{x} \mapsto \exp(i\omega \boldsymbol{d}_p \boldsymbol{x}) \rangle}_{\text{unstable basis}} , \quad r = 1, \dots, p \; .$$

$$b_j(\boldsymbol{x}) := (i\omega)^{-[j/2]} \sum_{k=1}^{p} (\boldsymbol{C}^{-1})_{kj} \exp(i\omega \boldsymbol{d}_k \cdot \boldsymbol{x}) \; , \qquad (1.4.1)$$

$$
\boldsymbol{C} := \begin{pmatrix}
1 & 1 & 1 & \cdots & \cdots & 1 \\
\cos(\varphi_1) & \cos(\varphi_2) & \cos(\varphi_3) & \cdots & \cdots & \cos(\varphi_p) \\
\sin(\varphi_1) & \sin(\varphi_2) & \sin(\varphi_3) & \cdots & \cdots & \sin(\varphi_p) \\
\cos(2\varphi_1) & \cos(2\varphi_2) & \cos(2\varphi_3) & \cdots & \cdots & \cos(2\varphi_p) \\
\sin(2\varphi_1) & \sin(2\varphi_2) & \sin(2\varphi_3) & \cdots & \cdots & \sin(2\varphi_p) \\
\vdots & \vdots & \vdots & & & \vdots \\
\cos(m\varphi_1) & \cos(m\varphi_2) & \cos(m\varphi_3) & \cdots & \cdots & \cos(m\varphi_p) \\
\sin(m\varphi_1) & \sin(m\varphi_2) & \sin(m\varphi_3) & \cdots & \cdots & \sin(m\varphi_p)
\end{pmatrix} ,
$$

✎ directional angles: $\boldsymbol{d}_j := \begin{pmatrix} \cos(\varphi_j) \\ \sin(\varphi_j) \end{pmatrix}$, $j = 1, \dots, p$

Observation: coefficients $(\boldsymbol{C}^{-1})_{kj}$ do not depend on ω **!**

$$
b_j(\boldsymbol{x}) = \frac{2^{-[j/2]}}{[j/2]!} \left\{ \begin{matrix} \mathrm{Re} \\ \mathrm{Im} \end{matrix} \right\} (x+iy)^{[j/2]} + \omega R_j(\omega, \boldsymbol{x}) \quad \begin{matrix} \text{for even } j\ , \\ \text{for odd } j\ , \end{matrix} \tag{1.4.2}
$$

$R_j(\omega, \boldsymbol{x})$ uniformly bounded on $\mathbb{R}^+ \times \Omega$.

For $\omega \longrightarrow 0$ plane waves approach harmonic polynomials

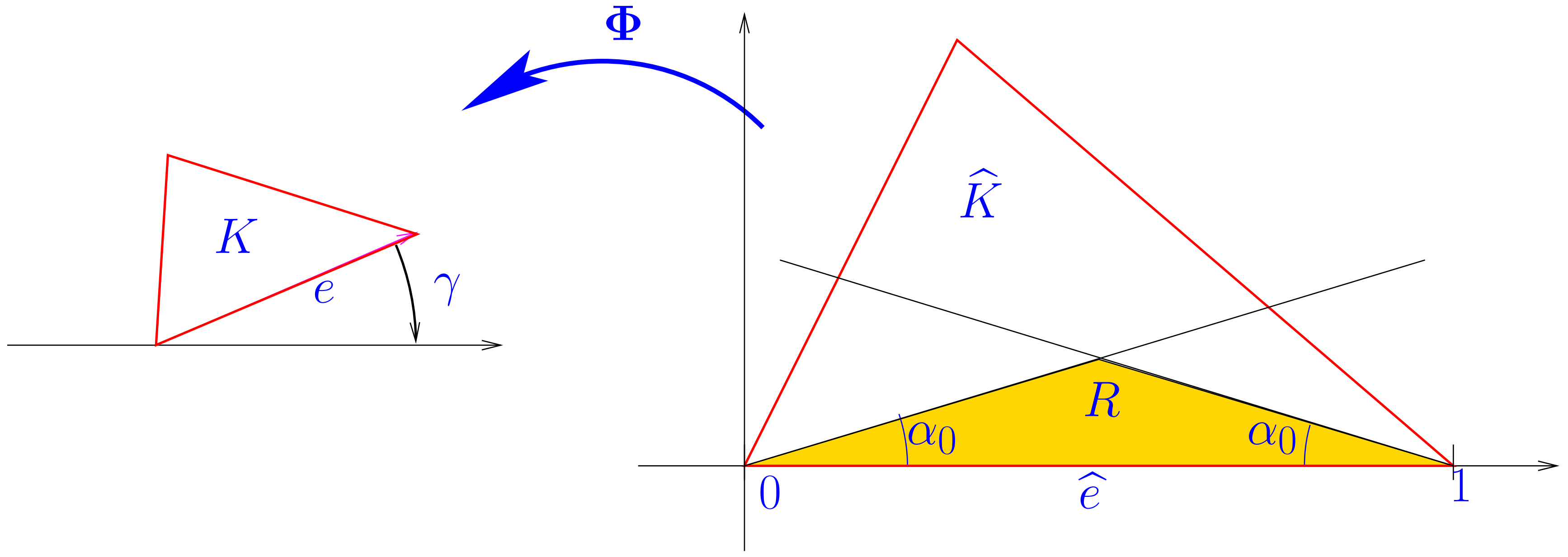
Lemma 1.4.3 (Inverse trace inequality for plane waves)**.**
Assume "shape regularity"of triangle K: minimal angle condition $\alpha_{\min}(K) \geq \alpha_0$
Then $\exists\, C > 0$ *only depending on* p

$$\|v\|_{0,\partial K} \leq C h_K^{-1/2} \|v\|_{0,K} \qquad \forall v \in W_p(K),\ \forall\omega\ .$$

Proof. ($e \mathrel{\hat{=}}$ edge of K)

① similarity mapping $\mathbf{\Phi} : \widehat{K} \mapsto K$, $\Phi^{-1}(e) = \widehat{e} := \left[\binom{-1}{0}, \binom{1}{0}\right]$

② prove inverse trace estimate for $\widehat{e}$ and R



③ C^2 = largest eigenvalue $\lambda_{\max}(\omega)$ of generalized eigenvalue problem

$$\mathbf{T}x = \lambda \mathbf{M}x \ ,$$

$$\mathbf{T}_{jk} = \int_{\widehat{e}} w_j(\boldsymbol{x})\overline{w}_k(\boldsymbol{x})\,\mathrm{d}S \quad , \quad \mathbf{M}_{jk} = \int_R w_j(\boldsymbol{x})\overline{w}_k(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \ ,$$

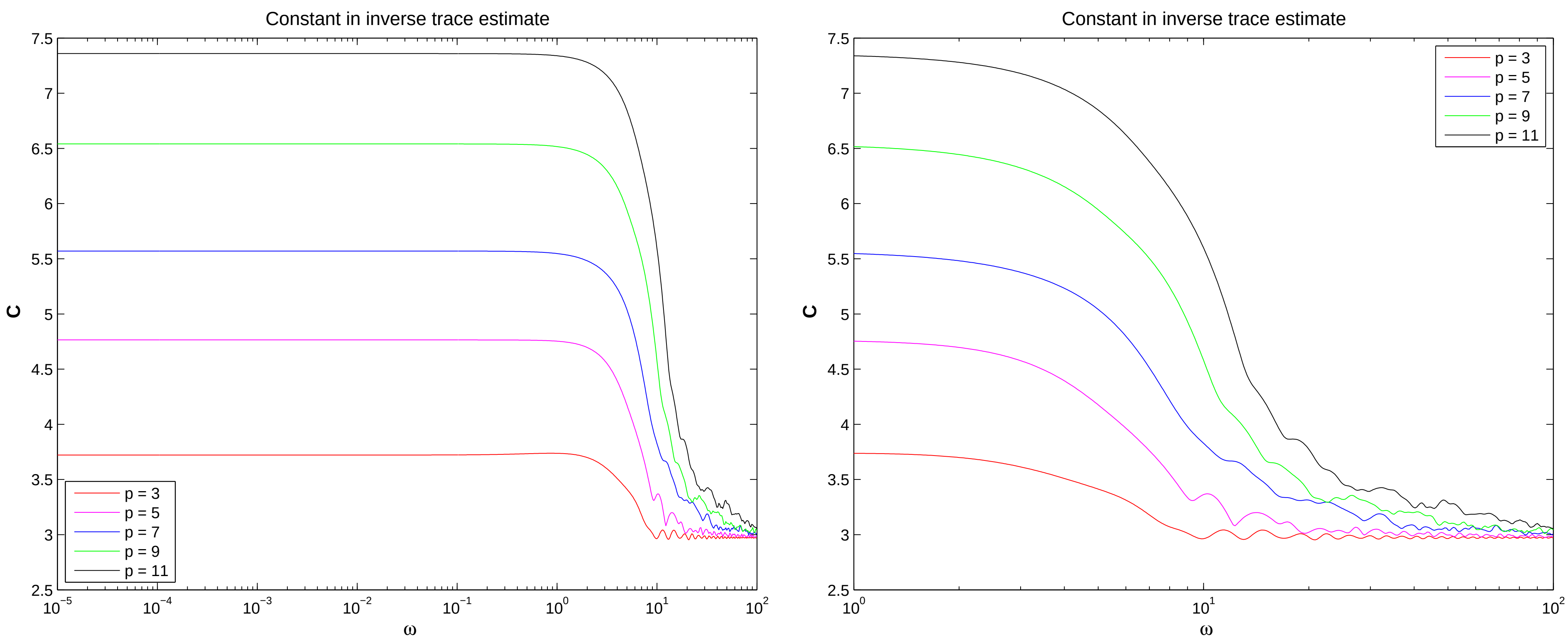
$\{w_1, \ldots, w_p\} \hat{=}$ any basis of W_p ($\rightarrow$ does not affect eigenvalues !)

④ use stable basis for small ω: ➢ positive limit $\lim_{\omega \rightarrow 0} \lambda_{\max}(\omega)$

⑤ use exponential basis for large ω: ➢ $\lim_{\omega \rightarrow 0} \lambda_{\max}(\omega) < \infty$

⑥ $\omega \to \lambda_{\max}(\omega)$ continuous $\Rightarrow$ $C^2 := \sup_{\omega \geq 0} \lambda_{\max}(\omega) < \infty$ □

Numerical experiment: inverse trace estimate for unit triangle



Theorem 1.4.4 (Inverse estimate for plane waves)**.**

$$\exists C = C(\alpha_0, p): \quad \|\nabla v\|_{0,K} \leq C h_K^{-1} \|v\|_{0,K} \quad \forall v \in W_p \, .$$

Proof. transformation & estimate (using inverse trace estimate) on reference triangle K

$$\begin{aligned}\int_{\hat{K}} |\nabla v|^2 \,\mathrm{d}\boldsymbol{x} &= -\int_{\hat{K}} \Delta v \cdot \bar{v} \,\mathrm{d}\boldsymbol{x} + \int_{\partial\hat{K}} \nabla v \cdot \boldsymbol{n}\, \bar{v} \,\mathrm{d}S \\ &\leq \omega^2 \int_{\hat{K}} |v|^2 \,\mathrm{d}\boldsymbol{x} + \|\nabla v\|_{0,\partial\hat{K}} \|v\|_{0,\partial\hat{K}} \\ &\leq \omega^2 \|v\|^2_{0,\hat{K}} + C \|\nabla v\|_{0,\hat{K}} \|v\|_{0,\hat{K}} \\ &\leq \left(\omega^2 + \frac{C^2}{2}\right) \|v\|^2_{0,\hat{K}} + \frac{1}{2} \|\nabla v\|^2_{0,\hat{K}} \;.\end{aligned}$$

Theorem 1.4.5 (Projection estimates for plane waves)**.**
$\mathsf{Q}_\omega \hat{=} L^2(K)$*-orthogonal projection onto* $V_h(K)$*. For* odd $p \geq 5$ *and* $l \in \{0,1,2\}$

$$\exists C = C(p,\alpha_0): \quad \|(Id - \mathsf{Q}_\omega)u\|_{l,K} \leq C h_K^{2-l} (\omega h_K + 1)^l (|u|_{1,K} + \omega^2 \|u\|_{0,K}) \quad \forall u \in H^2(K) \,.$$

1.5 Coercivity

Now: $V_h \hat{=}$ piecewise plane wave space on triangular *mesh* $\mathcal{T}_h$

$$\int_K \nabla u_h \cdot \nabla \overline{v}_h - \omega^2 u_h \overline{v}_h \,\mathrm{d}V - \int_{\partial K} (u_h - \hat{u}_h)\, \overline{\nabla v_h \cdot \boldsymbol{n}}\,\mathrm{d}S - \int_{\partial K} i\omega \hat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{n}\, \overline{v}_h \,\mathrm{d}S = \int_K f\, \overline{v}_h \,\mathrm{d}V \;. \tag{1.3.2}$$

+ generalized UWF fluxes:
(boundary fluxes ignored)

$$\begin{aligned} \hat{\boldsymbol{\sigma}}_h &= \frac{1}{i\omega}\{\!\{\nabla_h u_h\}\!\} - \alpha\, [\![u_h]\!] \;, \\ \hat{u}_h &= \{\!\{u_h\}\!\} - \frac{\beta}{i\omega}\, [\![\nabla_h u_h]\!] \;. \end{aligned} \qquad \alpha, \beta > 0 \;. \tag{1.5.1}$$

+ "DG magic formula"

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\partial u}{\partial \boldsymbol{n}_K} v \,\mathrm{d}S = \int_{\mathcal{F}_h} \{\!\{\nabla u\}\!\} [\![v]\!] \,\mathrm{d}S + \int_{\mathcal{F}_h^I} [\![\nabla u]\!] \{\!\{v\}\!\} \,\mathrm{d}S \;. \tag{1.5.2}$$

✎ notation: $\mathcal{F}_h$/$\mathcal{F}_h^I \hat{=}$ edges/interior edges of $\mathcal{T}_h$

(1.3.2) $\Leftrightarrow$ $u_h \in V_h$: $a_h(u_h, v_h) - \omega^2 (u_h, v_h) = (f, v_h) + \{$boundary terms$\}$ $\forall v_h \in V_h$.

$$
\begin{aligned}
a_h(u,v) =& (\nabla_h u, \nabla_h v) - \int_{\mathcal{F}_h^{\mathcal{I}}} [\![u]\!] \cdot \{\!\{\overline{\nabla_h v}\}\!\} \, \mathrm{d}S - \int_{\mathcal{F}_h^{\mathcal{I}}} \{\!\{\nabla_h u\}\!\} \cdot [\![\overline{v}]\!] \, \mathrm{d}S \\
& - \int_{\mathcal{F}_h^{\mathcal{B}}} (1-\delta)\, u \, \overline{\nabla_h v} \, \mathrm{d}S - \int_{\mathcal{F}_h^{\mathcal{B}}} (1-\delta)\, \nabla_h u \cdot \boldsymbol{n} \, \overline{v} \, \mathrm{d}S \\
& - \frac{1}{i\omega} \int_{\mathcal{F}_h^{\mathcal{I}}} \beta [\![\nabla_h u]\!] [\![\overline{\nabla_h v}]\!] \, \mathrm{d}S - \frac{1}{i\omega} \int_{\mathcal{F}_h^{\mathcal{B}}} (1-\delta)\, \nabla_h u \cdot \boldsymbol{n} \, \overline{\nabla_h v \cdot \boldsymbol{n}} \, \mathrm{d}S \\
& + i \int_{\mathcal{F}_h^{\mathcal{I}}} \alpha \, [\![u]\!] \cdot [\![\overline{v}]\!] \, \mathrm{d}S + i\omega \int_{\mathcal{F}_h^{\mathcal{B}}} \delta \, u \, \overline{v} \, \mathrm{d}S \,, \quad \alpha, \beta, \delta > 0 \,.
\end{aligned}
$$

$u_h \mapsto |\operatorname{Im} a_h(u_h, u_h)|$ is a norm on V_h

- existence/uniqueness of solutions of discretized problem

Lemma 1.4.3. [Inverse trace estimate for plane waves]

$$
\exists C_{\mathrm{inv}} > 0\colon \quad \|\nabla v_h\|_{0,\partial K} \leq C_{\mathrm{inv}} h_k^{-\frac{1}{2}} \|\nabla v_h\|_{0,K} \quad \forall v_h \in W_p, \forall K \in \mathcal{T}_h \,,
$$

with C_{inv} only depending on shape regularity of K (independent of $\omega > 0$).

Tool: mesh dependent "DG-norm" (without boundary terms):

$$\|v\|_{DG}^2 = \|\nabla_h v\|_{0,\Omega}^2 + \omega^{-1} \left\| \beta^{1/2} [\![\nabla_h v]\!] \right\|_{0,\mathcal{F}_h^{\mathcal{I}}}^2 + \left\| \alpha^{1/2} [\![v]\!]_N \right\|_{0,\mathcal{F}_h^{\mathcal{I}}}^2 + \omega^2 \|v\|_{0,\Omega}^2$$

does not match original UWVF !

Theorem 1.5.1. *If* $\alpha := \frac{\mathsf{a}}{h}$ *(*$h \hat{=}$ *local meshwidth),* $\mathsf{a} > C_{\mathrm{inv}}$*,* $\delta > \frac{1}{2}$

$$\exists C > 0: \quad |a_h(u_h, u_h) + \omega^2 (u_h, u_h)| \geq C \|u_h\|_{DG}^2 \quad \forall u_h \in V_h \ ,$$

with $C > 0$ *only depending on shape regularity of* $\mathcal{T}_h$*.*

Assumed below: $\alpha := \frac{\mathsf{a}}{h}$

1.6 UWVF estimate of Buffa & Monk (2007)

Error estimate in “energy norm”: $\mathcal{E}_j := u - u_{h|\partial K_j}$

$$|\mathsf{a}(\vec{\mathcal{E}}, \vec{\mathcal{E}}) \leq 2 \inf_{\vec{\mathcal{V}}_h \in X_h} \left\| \vec{\mathcal{X}} - \vec{\mathcal{V}}_h \right\|_X^2 .$$

Theorem 1.6.1 (L^2 error estimate)**.**
Suppose that the mesh is regular and quasi-uniform and that Ω *is a convex polygon. Then*

$$\|u - u_h\|_{0,\Omega} \leq Ch^{-1/2} \inf_{\vec{\mathcal{V}}_h \in X_h} \left\| \vec{\mathcal{X}} - \vec{\mathcal{V}}_h \right\|_X .$$

optimal ?

1.7 Duality techniques

Tool: strong mesh dependent “DG norm”

$$\|v\|_{DG^+}^2 = \|v\|_{DG}^2 + \omega \left\| \beta^{-1/2} \{\!\{v\}\!\} \right\|_{0,\mathcal{F}_h^{\mathcal{I}}}^2 + \left\| \mathsf{a}^{-1/2} h^{1/2} \{\!\{\nabla_h v\}\!\} \right\|_{0,\mathcal{F}_h^{\mathcal{I}}}^2 .$$

A. Schatz (1974), ‘An observation concerning Ritz-Galerkin methods with indefinite bilinear forms’, *Math. Comp.* **28**, 959–962.

L. Banjai and S. Sauter (2006), A refined Galerkin error and stability analysis for highly indefinite variational problems, Report 03-06, Institut für Mathematik, Universität Zürich, Zürich, Switzerland.

Lemma 1.7.1. *With $C > 0$ only depending on shape regularity and p*

$$\|u - u_h\|_{DG} \leq C \inf_{v_h \in V_h} \|u - v_h\|_{DG^+} + C\omega \sup_{0 \neq w_h \in V_h} \frac{|(u - u_h, w_h)|}{\|w_h\|_{0,\Omega}} .$$

attack with duality estimates

adjoint problem : $w_h \in V_h$: $$\begin{aligned} -\Delta\phi - \omega^2\phi &= w_h && \text{in } \Omega, \\ -\nabla\phi\cdot\boldsymbol{n} + i\omega\,\phi &= 0 && \text{on } \partial\Omega \end{aligned} \tag{1.7.1}$$

[consistency **&** adjoint consistency]

⋮

$$|(u-u_h, w_h)| \leq C\,\|u-u_h\|_{DG}\,\|\phi - \mathsf{Q}_\omega\phi\|_{DG^+} + |(f-\mathsf{Q}_\omega f, \phi - \mathsf{Q}_\omega\phi)|\ .$$

$\mathsf{Q}_\omega \hat{=} L^2(\Omega)$-orthogonal projection onto V_h (piecewise definition)

Theorem 1.7.2 (Proposition 8.1.4 in (Melenk 1995))**.**

Ω *convex* $\Rightarrow$
$$|\phi|_{1,\Omega} + \omega\,\|\phi\|_{0,\omega} \leq C(\Omega)\,\|w_h\|_{0,\Omega}\ ,$$
$$|\phi|_{2,\Omega} \leq C(\Omega)(1+\omega)\,\|w_h\|_{0,\Omega}\ .$$

Lemma 1.7.3 (Approximation in DG^+-norm)**.**
Provided that $h\omega^2 \leq c_0$ *with* c_0 *sufficiently small,*

$$\omega^2 \left\| \phi - \mathsf{Q}_\omega \right\|^2_{0,\Omega} \leq C h^2 c_0 (h + c_0) \left\| w_h \right\|_{0,\Omega} ,$$
$$\left\| \phi - \mathsf{Q}_\omega \phi \right\|^2_{DG^+} \leq C h (h + c_0) \left\| w_h \right\|^2_{0,\Omega} ,$$
$$\omega^2 \left\| \phi - \mathsf{Q}_\omega \phi \right\|^2_{DG^+} \leq C c_0 (h + c_0) \left\| w_h \right\|^2_{0,\Omega} .$$

$$[f = 0]: \qquad Ch\omega^2 < 1 \quad \Rightarrow \quad \left\| u - u_h \right\|_{DG} \leq C \inf_{v_h \in V_h} \left\| u - v_h \right\|_{DG^+} .$$

with $C > 0$ independent of the meshwidth and ω.

inacceptable threshold

Theorem 1.7.4 (A priori error estimates for primal plane wave DG method (h-version))**.**
If Ω *is convex,* $\omega^2 h \leq c_0$ *with* c_0 *sufficiently small, then*

$$\left\| u - u_h \right\|_{DG} \leq C h \left(|u|_{2,\Omega} + \omega^2 \left\| u \right\|_{0,\Omega} + (c_0(h + c_0))^{1/2}) \left\| f - \mathsf{Q}_\omega f \right\|_{0,\Omega} \right) .$$

1.7.1 1D plane wave duality estimate

- 1D adjoint problem:

$$-\phi'' - \omega^2\phi = w_h \quad \text{in }]-1,1[\,, \quad \pm\phi'(\pm 1) - i\omega\phi(\pm 1) = 0\,. \qquad \textbf{(1.1.1)}$$

- 1D mesh: $\mathcal{T}_h = \{]x_{j-1}, x_j[\}_{j=1}^{M}$, $M \in \mathbb{N}$, meshwidth h
- 1D plane wave space $\quad V_h(K) = \langle x \mapsto \cos(\omega x), x \mapsto \sin(\omega x)\rangle$

Fix $K =]x_{j-1}, x_j[$, $h_K := x_j - x_{j-1}$:

$$-\phi'' - \omega^2\phi = \chi_K(x)\,(\beta\cos(\omega x) + \gamma\sin(\omega x)) \quad \text{in }]-1,1[\,, \quad \pm\phi'(\pm 1) - i\omega\phi(\pm 1) = 0\,.$$

$$\blacktriangleright \quad \begin{cases} \phi(x) \in \beta x\dfrac{sin(\omega x)}{2\omega} - \gamma x\dfrac{\cos(\omega x)}{2\omega} + V_h(K) & \text{for } x \in K\,, \\ \phi \in \langle\cos(\omega x), \sin(\omega x)\rangle & \text{for } x \notin K \end{cases}$$

$$w_h(x) = \sum_{j=1}^{N} \chi_{[x_{j-1},x_j]}(x)\left(\beta_j \cos(\omega x) + \gamma_j \sin(\omega x)\right) \;, \quad -1 \leq x \leq 1$$

$$\Downarrow$$

$$(Id - \mathsf{Q}_\omega)\phi = \sum_{j=1}^{N} \chi_{[x_{j-1},x_j]}(Id - \mathsf{Q}_\omega^j)(\beta_j \left[x\frac{sin(\omega x)}{2\omega}\right] + \gamma_j \left[x\frac{\cos(\omega x)}{2\omega}\right]) \;.$$

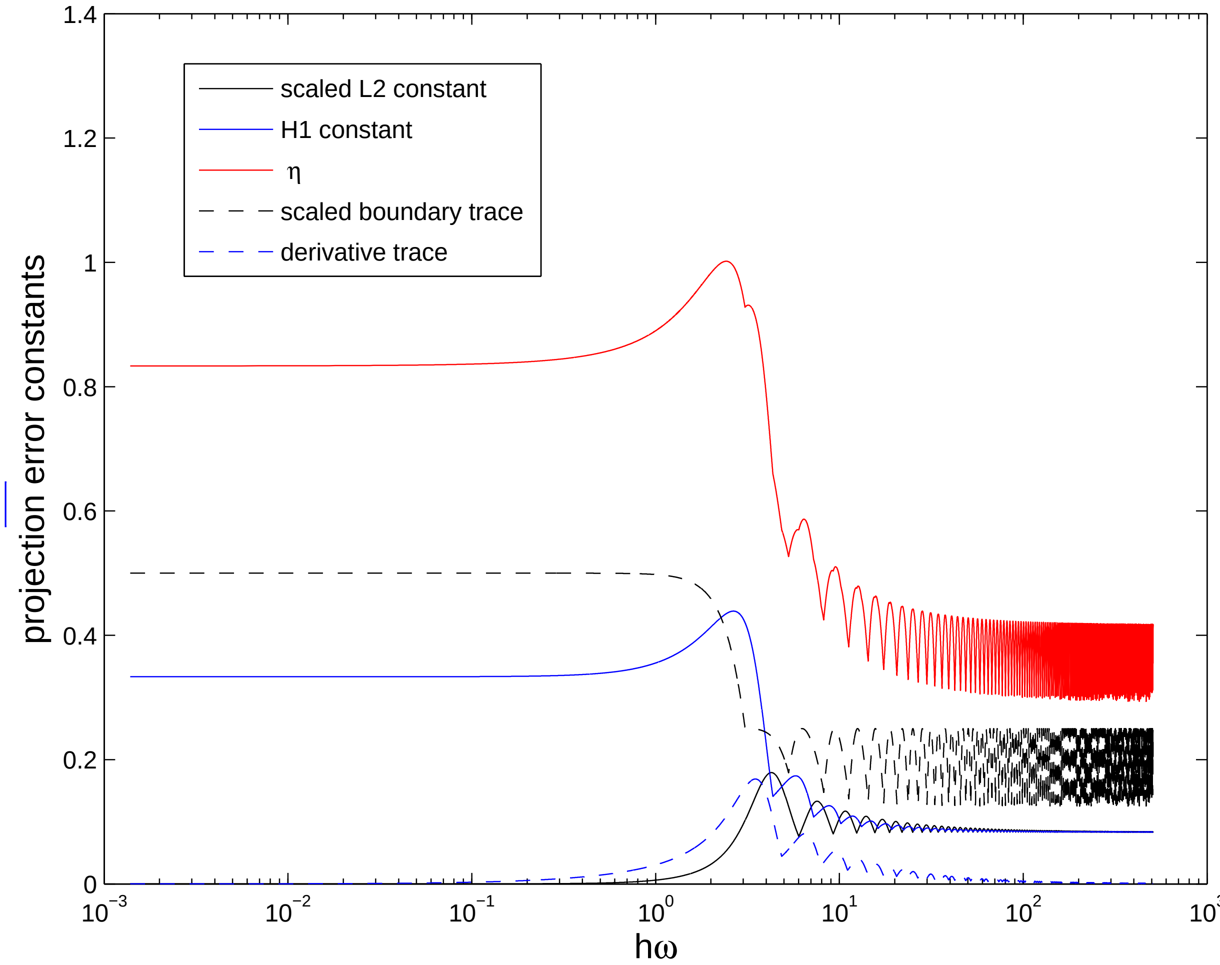
study local projection estimates for funtions $x \mapsto x\frac{sin(\omega x)}{2\omega}$, $x \mapsto x\frac{\cos(\omega x)}{2\omega}$ **!**

← MAPLE & MATLAB

$\|\phi - \mathsf{Q}_\omega\phi\|^2_{DG^+} \leq \eta(h\omega)h^2 \, \|w_h\|^2_{0,]-1,1[}$

with bounded $\eta : \mathbb{R}^+ \mapsto \mathbb{R}^+$.

η ▷

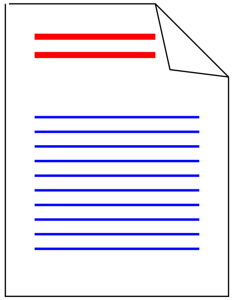


“quasi-optimality threshold” $Ch\omega < 1$

Challenge: extension to higher dimensions **?**

1.8 Hybrid plane wave method

C. FARHAT, I. HARARI, AND U. HETMANIUK, *A discontinuous Galerkin method with lagrange multipliers for the solution of Helmholtz problems in the mid-frequency regime*, Computer Methods in Applied Mechanics and Engineering, 192 (2003), pp. 1389–1419.



C. FARHAT, R. TEZAUR, AND P. WEIDEMANN-GOIRAN, *Higher-order extensions of a discontinuous Galerkin method for mid-frequency Helmholtz problems*, Int. J. Numer. Meth. Engr., 61 (2004).

R. TEZAUR AND C. FARHAT, *Three-dimensional discontinuous Galerkin elements with plane waves and Lagrange multipliers for the solution of mid-frequency Helmholtz problems*, Int. J. Numer. Meth. Engr., 66 (2006), pp. 796–815.

Multiplier space: $M := L^2(\Sigma)$, $\Sigma := \bigcup_j \partial K_j$,

Broken Sobolev space: $H^1(\mathcal{T}_h) = \bigotimes_j H^1(K_j)$

▶ Hybrid (2-field) variatoinal formulation: seek $(u,\mu) \in^1 (\mathcal{T}_h) \times M$

$$\sum_{j=1}^{N} \int_{K_j} \nabla u \cdot \nabla v \,\mathrm{d}V + \int_{\Sigma} [v] \cdot \mu \,\mathrm{d}S + \ldots = \int_{\Omega} f v \,\mathrm{d}V + \ldots \quad \forall v \in H^1(\mathcal{T}_h) \,,$$

$$\int_{\Sigma} [u] \cdot \nu \,\mathrm{d}S = 0 \quad \forall \nu \in M \,.$$

Trial- and test spaces:

- $H^1(\mathcal{T}_h \;\longrightarrow\; \bigotimes_j V_h(K_j) + \mathcal{P}_p(\mathcal{T}_h)$

  local plane wave space (→ $V_h(K_j)$)   p.w. polynomial space $\subset H^1(\Omega)$ (→ $\mathcal{P}_p(\mathcal{T}_h)$) $\longrightarrow$ "discontinuous enrichment"

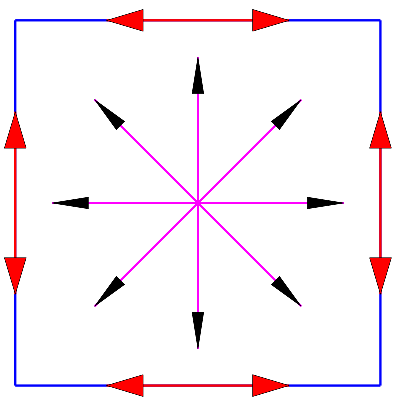
- $M \;\longrightarrow\;$ traces of plane wave spaces onto interior faces

Choice of spaces for acoustic scattering

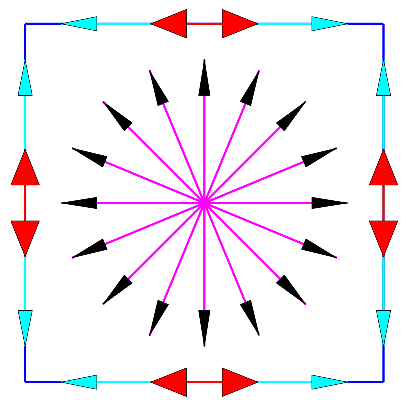
- Superposition of planar waves

$$u^E = \sum_{p=1}^{n_\theta} e^{ik\boldsymbol{\theta}_p \cdot \mathbf{x}} u_p^E$$

- Lagrange multipliers
  - Accuracy considerations $\Longrightarrow$ $\lambda \approx \dfrac{\partial u^E}{\partial \nu}$
  - Stability considerations (inf-sup condition)

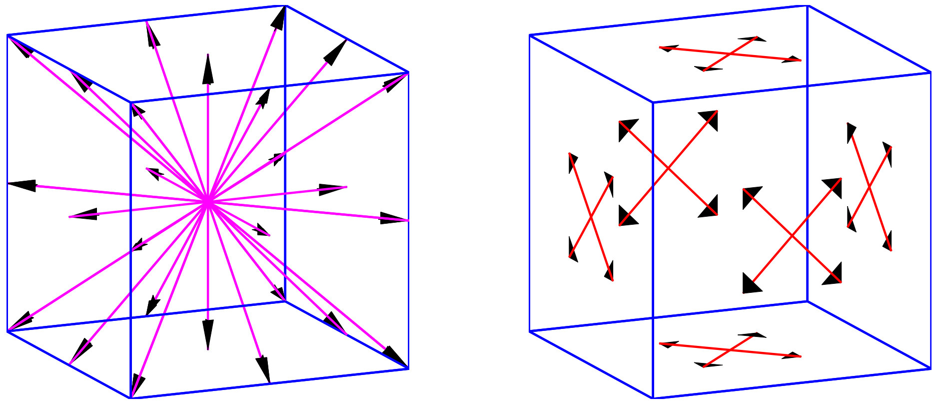


Element Q-8-2: $n_\theta = 8$, $n_\lambda = 2$

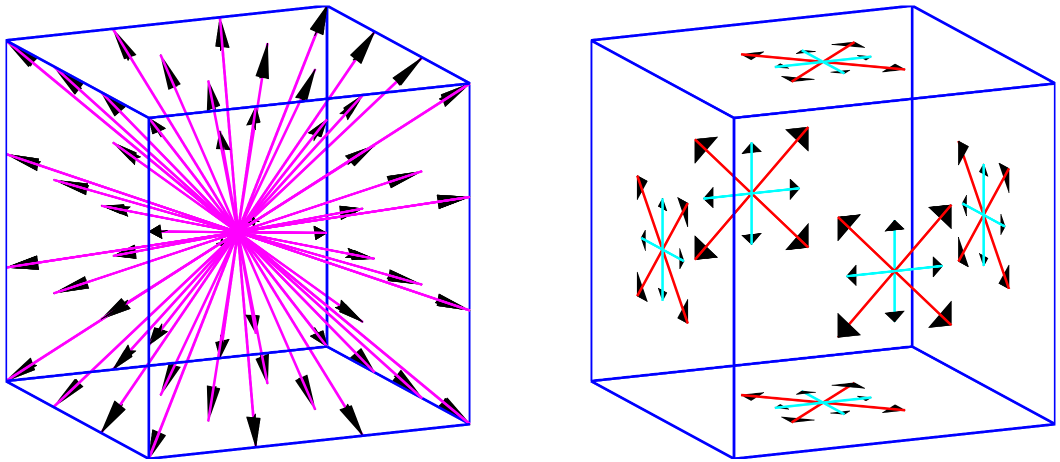


Element Q-16-4: $n_\theta = 16$, $n_\lambda = 4$

Discretization pairs for 3-d Helmholtz problems



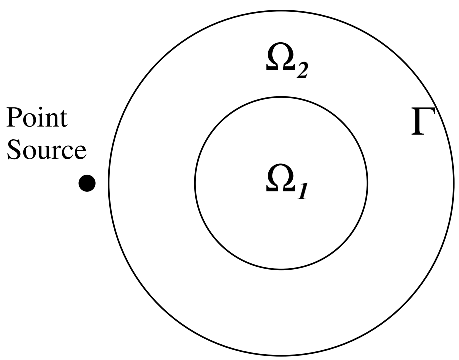
Hexahedral element H-26-4: $n_\theta = 26$, $n_\lambda = 4$



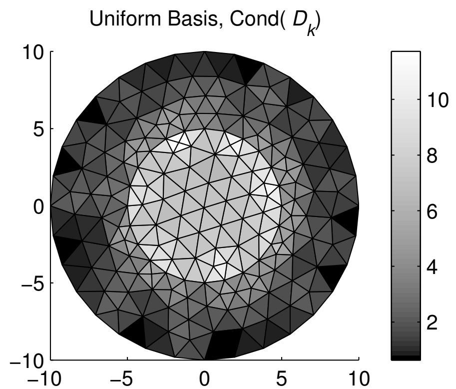
Hexahedral element H-56-8: $n_\theta = 56$, $n_\lambda = 8$

Conditioning

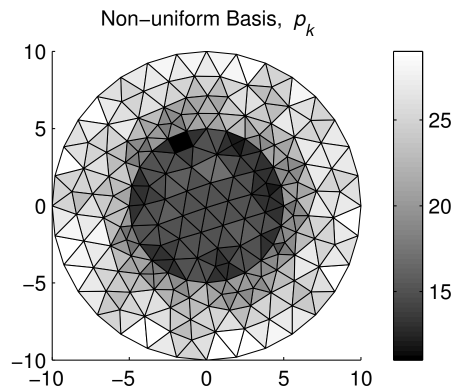
- Basic UWVF uses p directions/element. This can cause bad conditioning for B (e.g. on small elements, if κ changes,...)
- We use different p_k for element Ω_k. One possibility: chose p_k so that the condition number of the submatrix corresponding to $\int_{\partial K} \mathcal{X}\overline{\mathcal{Y}}\, ds$ is a desired maximum value.



Domain



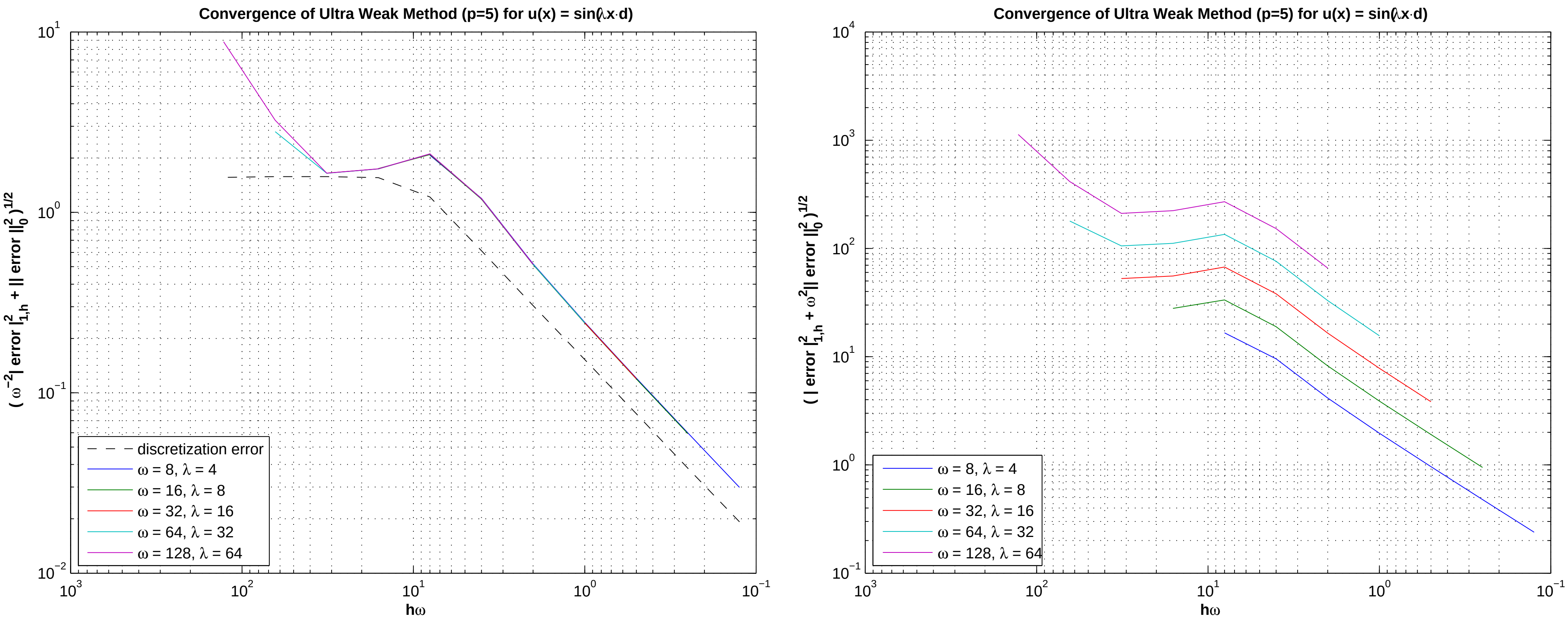
Cond. No. Uniform M



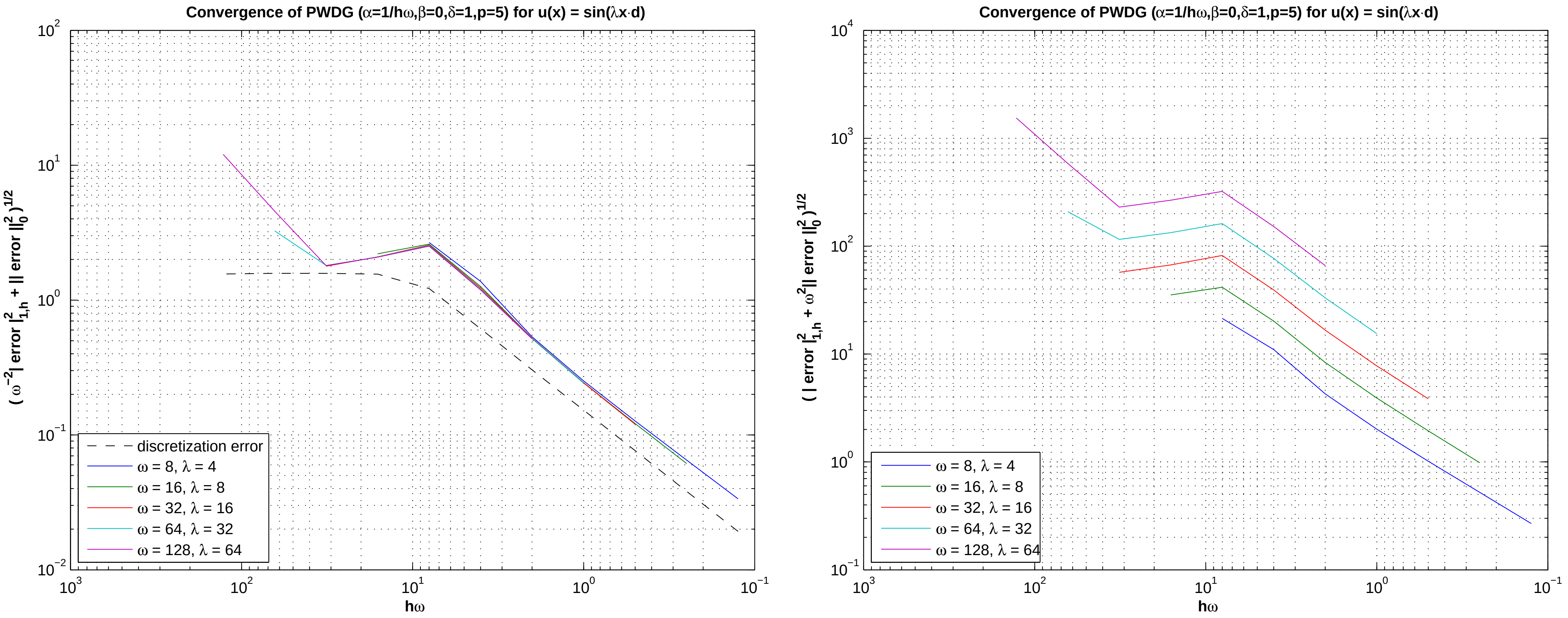
Non uniform M

1.9 Numerical experiments

- $\Omega =]0,1[^2$, unstructured triangular grids, $p = 7$
- boundary conditions & right hand side ➥ analytically prescribed u
- $u(\mathbf{x}) = \sin(\frac{1}{2}\omega\mathbf{d}\cdot\mathbf{x})$, $\mathbf{d} = \frac{1}{2}\sqrt{2}\binom{1}{1}$, (no homogeneous Helmholtz solution)

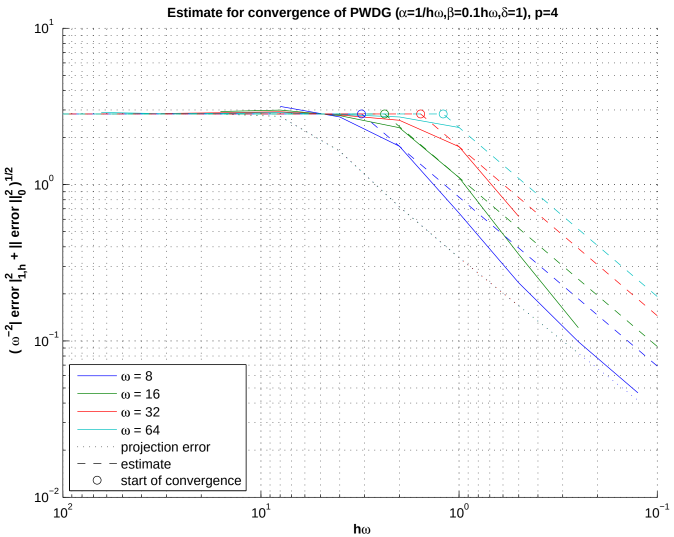


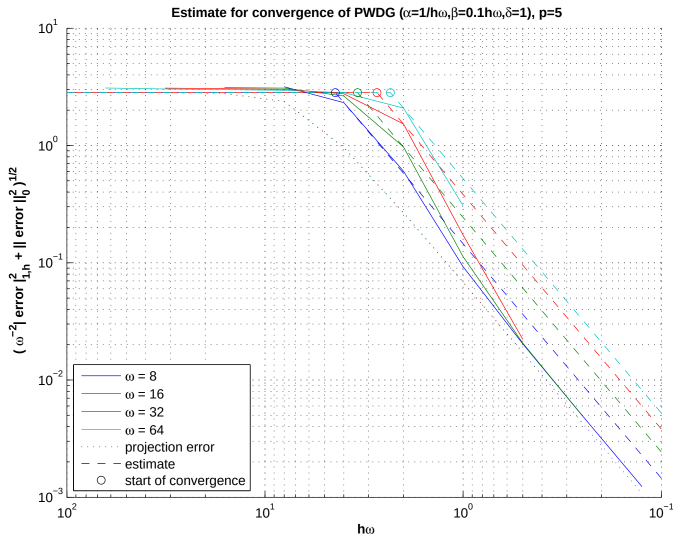
Discretization error of the original ultra-weak discretization of Cessenat and Despres

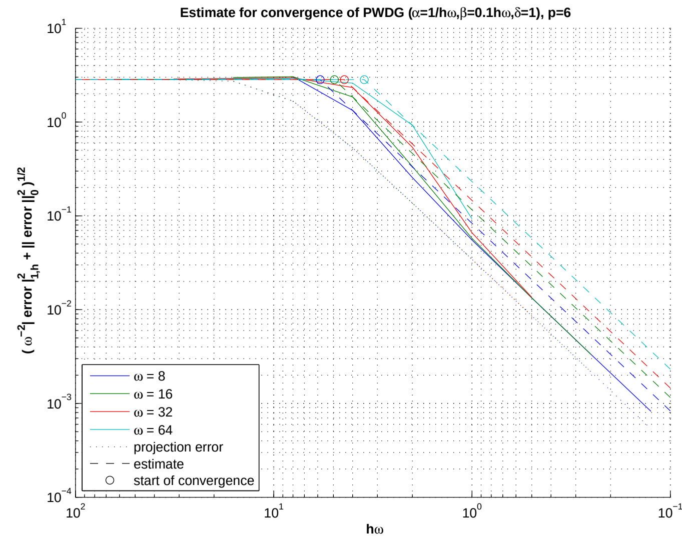


Discretization error of the plane wave DG method with $\alpha = (h\omega)^{-1}$, $\beta = 0$, $\delta = 1$

Now: u = plane wave with wave number ω ($f = 0$)







Discretization error of the plane wave DG method with $\alpha = (h\omega)^{-1}$, $\beta = 0.1h\omega$, $\delta = 1$

▶ Numerical dispersion !

Inevitable for local discretizations of Helmholtz equation **?**

I. BABUŠKA AND S. SAUTER, *Is the pollution effect of the FEM avoidable for the Helmholtz equation?*, SIAM Review, 42 (2000), pp. 451–484.

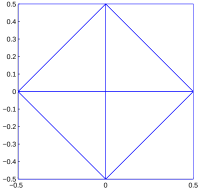
Bibliography

J. Melenk (1995), On Generalized Finite Element Methods, PhD thesis, University of Maryland, USA.

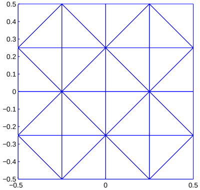
Results due to T. Huttunen

$\Omega = [-0.5, 0.5]^2$ and use the three grids below. The exact solution is $u(\boldsymbol{x}) = \frac{i}{4} H_0^{(1)}(\kappa|\boldsymbol{x} - \boldsymbol{x}_0|)$ with $\boldsymbol{x}_0 = (-0.75, 0)^T$.

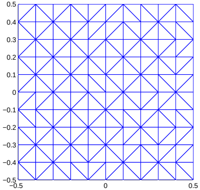
$h = 0.5$



$h = 0.25$



$h = 0.1$



First Results

In the first test case we choose $\kappa = 20$ (so the domain is slightly over 3 wavelengths across) and $p = 15$ (or $\mu = 7$) directions on each triangle.

The predicted rate of convergence in the $L^2(\Omega)$ norm is $O(h^6)$.

Mesh size h	# DoF	N_{lam}	$L^2(\Omega)$ Error (%)	Order	cond(D)	Order
0.50	120	4.6	4.38	-	0.64×10^2	-
0.25	480	9.2	0.01873	7.9	0.20×10^6	-11.6
0.10	3000	22.9	1.51×10^{-5}	7.8	0.22×10^{11}	-12.7

Some more results: $\kappa = 40$, $p = 21$

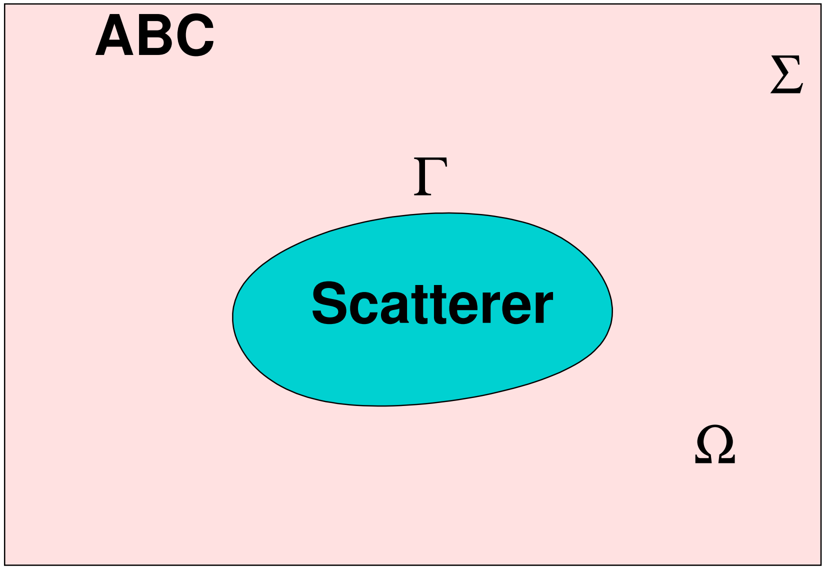
Since $\mu = 10$, we expect 9th order convergence in the error, but we observe roughly 10th order.

Mesh size h	# DoF	N_{lam}	$L^2(\Omega)$ Error (%)	Order	cond(D)	Order
0.50	168	2.0	25.2	-	8.7	-
0.25	672	4.1	0.0337	9.55	0.94×10^5	-13.4
0.1	4200	10.2	2.32×10^{-6}	10.5	0.14×10^{13}	-18

A Model Scattering Problem

Let $\Omega \subset \mathbb{R}^3$ (or $\mathbb{R}^2$) with disjoint boundaries Γ and Σ. Approximate u which satisfies

$$\begin{aligned} \Delta u + \kappa^2 u &= 0 \text{ in } \Omega \\ u &= g \text{ on } \Gamma \; (Q = 1) \\ \frac{\partial u}{\partial \nu} - iku &= 0 \text{ on } \Sigma \; (Q = 0) \end{aligned}$$

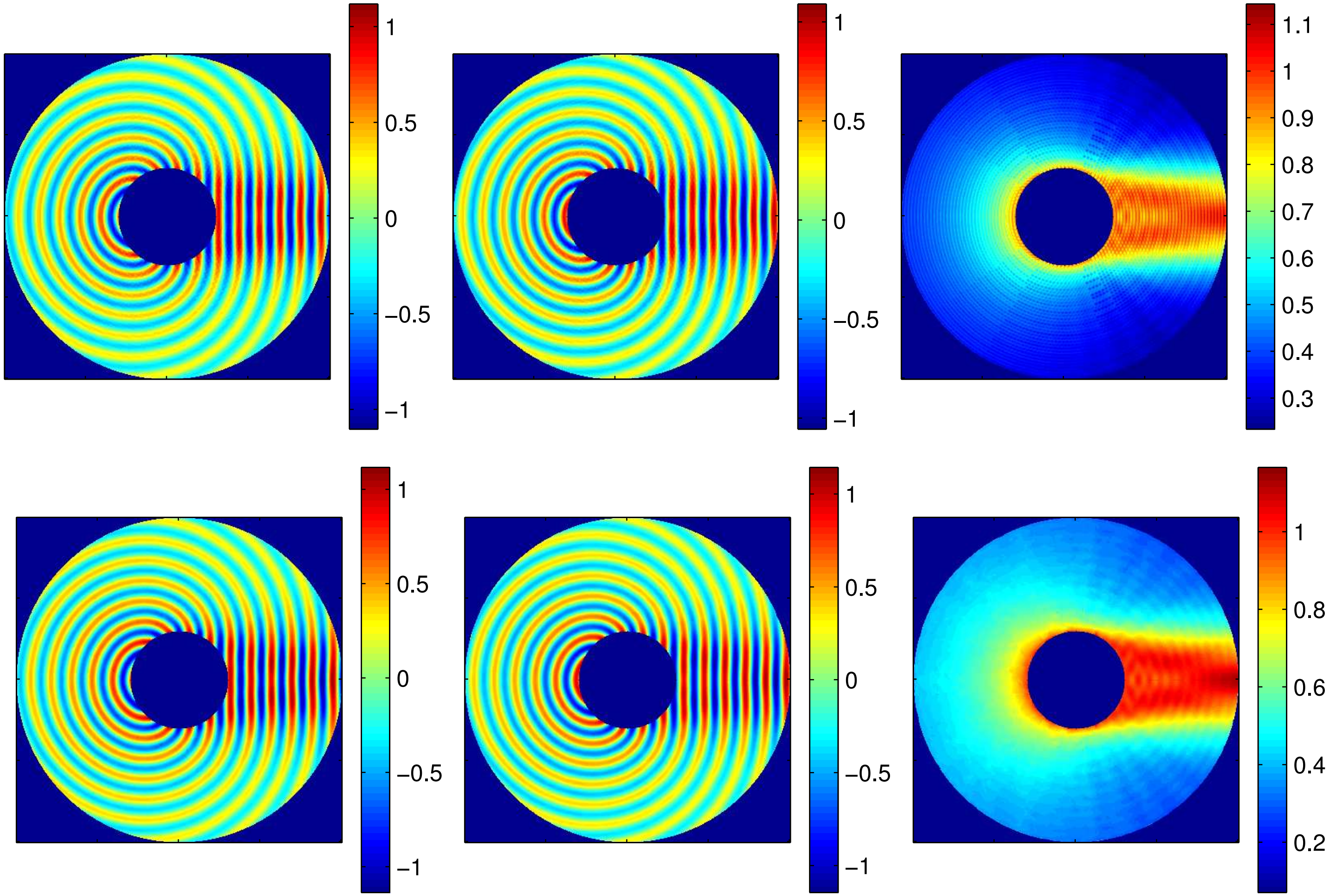


where g describes the incoming plane wave. The region Ω is meshed with tetrahedra and the UWVF applied there.

ABC = Absorbing Boundary Condition

UWVF results in 2D

Circle radius 0.3, k=50, $h_{max} = 0.2399$, $p_k = 13$: l_2 error: 3.7e-2.



Exact:

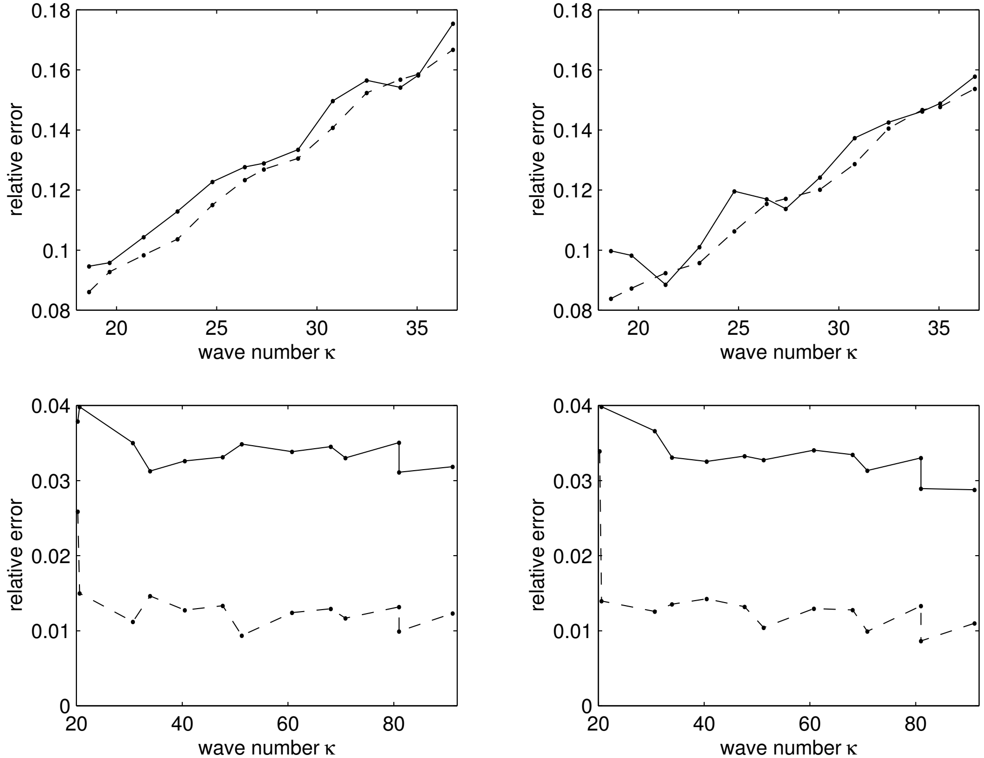
UWVF:

More UWVF results in 2D

Domain Ω is annulus $0.4 \leq r \leq 1$

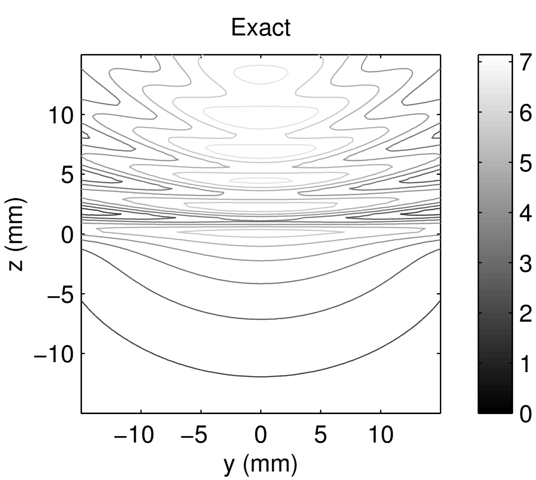
Remesh at each κ to keep $\lambda/h \approx 8$ (FEM)

Remesh at each κ to keep $\sqrt{2p}\lambda/h \approx 4.5$ (UWVF)

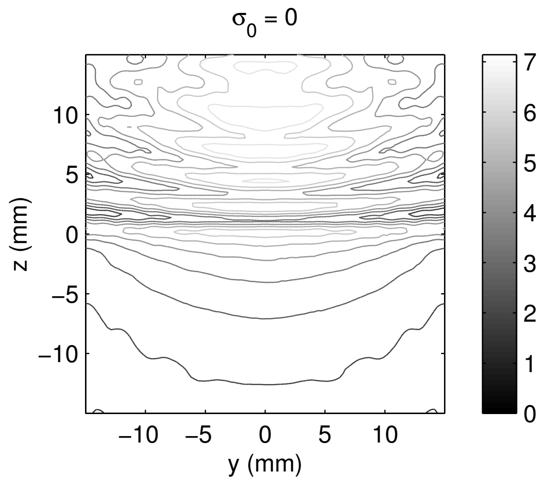


Dirichlet   Neumann

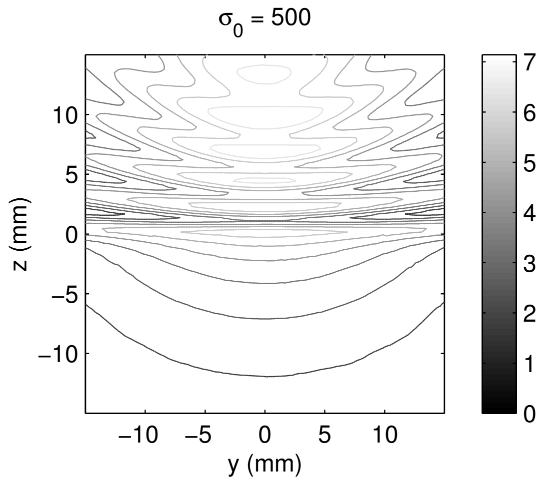
Layered medium



Exact



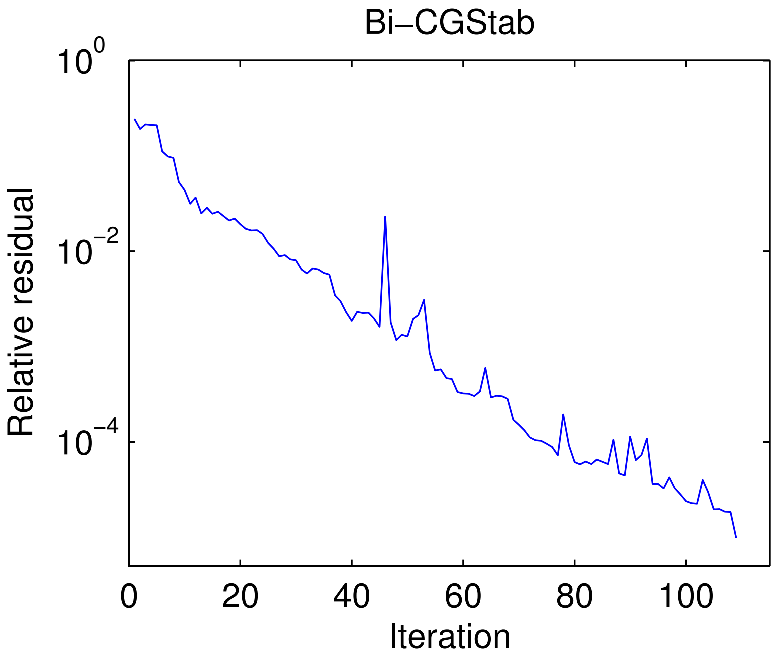
$\sigma_0 = 0$



$\sigma_0 = 500$

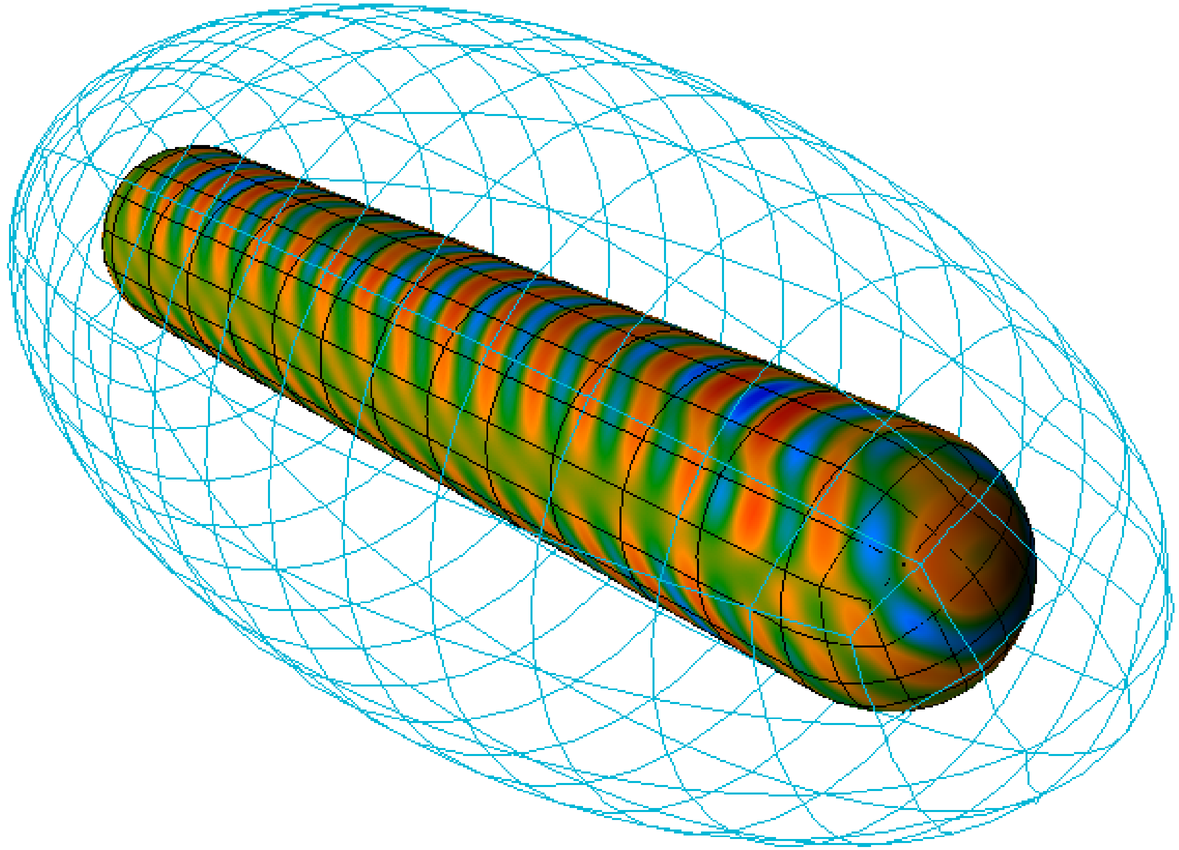
Iterative solution of the linear system

The UWVF linear system can be solved by simple iterative scheme. We use BiCGStab.



BiCGStab convergence for a problem having 3,474,770 degrees of freedom using a 24 processor cluster (2.8GHz P4, 48Gb memory total, 1000BaseT). Solution time is 451s using 25.3 GB memory (109 iterations).

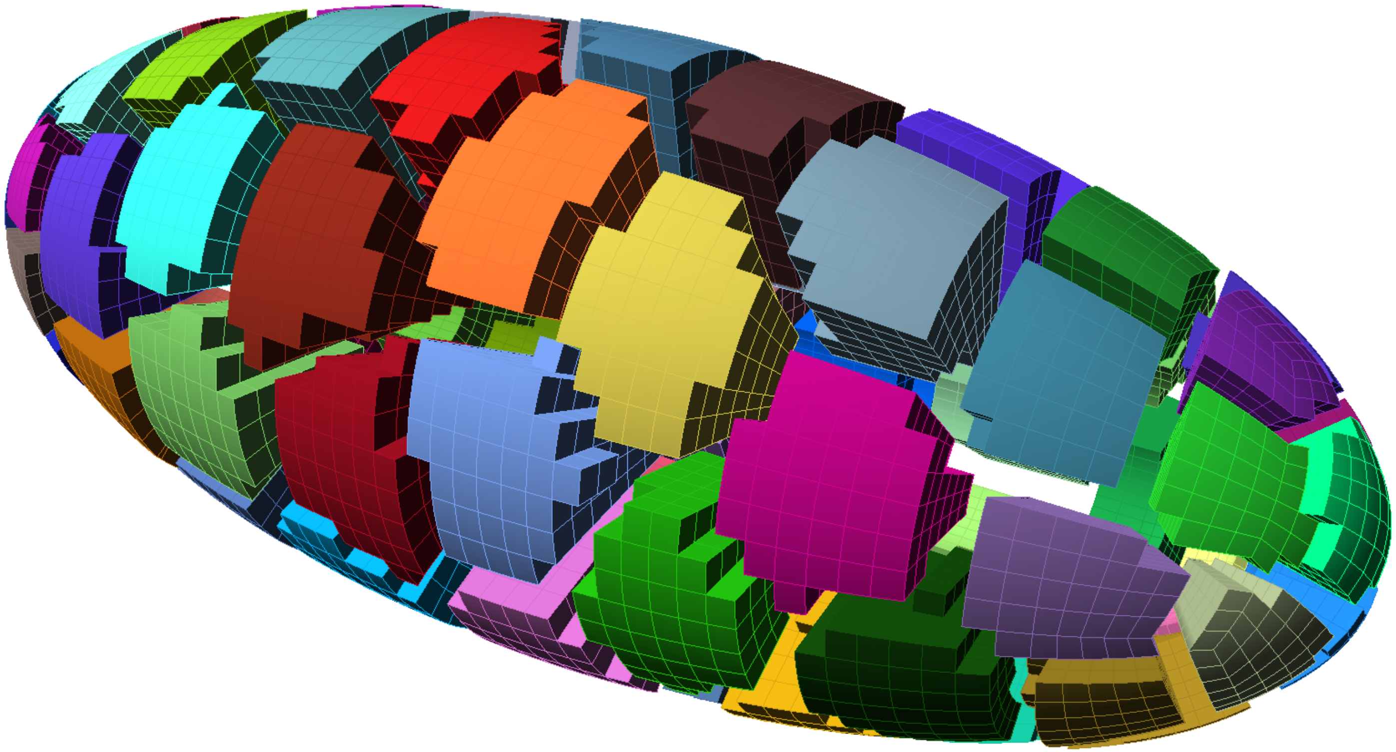
Scattering from a capped cylindrical hull



Sound-hard capped cylinder with the length 12 and the radius 1.

The incident field is $g = e^{ik(0.2\ 0.6\ 1)\cdot\mathbf{x}/\sqrt{1.4}}$.

Unstructured domain decompositions



81 subdomains

Discretization errors at $k \approx 2\pi$ and 4π