Computing complex resonances

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Workshop on
High-order methods for computational wave propagation and scattering

American Institute of Mathematics
Palo Alto, CA, USA

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outline

1. What are resonances?
2. examples
3. mathematical properties
4. numerical methods
5. Hardy space infinite elements

definition of resonances

Let $K \subset \mathbb{R}^d$ be smooth, compact, and $\Omega := \mathbb{R}^d \setminus K$ connected.

Definition

A complex $k \in \mathbb{C}^+ := \{z \in \mathbb{C} : \mathrm{Re}\,(z) > 0\}$ is called a resonance or scattering pole of $-\Delta$ in Ω with Dirichlet boundary conditions, if there exists a non-trivial solution $u \in H^2_{\text{loc}}(\Omega)$ to the eigenvalue equation

$$\begin{array}{rcll} -\Delta u &=& k^2 u & \text{in } \Omega \\ u &=& 0 & \text{on } \partial\Omega \end{array}$$

which satisfies a radiation condition at infinity.

radiation condition

Let $d = 2$ for simplicity, and assume $K \subset B_a := \{x : |x| < a\}$. A solution u to the Helmholtz equation $\Delta u + k^2 u = 0$ is called outgoing if it satisfies one of the following equivalent radiation conditions:

- In polar coordinates (r, ϕ) ($r > a$, $0 \leq \phi < 2\pi$) u has a series representation

  $$u(r,\phi) = \sum_{n=-\infty}^{\infty} c_n e^{in\phi} H^{(1)}_{|n|}(kr).$$

  Here $H^{(1)}_{|n|}$ is the Hankel function of the first kind of order $|n|$.
- u has an integral representation

  $$u(x) = \int_{|y|=a} \left\{ \frac{\partial \Phi(x,y,k)}{\partial n(y)} u(y) - \Phi(x,y,k) \frac{\partial u}{\partial n}(y) \right\} ds(y)$$

  in terms of the outgoing fundamental solution $\Phi(x,y,k) := (i/4) H^{(1)}_0(k\,|x-y|)$.

exponential growth, Sommerfeld radiation condition

Recall the asymptotic formulas

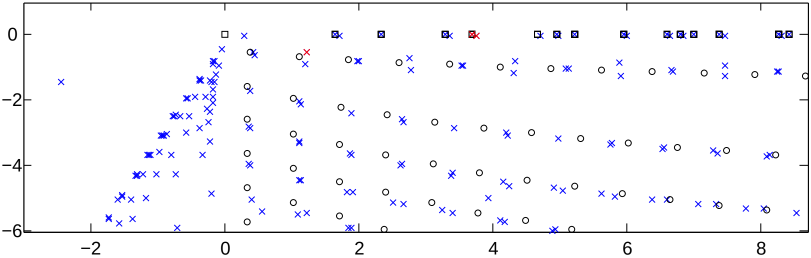
$$H_n^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}} e^{i(z - \frac{n\pi}{2} - \frac{\pi}{4})}$$

$$H_n^{(2)}(z) \sim \sqrt{\frac{2}{\pi z}} e^{-i(z - \frac{n\pi}{2} - \frac{\pi}{4})}$$

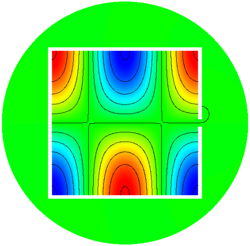
for $|z| \to \infty$, $|\arg z| < \pi/2$. Hence, for $\operatorname{Im}(k) < 0$

- $|H_n^{(1)}(kr)|$ is exponentially growing whereas $|H_n^{(2)}(kr)|$ is exponentially decaying as $r \to \infty$.
- Sommerfeld's radiation condition $\sqrt{r}(\frac{\partial u}{\partial r} - iku) \to 0$ holds true for incoming, but not for outgoing solutions!

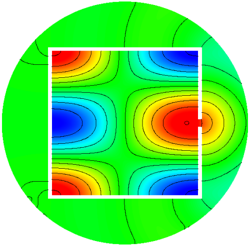
resonances of an open square



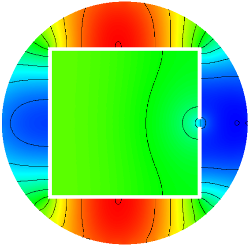
$|\,\mathrm{Re}\,k/\,\mathrm{Im}\,k| = 1,98\cdot 10^7$



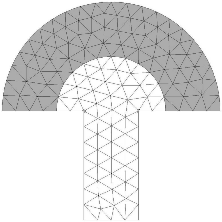
$|\,\mathrm{Re}\,k/\,\mathrm{Im}\,k| = 199,6$

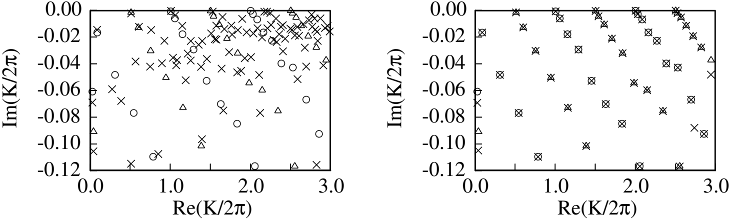


$|\,\mathrm{Re}\,k/\,\mathrm{Im}\,k| = 2,24$



spurious resonances





p = 2

p = 12

W. Koch. Acoustic resonances in rectangular open cavities. *AIAA Paper 2004-2843*, 2004.

outline

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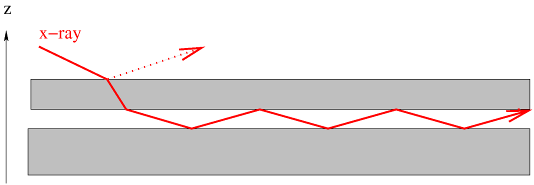
multi-layer x-ray resonator



$$u''(z) + k_0^2 n^2(z) = \gamma^2 u(z)$$

$$u(z) = u_s(z) + \exp\left(-iz\sqrt{k_0^2 - \gamma^2}\right)$$

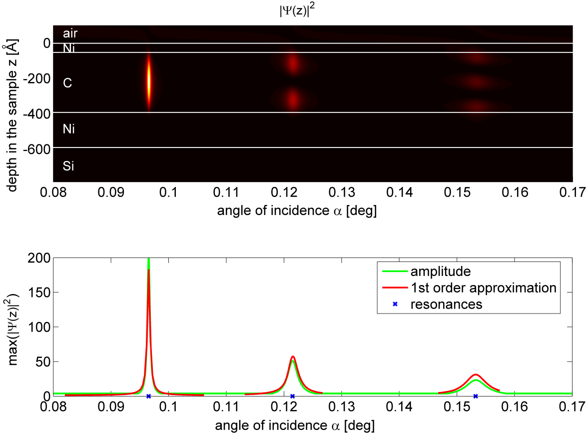
u_s outgoing for$z > 0$

u outgoing for $z < -a$

$\gamma = k_0 \cos(\textit{angle of incidence})$

collaborator: T. Salditt, Dep. X-Ray Physics, Univ. Göttingen

multi-layer x-ray resonator, cont’d



F. Schenk. Asymptotische Entwicklung von Streulösungen um Resonanzen am Beispiel von Röntgen– Mehrschicht–Resonatoren. *Diploma thesis, Univ. Göttingen*, 2007.

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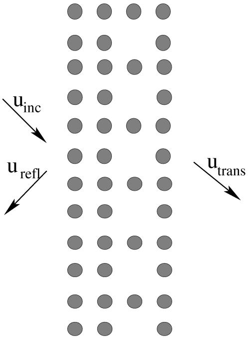
photonic crystal slabs

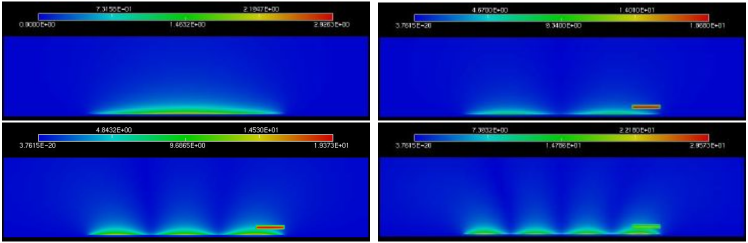


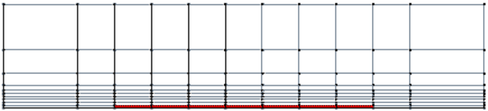
M. A. Haider, S. P. Shipman, and S. Venakides. Boundary-integral calculations of two-dimensional electromagnetic scattering in infinite photonic crystal slabs: channel defects and resonances. *SIAM J. Appl. Math.*, 62(6):2129–2148, 2002.

S. P. Shipman and S. Venakides. Resonance and bound states in photonic crystal slabs. *SIAM J. Appl. Math.*, 64(1):322–342, 2003.

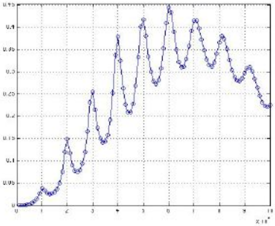
S. Venakides, M. A. Haider, and V. Papanicolaou. Boundary integral calculations of two-dimensional electromagnetic scattering by photonic crystal Fabry-Perot structures. *SIAM J.*

thin rods: electromagnetic resonances





thin rods: electromagnetic resonances

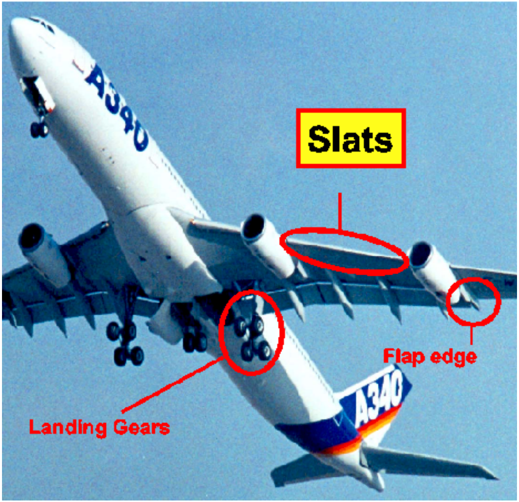


P. Boissoles, G. Caloz, M. Costabel, M. Dauge.
`http://patrice.boissoles.free.fr/Monique/resultats.html`

A related time-dependent problem was studied by

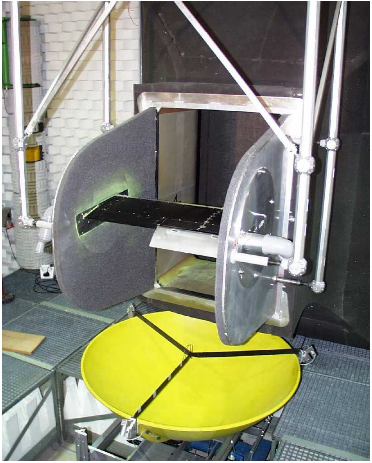
P. Davies, D. B. Duncan, B. Zubik-Kowl. The stability of numerical approximations of the time domain current induced on thin wire and strip antennas. *Applied Num. Math.* 55:48–68, 2005.

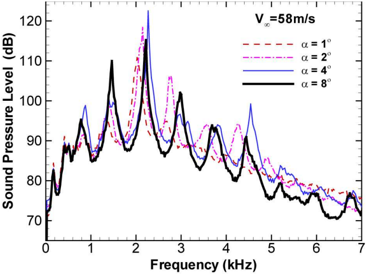
major sources of airframe noise



Source: U. Michel *International Symposium Arcachon, France* (2002)

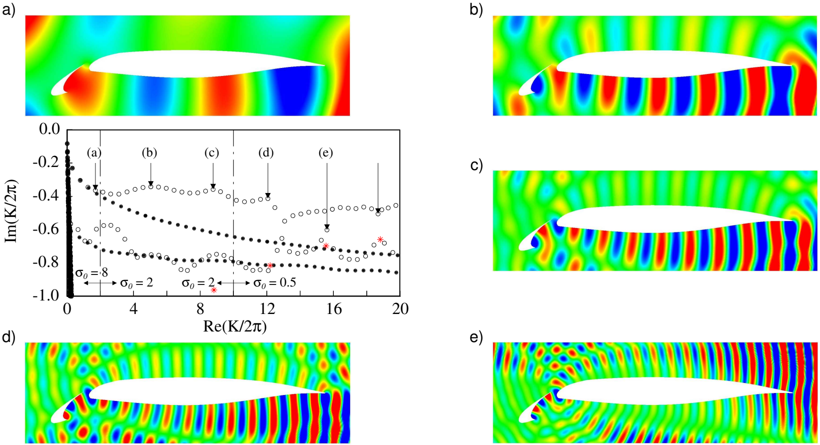
wind tunnel experiment





M. Pott-Pollenske *et al.*, *AIAA Paper 2003-3228*, 2003.

resonances of the high lift configuration



S. Hein, T. Hohage, W. Koch, J Schöberl. Acoustic Resonances in High Lift Configuration. *J. Fluid Mech.* 582:179-202, 2007.

outline

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2. examples
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resolvent of $-\Delta$

Recall that the resolvent $R(z) := (zI + \Delta)^{-1}$ of the Dirichlet-Laplacian $-\Delta$ on Ω can be written as an integral operator

$$\left(R(k^2)f\right)(x) = \int_\Omega G(x,y,k)f(y)\,dy, \qquad x \in \Omega,$$

with Green's function $G(x,y,k)$ of the differential equation $\Delta u + k^2 u = 0$ on Ω with Dirichlet boundary conditions. For $\mathrm{Im}\,(k) < 0$, the function $x \mapsto G(x,y,k)$ grows exponentially as $x \to \infty$.

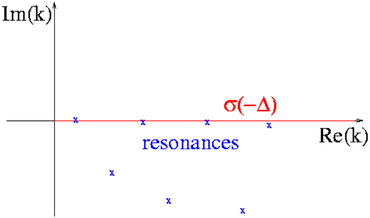
definition via resolvent

Theorem

A complex number k *with* $\mathrm{Re}\,(k) > 0$ *is a resonance of* $-\Delta$ *on* Ω *if and only if there exist* $f, g \in C_0^\infty(\Omega) \setminus \{0\}$ *such that the meromorphic extension of the "matrix element"*

$$k \mapsto \langle f, R(k^2) g \rangle$$

initially defined on $\{k \in \mathbb{C} : \mathrm{Re}\,(k) > 0, \mathrm{Im}\,(k) > 0\}$ *has a pole at* k.



definition via single layer operator

Let $\Phi(\cdot,\cdot,k)$ denote the outgoing fundamental solution to the Helmholtz equation, e.g. $\Phi(x,y,k) = \frac{\exp(ik|x-y|)}{4\pi|x-y|}$ for $d = 3$ and consider the single layer potential operator

$$(V(k)f)(x) := \int_{\partial K} \Phi(x,y,k) f(y) ds(y), \qquad x \in \partial K.$$

Theorem

For any $s \in \mathbb{R}$ the mapping

$$\mathbb{C}^+ \to L\left(H^s(\partial K), H^{s-1}(\partial K)\right), \quad k \mapsto V(k)^{-1}$$

is meromorphic. Moreover, on

- $\{k : \mathrm{Re}\, k > 0, \mathrm{Im}\, k > 0\}$*: V^{-1} is holomorphic.*
- $\mathbb{R}^+$*: The poles of V^{-1} coincide with the square roots of the eigenvalues of the negative* **interior** *Dirichlet-Laplacian.*
- $\{k : \mathrm{Re}\, k > 0, \mathrm{Im}\, k < 0\}$*: The poles of V^{-1} coincide with the resonances of the negative* **exterior** *Dirichlet-Laplacian.*

definition via scattering matrix

Let $u_s(\cdot, \omega, k)$ denote the scattered field for the incident field $\exp(ik\langle\cdot, \omega\rangle)$ ($\omega \in S^{d-1}$), and let $u_\infty(\cdot, \omega, k)$ denote its far field pattern. Then $F(k) : L^2(S^2) \to L^2(S^2)$,

$$(F(k)g)(\hat{x}) := \int_{S^2} u_\infty(\hat{x}, \omega, k) g(\omega) ds(\omega), \qquad \hat{x} \in S^2$$

is called the far-field operator, and $A(k) := I + \frac{ik}{2\pi} F(k)$ is called the scattering matrix. $A(k)$ is unitary for $k > 0$.

Theorem

The mapping $\mathbb{R}^+ \to L(L^2(S^2))$, $k \mapsto A(k)$ *has a meromorphic extension to* $\mathbb{C}^+$, *and the poles coincide with the resonances of* $-\Delta$ *on* Ω.

resonances in the time domain

assumptions: $K \subset \mathbb{R}^d$ compact and "non-trapping", d odd, resonances k_j of $-\Delta$ simple with resonance functions w_j.

Let U denote the solution of the initial value problem

$$\begin{aligned} (\partial_t^2 - \Delta) U(t,x) &= 0 \qquad && x \in \mathbb{R}^d \setminus K, \\ U(t,x) &= 0 && x \in \partial K, \\ U(0,\cdot) &= f && f \in C_0^\infty(\mathbb{R}^d \setminus K), \\ \partial_t U(0,\cdot) &= g && g \in C_0^\infty(\mathbb{R}^d \setminus K) \end{aligned}$$

and let $\Omega' \subset \mathbb{R}^d \setminus K$ be compact. Then for all $C > 0$ and $\epsilon > 0$, U has the asymptotic behavior

$$U(t,x) = \operatorname{Re} \Big(\sum_{\{j:\, \operatorname{Im}(k_j) \geq -C\}} \alpha_j w_j(x) e^{-ik_j t} \Big) + O(e^{-(C-\epsilon)t}), \qquad t \to \infty.$$

uniformly for all $x \in \Omega'$.

some references

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- P. Lax and R. Phillips. *Scattering theory*. Academic Press, 1967.
- S.-H. Tang and M. Zworski. Resonance expansions of scattered waves. *Comm. Pure and Applied Math.*, LIII:1305–1334, 2000.
- M. Taylor. *Partial Differential Equations: Qualitative Studies of Linear Equations*, volume 2. Springer, 1996.
- M. Zworski. Resonances in physics and geometry. *Notices of the AMS*, 46:319–328, 1999.

outline

1. What are resonances?
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boundary integral equation methods

Recall that resonances can be characterized as poles of the single layer potential V^{-1}, or equivalently solutions to

$$\det(V(i)^{-1}V(k)) = 0.$$

This equation may be solved by Newton's method or by complex contour integrations (Rouché's theorem and extensions).

O. Poisson. Étude numérique des pôle de résonance associés d'ondes acoustique et élastique par un obstacle en dimension 2. *Math. Mod. Num. Anal.*, 7:819–815, 1995.

C. Labreuche. Problème inverses en diffraction d'ondes basés sur la notion de résonances. PhD thesis, Université de Paris IX, 1997.

finite elements with DtN operator

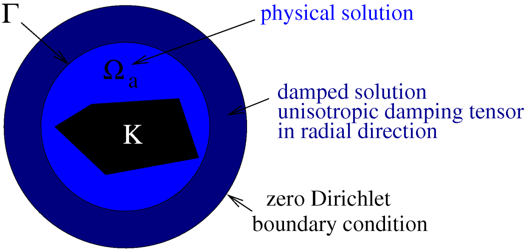
Let $\mathrm{DtN}(k) : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ denote the exterior Dirichlet-to-Neumann map for the ball of radius a.
Then a resonance pair (u, k) satisfies the variational equation

$$\int_{\Omega_a} (\nabla u \nabla v - k^2 uv)\, dx - \int_{\Gamma} \mathrm{DtN}(k) u\, v\, ds = 0, \quad v \in H^1(\Omega_a), v|_{\partial K} = 0$$

if k is not an exterior resonance of B_a. Short: $\mathcal{B}(k)u = 0$.
It can be shown that $k \mapsto \mathcal{B}(k)^{-1}$ is meromorphic, and its poles are the resonances. $\rightsquigarrow$ Solve $\det \mathcal{B}(k) = 0$.

M. Lenoir, M. Vullierme-Ledard, C. Hazard. Variational formulations for the determination of resonant states in scattering problems. *SIAM J. Math. Anal.* 23:579–608, 1992.

Perfectly Matched Layers



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N. Moiseyev. Quantum theory of resonances: Calculating energies, width and cross-sections by complex scaling. *Physics reports*, 302:211–293, 1998.

B. Simon. The theory of resonances for dilation analytic potentials and the foundations of time dependent perturbation theory. *Ann. Math.*, 97:247–274, 1973.

discussion

- The eigenvalue structure is destroyed on continuous level by the DtN operator, i.e. by truncation to a bounded domain.
- Many standard methods for approximating DtN, e.g. by series expansions (infinite elements) or boundary integral operators (FEM/BEM coupling) also destroy the eigenvalue structure.
- Local transparent boundary conditions such as $\frac{\partial u}{\partial n} = iku$ on Γ cannot be used since Sommerfeld's radiation condition is not valid.
- PML preserves the eigenvalue structure of the problem.

discussion, cont'd

- PML can be analyzed by introducing a holomorphic operator $\mathrm{DtN}_\rho^{PML}(k)$ in analogy to $\mathrm{DtN}_a(k)$.
- If k is not a resonance of B_a, then $\|\mathrm{DtN}_\rho^{PML}(k) - \mathrm{DtN}\| \to 0$ exponentially as the thickness ρ of the PML layer tends to infinity.

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convergence theorem for resonances

Consider

- a bounded domain $D \subset \mathbb{C}^+$ containing exactly N simple resonances k_j with normalized resonance functions w_j, $j = 1, \ldots, N$. There exists a neighborhood of D containing no further resonances.
- a series of finite-dimensional subspaces $V_h \subset H^1(\Omega_a)$ such that
$$\inf_{u \in V_h} \|u - w_j\| \leq \epsilon_1(h), \quad j = 1, \ldots, N\ , \ \epsilon_1(h) \to 0.$$
- A sequence of holomorphic approximations $\mathrm{DtN}_h(k)$ to $\mathrm{DtN}(k)$ such that
$$\|\mathrm{DtN}(k) - \mathrm{DtN}_h(k)\| \leq \epsilon_2(h), \quad \epsilon_2(h) \to 0.$$

convergence theorem for resonances, cont'd

$k \in \mathbb{C}^+$ is called a numerical resonance if there exists a nontrivial $u_h \in V_h$ such that

$$\int_{\Omega_a} (\nabla u_h \nabla v_h - k^2 u_h v_h)\, dx - \int_\Gamma \mathrm{DtN}_h(k) u_h\, v_h\, ds = 0 \quad \forall v_h \in V_h.$$

Theorem (Hohage, Rapún)

Under the assumptions above there exists $h_0 > 0$ such that for all $h \leq h_0$ there exist N numerical resonances $k_{j,h}$ in D, and

$$|k_{j,h} - k_j| \leq C\left(\epsilon_1(h) + \epsilon_2(h)\right), \qquad j = 1, \dots, N.$$

Remark: A continuous resonance k_j of multiplicity r_j may split up into r_j numerical resonances $k_{j,h}$, each of which satisfies

$$|k_{j,h} - k_j| \leq C\left(\epsilon_1(h) + \epsilon_2(h)\right)^{1/r_j}.$$

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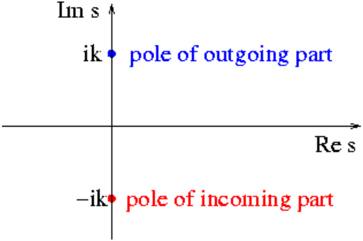
pole condition as radiation condition for $d = 1$

The general solution to $u''(r) + k^2 u(r) = 0$ is

$$u(r) = u_\infty^+ e^{ikr} + u_\infty^- e^{-ikr}.$$

Its **Laplace transform** $\hat{u}(s) := \int_0^\infty e^{-sr} u(r)\, dr$ is given by

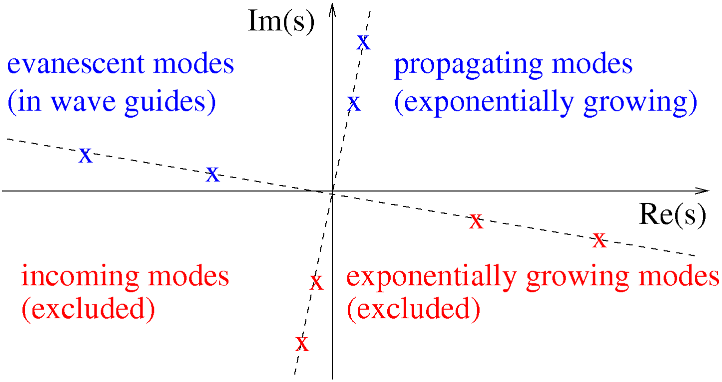
$$\hat{u}(s) = \frac{u_\infty^+}{s - ik} + \frac{u_\infty^-}{s + ik}.$$



Definition

u satisfies the **pole condition** if its Laplace transform $\hat{u}$ (initially defined on $\{s : \mathrm{Re}\, s > 0\}$) has a holomorphic extension to the lower complex half-plane.

pole condition for $\mathrm{Im}\, k < 0$



evanescent modes
(in wave guides)

propagating modes
(exponentially growing)

incoming modes
(excluded)

exponentially growing modes
(excluded)

Hardy space $\mathcal{H}_-(\mathbb{R})$

Definition

A function u, which is holomorphic in the lower complex half-plane $\{s \in \mathbb{C} : \operatorname{Im}(s) < 0\}$ *has* L^2 *boundary values* $v = u|_{\mathbb{R}} \in L^2(\mathbb{R})$ *if*

$$\int_{-\infty}^{\infty} |u(x - i\epsilon) - v(x)|^2 \, dx \xrightarrow{\epsilon \searrow 0} 0.$$

$$\mathcal{H}_-(\mathbb{R}) := \left\{ v \in L^2(\mathbb{R}) : \exists u : \mathbb{C}^- \to \mathbb{C} \text{ holomorphic with } v = u|_{\mathbb{R}} \right\}.$$

- $\mathcal{H}_-(\mathbb{R})$ equipped with the L^2 inner product is a Hilbert space.
- pole condition: $\hat{u}|_{\mathbb{R}} \in \mathcal{H}_-(\mathbb{R})$
- idea: Galerkin method in $\mathcal{H}_-(\mathbb{R})$
- problem: appropriate basis of $\mathcal{H}_-(\mathbb{R})$

Hardy space $\mathcal{H}_+(S^1)$

Definition

Let $B^1 := \{z \in \mathbb{C} : |z| < 1\}$ *and* $S^1 := \partial B$. *A holomorphic function* $u : B^1 \to \mathbb{C}$ *has* L^2 *boundary values* $v = u|_{S^1} \in L^2(S^1)$ *if*

$$\int_{S^1} |u(rz) - v(z)|^2 \, |dz| \xrightarrow{r \nearrow 1} 0.$$

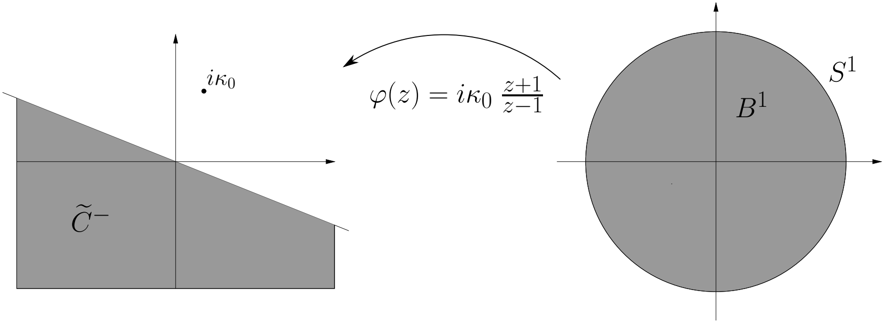
$$\mathcal{H}_+(S^1) := \left\{ v \in L^2(S^1) : \exists u : B^1 \to \mathbb{C} \text{ holomorphic with } v = u|_{S^1} \right\}.$$

Lemma

$\mathcal{H}_+(S^1)$ *equipped with the* L^2 *inner product is a Hilbert space with orthonormal basis*

$$z \mapsto \frac{1}{\sqrt{2\pi}} z^j, \qquad j = 0, 1, 2, \ldots$$

Möbius transform



Lemma

The mapping $\mathcal{M}u := (u \circ \varphi) \cdot \sqrt{\varphi'}$ is a unitary operator from $\mathcal{H}_-(\mathbb{R})$ to $\mathcal{H}_+(S^1)$.

pole condition for $d \geq 2$

Consider Laplace transform in radial direction:

$$U(s,\hat{x}) := \int_0^\infty e^{-sr}(r+a)^{(d-1)/2} u\left(\frac{r+a}{a}\hat{x}\right)\, dr$$

for $\mathrm{Re}\, s > 0$ and $\hat{x} \in \Gamma$.

Definition

u satisfies the pole condition if the mapping $s \to \hat{U}(s,\cdot)$ *defined on* $\{s \in \mathbb{C} : \mathrm{Re}\, s > 0\}$ *with values in* $L^2(\Gamma)$ *has a holomorphic extension to the lower complex half-plane* $\{s \in \mathbb{C} : \mathrm{Im}\, s < 0\}$.

equivalence to Sommerfeld radiation condition for $k > 0$

Theorem

A bounded solution to the Helmholtz equation for $k > 0$ *satisfies the pole condition if and only if it satisfies the Sommerfeld radiation condition.*

T. Hohage, F. Schmidt, L. Zschiedrich. Solving time-harmonic scattering problems based on the pole condition. I: Theory. *SIAM J. Math. Anal.*, 35:183-210, 2003.

variational formulation

With $X := H^1(\Omega_{\text{int}}) \times (\mathcal{H}_+(S^1) \otimes H^{1/2}(\Gamma))$ the complete problem is given by:

Find nontrivial eigenpairs $(\,(u, \widetilde{U})\,,\ k^2) \in X \times \mathbb{C}$ satisfying

$$a(\,(v, \widetilde{V})\,,\,(u, \widetilde{U})\,) = k^2 b(\,(v, \widetilde{V})\,,\,(u, \widetilde{U})\,) \quad \forall\ (v, \widetilde{V}) \in X.$$

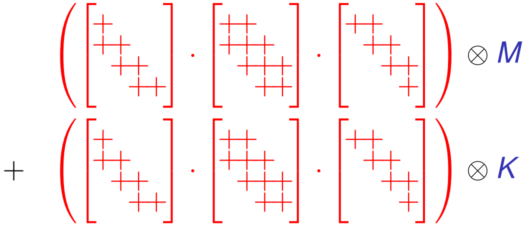
$$\begin{aligned}
a(\,(v, \widetilde{V})\,,\,(u, \widetilde{U})\,) &= \int_{\Omega_a} \nabla v \nabla u \, dx + \int_\Gamma u|_\Gamma(\hat{x} v|_\Gamma(\hat{x} \, d\hat{x} \\
&\quad - \frac{i\kappa_0}{\pi} \int_\Gamma \int_{S^1} [\ldots] \left[u|_\Gamma(\hat{x}) + (z+1)\widetilde{U}(z, \hat{x}) \right] |dz| d\hat{x} \\
&\quad + \frac{i}{\kappa_0 \pi} \int_\Gamma \int_{S^1} \widetilde{I}_a \nabla_{\hat{x}} [\ldots] \widetilde{I}_a \nabla_{\hat{x}} \left[u|_\Gamma(\hat{x}) + (z-1)\widetilde{U}(z, \hat{x}) \right] |dz| d\hat{x}, \\
b(\,(v, \widetilde{V})\,,\,(u, \widetilde{U})\,) &= \int_{\Omega_a} v \, u \, dx \\
&\quad + \frac{i\kappa_0 a^2}{4\pi} \int_\Gamma \int_{S^1} [\ldots] \left[u|_\Gamma(\hat{x}) + (z-1)\widetilde{U}(z, \hat{x}) \right] |dz| d\hat{x}.
\end{aligned}$$

Hardy space infinite elements

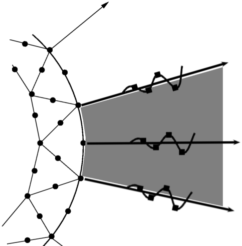
Galerkin discretization:

$\mathrm{span}\{z^0,\dots,z^N\}\otimes\mathcal{P}^p(\Gamma_h)\subset\mathcal{H}_+(S^1)\otimes H^{1/2}(\Gamma)$

⇝ "Hardy–space infinite elements". local element matrices:



M	boundary mass matrix corresponding to $\int_\Gamma uv\,ds$
K	boundary stiffness matrix corresponding to $\int_\Gamma \nabla_{\hat{x}}u\nabla_{\hat{x}}v\,ds$



separation of variables

For given k consider the exterior boundary value problem

$$\begin{aligned} &\Delta u + k^2 u = 0 \qquad \text{in } \{x : |x| > a\}, \\ &u = u_0 \qquad \text{on } \Gamma, \end{aligned}$$

which we solve using the Hardy space method.
Let $\{\varphi_j : j \in \mathbb{N}\} \subset L^2(\Gamma)$ be a complete orthonormal system of the Laplace-Beltrami operator $\Delta_{\hat{x}}$ on Γ (e.g. trigonometric monomials). Then the equations of the Hardy space method can be separated into

$$A_j \tilde{U}_j = F_j(u_0), \qquad j \in \mathbb{N}$$

with operators $A_j \in L(\mathcal{H}_+(S^1))$ and right hand sides $F_j(u_0) \in \mathcal{H}_+(S^1)$.

Toeplitz operators

Definition

Let $f : S^1 \to \mathbb{C}$ *be continuous, and* $P : L^2(S^1) \to H^2_-(S^1)$ *the orthogonal projection. The Toeplitz operator* $T_f : \mathcal{H}_+(S^1) \to \mathcal{H}_+(S^1)$ *with symbol* f *is defined by*

$$T_f\varphi := P(f \cdot \varphi), \qquad \varphi \in \mathcal{H}_+(S^1).$$

Theorem

If $f(z) \neq 0$ *for all* $z \in S^1$*, then* T_f *is a Fredholm operator with* $\mathrm{index}(T_f) = -\mathrm{wn}(f)$*, where* $\mathrm{wn}(f)$ *is the winding number of* f *around* 0*.*

Lemma

The operators A_j *are complex perturbations of the Toeplitz operator with symbol* $f(z) = -s_0^2|z+1|^2 + k^2|z-1|^2$*. Moreover, they are one-to-one. Hence,* A_j *is boundedly invertible.*

convergence theorem

Let $P_n : \mathcal{H}_+(S^1) \to \mathrm{span}\{z^0, \ldots, z^n\}$ denote the orthogonal projection. We approximate the exact equation by

$$P_n A_j P_n \tilde{U}_j^{(n)} = P_n F_j(u_0).$$

Theorem (Hohage, Nannen)

There exists n_0 such that the discrecte equation has a unique solution $\tilde{U}_j^{(n)}$ for all $n \geq n_0$, and $\left\| \tilde{U}_j^{(n)} - \tilde{U}_j \right\|_{L^2}$ converges super–algebraically to 0 for $n \to \infty$.

sketch of the proof

- **stability:** $\|(P_n A_j P_n)^{-1}\| \leq C$ follows from general results on shift-operators in

  Prössdorf & Silbermann. *Numerical Analysis for Integral and Related Operator Equations*, 1991

- **consistency:** It follows from results in the reference below that $\tilde{U}_j \in C^\infty(S^1)$. This entails super-algebraic convergence of the approximation error $\inf_{v \in R(P_n)} \|v - \tilde{U}_j\|$.

  T. Hohage, F. Schmidt, L. Zschiedrich. Solving time-harmonic scattering problems based on the pole condition. I: Theory. *SIAM J. Math. Anal.*, 35:183-210, 2003.

numerical convergence