Numerical solution of time domain boundary integral equations

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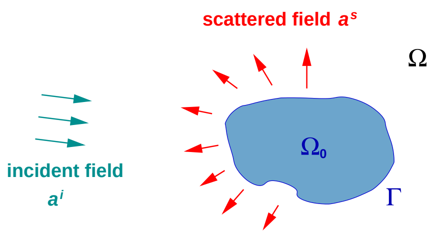
Overview

- The problem (mainly acoustic scattering with Dirichlet BC)
- Collocation approximation (Dundee group)
- Galerkin approximation (French group)
- Convolution quadrature (Lubich et al)
- Fast plane wave methods (Michielssen et al)
- Fourier analysis
- Higher order methods
- **Good survey articles on TD BIEs:**
  - Tuong Ha Duong (2003)
  - Martin Costabel (2004)

Acoustic wave equation

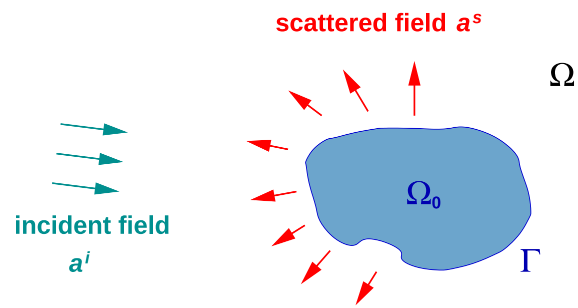
Problem: $a^i(x,t)$ is incident on Γ — find the scattered field $a^s(x,t)$



Incident field a^i:

- $a^i_{tt} = \Delta a^i$ in $\mathbb{R}^3$ (wave speed is $c = 1$);
- e.g. $a^i(x,t) = f(\nu \cdot x - t)$ where f has compact support and $|\nu| = 1$ (often approximate by a Gaussian);
- assume a^i does not reach obstacle before time $t = 0$.

Equations for scattered field



- $a^s_{tt} = \Delta a^s$ in $\mathbb{R}^3/\Gamma$;
- $a^s(x,t) = 0$ for $t \le 0$ (causality);
- Dirichlet BC: $a^s + a^i = 0$ on Γ (“sound soft” case).

Boundary integral formulation

- Scattered field (exterior):

$$a^s(\boldsymbol{x},t) = \frac{1}{4\pi}\int_\Gamma \frac{u(\boldsymbol{x}',t-|\boldsymbol{x}'-\boldsymbol{x}|)}{|\boldsymbol{x}'-\boldsymbol{x}|}\, dS_{\boldsymbol{x}'} \quad \boldsymbol{x}\in\Omega,\ t>0$$

- The surface potential u satisfies:

$$\frac{1}{4\pi}\int_\Gamma \frac{u(\boldsymbol{x}',t-|\boldsymbol{x}'-\boldsymbol{x}|)}{|\boldsymbol{x}'-\boldsymbol{x}|}\, dS_{\boldsymbol{x}'} = -a^i(\boldsymbol{x},t) \quad \boldsymbol{x}\in\Gamma,\ t>0. \qquad (*)$$

IC: $u\equiv 0$, $a^i\equiv 0$ on Γ for all $t\le 0$ (causality).

- Need to solve BIE $(*)$ for $u(\boldsymbol{x},t)$ on $\Gamma\times(0,T)$.

Problem: BIE

Find u given a on $\Gamma\times(0,T)$:

$$\int_\Gamma \frac{u(\boldsymbol{x}',t-|\boldsymbol{x}'-\boldsymbol{x}|)}{|\boldsymbol{x}'-\boldsymbol{x}|}\, dS_{\boldsymbol{x}'} = a(\boldsymbol{x},t) \quad \boldsymbol{x}\in\Gamma,\ t>0.$$

IC: $u\equiv 0$, $a\equiv 0$ on Γ for all $t\le 0$ (causality).

- This is the **single layer potential equation** for acoustic scattering the (open or closed) surface Γ.
- It is a **retarded potential integral equation (RPIE)**.
- **EM:** equations much more complicated but EFIE formulation for a perfect conductor also involves this RPIE.

Existence and uniqueness for BIE . . .

. . . when Γ is **smooth and closed** (Bamberger and Ha Duong 1986, Lubich 1994) or **flat** (Ha Duong 1990, Lubich 1994).

$$\textbf{BIE:}\quad \int_\Gamma \frac{u(\boldsymbol{x}',t-|\boldsymbol{x}'-\boldsymbol{x}|)}{|\boldsymbol{x}'-\boldsymbol{x}|}\, dS_{\boldsymbol{x}'} = a(\boldsymbol{x},t) \quad \boldsymbol{x}\in\Gamma,\ t>0.$$

Flat plate result: If $a(\cdot,\,t)\in H^{1/2}(\Gamma)$ is smooth in time and vanishes near $t=0$, there is a unique smooth solution $u(\cdot,\,t)\in H^{-1/2}(\Gamma)$ with

$$\|u\|_{H^m_*(0,T;H^{-1/2}(\Gamma))} \le C\,\|a\|_{H^{m+1}_*(0,T;H^{1/2}(\Gamma))} \quad (m\in\mathbb{R}).$$

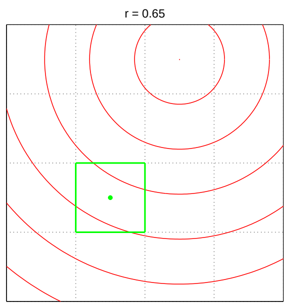
Notation: $H^m_*(0,T)=\{f|_{(0,T)} : f\in H^m(\mathbb{R})$ with $f\equiv 0$ on $(-\infty,0)\}$

$$\|f\|^2_{H^m_*(0,T;X)} = \sum_{k=0}^m \|\partial^k f/\partial t^k\|^2_{L^2(0,T;X)}, \qquad \|f\|^2_{L^2(0,T;X)} = \int_0^T \|f\|^2_X\, dt$$

Numerical approx: collocation

- Approx u in time and space: $u(\boldsymbol{x},t)\approx U(\boldsymbol{x},t)=\sum_{k,m} U_k^m\,\phi_k(\boldsymbol{x})\,\psi$
- Suppose approx BIE holds at N_S points $\boldsymbol{x}_\beta\in\Gamma$ and times $t^n=n\Delta$

$$\int_\Gamma \frac{U(\boldsymbol{x}',t^n-|\boldsymbol{x}'-\boldsymbol{x}_\beta|)}{|\boldsymbol{x}'-\boldsymbol{x}_\beta|}\, dS_{\boldsymbol{x}'} = a(\boldsymbol{x}_\beta,t^n)\,, \quad n\le N_T$$



$$\int_{\Gamma_\alpha} \frac{U(\boldsymbol{x}',t^n-|\boldsymbol{x}'-\boldsymbol{x}_\beta|)}{|\boldsymbol{x}'-\boldsymbol{x}_\beta|}\, dS_{\boldsymbol{x}'} = a(\boldsymbol{x}_\beta,t^n)$$

Collocation – 2

- Approx BIE:

$$\sum_{k,m} \int_{\Gamma} \frac{U_k^m \, \phi_k(\boldsymbol{x}') \, \psi_m(t^n - |\boldsymbol{x}' - \boldsymbol{x}_\beta|)}{|\boldsymbol{x}' - \boldsymbol{x}_\beta|} \, dS_{\boldsymbol{x}'} = a(\boldsymbol{x}_\beta, t^n)$$

- Evaluate integrals:

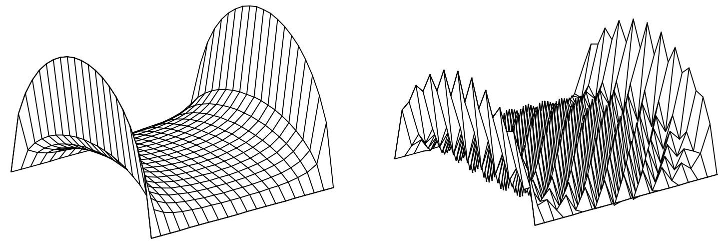
$$\sum_{m=0}^{n} Q_m \, \underline{U}^{n-m} = \underline{a}^n$$

- Rearrange sum to get time-stepping algorithm

$$Q_0 \, \underline{U}^n = \underline{a}^n - \sum_{m=1}^{n-1} Q_m \, \underline{U}^{n-m}$$

for $\underline{U}^n$ in terms of (very sparse) matrices $Q_m \in \mathbb{R}^{N_S \times N_S}$.

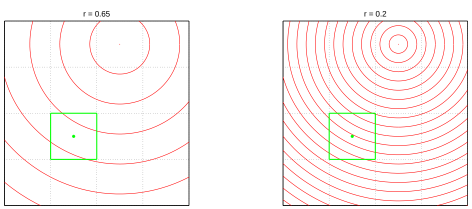
Numerical (in)stability



- Typical exponential oscillating instability (for EM) with $N_S = 20 \times 2$[...]
- This scheme appears **completely stable** for smaller N_S (nasty!)
- Stability depends crucially on:
  - **time basis functions** (low order generally better)
  - **integral evaluation** (needs to be accurate)

Integral evaluation

$$\int_{\Gamma_\alpha} \frac{U(\boldsymbol{x}', t^n - |\boldsymbol{x}' - \boldsymbol{x}_\beta|)}{\underbrace{|\boldsymbol{x}' - \boldsymbol{x}_\beta|}_{R}} \, dS_{\boldsymbol{x}'} = a(\boldsymbol{x}_\beta, t^n)$$



- Patchwise high order quadrature isn't sensible for low order approx
- Methods based on **local polar coordinates** centred at $\boldsymbol{x}_\beta$ can work well
- Can use exact integration (Stokes formula) on space–time patches

Galerkin approximations

- Some key names: Nédélec, Ha Duong, Terrasse
- Some key papers: Bamberger & Ha Duong (1986), Ha Duong (19[...] 2003), Terrasse (PhD thesis, 1993)
- Very nice theory, based on BIE as time convolution of integral oper[...] on Γ
- **Prove stability** (under exact integration) for an approx with piecew[...] linears in time, piecewise constants in space.

Galerkin – implementation

BIE: $$\int_\Gamma \frac{u(\boldsymbol{x}', t - |\boldsymbol{x}' - \boldsymbol{x}|)}{|\boldsymbol{x}' - \boldsymbol{x}|} \, dS_{\boldsymbol{x}'} = a(\boldsymbol{x}, t) \quad \boldsymbol{x} \in \Gamma, \, t > 0.$$

- Approx u in space (N_S basis functions, PWC) and time (N_T functions, PWL)
- Integrate BIE against test functions in time and space — 5D integrals over subregions of $\Gamma \times \Gamma \times (0, T)$ (use Stokes to reduce dimension)
- Again get time-stepping algorithm of the form

$$Q_0 \, \underline{U}^n = \underline{a}^n - \sum_{m=1}^{n-1} Q_m \, \underline{U}^{n-m}$$

with **very sparse** matrices Q_m

Convolution quadrature

- **Lubich** (1994). Assume: a is smooth in time, vanishes near $t = 0$
- Take the Laplace transform of the BIE — gives single layer potentia (SLP) equation for Helmholtz:

$$\int_\Gamma K(\boldsymbol{x} - \boldsymbol{x}', s) \, \overline{u}(\boldsymbol{x}', s) \, dS_{\boldsymbol{x}'} = \overline{a}(\boldsymbol{x}, s), \quad K(\boldsymbol{x}) = e^{i\,s\,|\boldsymbol{x}|}/|\boldsymbol{x}|$$

- Use Galerkin approximation in space for SLP.
- Convolution quadrature method: treat s like $\partial/\partial t$ and approx it by f difference quotient from an A–stable LMM (highest order is BDF2)
- Gives (usual) time-stepping algorithm $Q_0 \, \underline{U}^n = \underline{a}^n - \sum_{m=1}^{n-1} Q_m \, \underline{U}^{n}$ but now for **full** matrices Q_m.

  Operations count: $O(N_S^2 \, N_T^2)$

Speeded up versions

- "Raw" version:

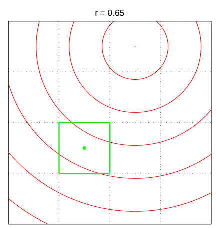
$$Q_0 \, \underline{U}^n = \underline{a}^n - \sum_{m=1}^{n-1} Q_m \, \underline{U}^{n-m}$$

  in $O(N_S^2 \, N_T^2)$ flops.

- Lubich (1994): can use FFT to speed up the recursive sums (Hairer, Lubich, Schlichte 1985) to give $O(N_S^2 \, N_T \, \log^2 N_T)$
- Hackbush, Kress, Sauter (preprint, 2006): introduce a 'cut-off' strategy to sparsify the matrices Q_m to facilitate panel clustering
- Note: Lubich shows that perturbing the matrices will still give a stable method (effect of perturbations not known for space–time Galerkin)

Operations count

- **Collocation or Galerkin** (unknowns: N_S in space, N_T in time)



$$Q_0 \, \underline{U}^n = \underline{a}^n - \sum_{m=1}^{n-1} Q_m \, \underline{U}^{n-m}$$

RHS takes $O(N_S^2)$ flops per time-step

Total is $\boldsymbol{O(N_S^2 \, N_T)}$

- **PDE approach** for acoustic wave equation: $\boldsymbol{O(N_S^{3/2} \, N_T)}$
- **Convolution quadrature**
  - "raw": $\boldsymbol{O(N_S^2 \, N_T^2)}$
  - with Hairer–Lubich–Schlichte speed-up: $\boldsymbol{O(N_S^2 \, N_T \, \log^2 N_T)}$
  - with sparsification + panel clustering(??): $\boldsymbol{O(\log^{11} N_S \, N_T^{9/2})}$

Fast plane wave methods

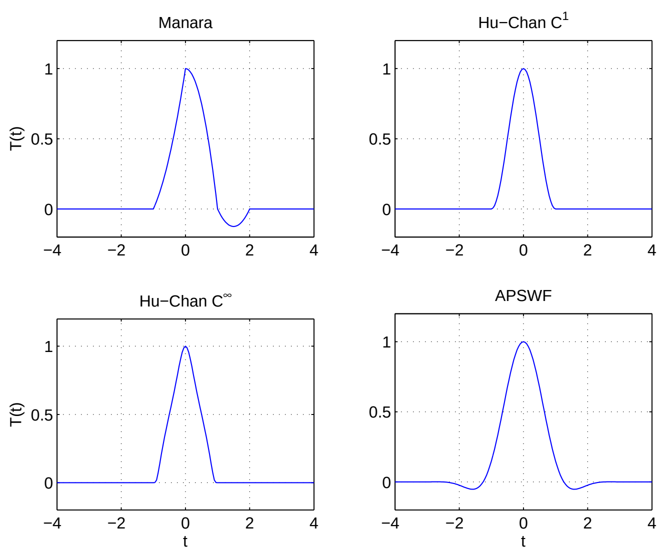
- Michielssen and team have developed and implemented the **plane wave time domain (PWTD)** algorithm
- Time domain counterpart of the fast multipole method
- Expand u in terms of band limited functions in time
- Complicated, but fast!
  - two-level: $O(N_S^{3/2}\, N_T)$
  - multi-level: $O(N_S\, \log^2 N_S\, N_T)$
  - (compare with $O(N_S^2\, N_T)$ for standard collocation or Galerkin)

Other temporal basis functions

- Many different approaches have been used by (mainly) engineers. much (any?) analysis.
- **global basis functions** e.g. Chung, Sarkar et al: Laguerre polyno
- **local basis functions**
  - Manara: piecewise quadratic
  - Hu and Chan: based on $\cos^2 t$
  - Hu and Chan: based on C^∞ exponential
  - Michielssen: approximate prolate spheroidal wave functions

$$T(t) = \left(\frac{\sin s\omega_0 t}{s\omega_0 t}\right) \frac{\sin\left(a\sqrt{(t/N\Delta t)^2 - 1}\right)}{\sinh(a)\sqrt{(t/N\Delta t)^2 - 1}}$$

Local temporal BFs



Fourier analysis of BIE: $\Gamma = \mathbb{R}^2$

- BIE:

$$\int_{\mathbb{R}^2} \frac{u(\boldsymbol{x}',\, t^n - |\boldsymbol{x}' - \boldsymbol{x}|)}{|\boldsymbol{x}' - \boldsymbol{x}|}\, dS' = a(\boldsymbol{x}, t^n)\,, \quad \boldsymbol{x} \in \mathbb{R}^2$$

- Continuous Fourier transform at frequency $\boldsymbol{\omega}$:

$$2\pi \int_0^{t^n} J_0(|\boldsymbol{\omega}|R)\, \widehat{u}(\boldsymbol{\omega}, t^n - R)\, dR = \widehat{a}(\boldsymbol{\omega}, t^n)\,, \qquad \boldsymbol{\omega} \in \mathbb{R}^2$$

- Approximation of BIE: $\sum_{m=0}^{n} Q_m\, \underline{U}^{n-m} = \underline{a}^n$
- Discrete Fourier transform (infinite mesh) at spatial frequency $\boldsymbol{\omega}$:

$$\sum_{m=0}^{n} q_m(\boldsymbol{\omega})\, \tilde{U}^{n-m}(\boldsymbol{\omega}) = \tilde{a}^n(\boldsymbol{\omega})\,, \qquad \boldsymbol{\omega} \in \left[\frac{-\pi}{\Delta x}, \frac{\pi}{\Delta x}\right]^2$$

Fourier analysis –2

- DFT convolution sum: $\sum_{m=0}^{n} q_m(\boldsymbol{\omega})\,\tilde{U}^{n-m}(\boldsymbol{\omega}) = \tilde{a}^n(\boldsymbol{\omega})$
- Invert (e.g. by using $Z-$transforms):

$$\tilde{U}^n = \sum_{m=1}^{n-1} p_m\, \tilde{a}^{n-m}\,, \quad \text{where} \quad p_n = -q_0^{-1} \sum_{m=1}^{n} q_m\, p_{n-m}$$

stability is determined by the coefficients $|p_n(\boldsymbol{\omega})|$

- Need $|p_n(\boldsymbol{\omega})|$ to be bounded for all $\boldsymbol{\omega} \in [-\pi/\Delta x, \pi/\Delta x]^2$ and all n.
- (Relatively) quick to compute, and seems to be a reliable test (ignoring edge effects)

Higher order approximations?

- Most (all?) existing approx are low order in time and space
- **In space:** could use higher order polynomials away from edges (d
seem to affect stability for collocation)
- **In time:** need to be careful – stability very sensitive to time approx
(collocation)
- Use Fourier transformed (in space) BIE to develop new time appro
(with **Dugald Duncan** and **Hermann Brunner**)

$$2\pi \int_0^{t^n} J_0(|\boldsymbol{\omega}|R)\, \widehat{u}(\boldsymbol{\omega}, t^n - R)\, dR = \widehat{a}(\boldsymbol{\omega}, t^n)\,, \quad \boldsymbol{\omega} \in \mathbb{R}^2$$

— first kind Volterra IE with smooth, oscillatory (convolution) kerne

Time approx for Volterra IE

$$2\pi \int_0^{t^n} J_0(|\boldsymbol{\omega}|R)\, \widehat{u}(\boldsymbol{\omega}, t^n - R)\, dR = \widehat{a}(\boldsymbol{\omega}, t^n)\,, \quad \boldsymbol{\omega} \in \mathbb{R}^2$$

- Analysed **discontinous Galerkin (DG)** approx – converges, but not quite as well as collocation...
- **DG result** for mth degree polynomials on intervals of size Δt:
  - global convergence: Δt^{m+1} for even m; Δt^m for odd m
  - local superconvergence: Δt^{m+2} for even m (needs extra condition on exact solution); Δt^{m+1} for odd m
- **Collocation result:** (Brunner) – can always choose collocation points to give Δt^{m+1} global convergence and Δt^{m+2} local superconvergence

DG in time for the BIE

- Approx the BIE in space only (collocation):

$$\int_0^t Q(s)\, \underline{U}(t - t')dt' = \underline{a}(t), \quad t > 0$$

— matrix Q comes from the space approx

- System of 1st kind VIEs: use mth degree DG to approx in time
- (Spatial) Fourier “Stability coefficients” $p_n(\boldsymbol{\omega})$ are now $(m+1) \times (m$
matrices
- $p_n(\boldsymbol{\omega})$ appear to bounded for all frequencies (i.e. **stable**) for $m = 0, 1, \ldots, 9$ (space approx is a Fourier interpolant)
- **But** scheme on a flat square plate looks **unstable** even for $m = 0$ (singular behaviour at corners?)

Galerkin in space, DG in time

- Approx u in space (Γ is unit square plate):

$$u(\boldsymbol{x},t) \approx U(\boldsymbol{x},t) = \sum_k U_k(t)\,\phi_k(\boldsymbol{x})$$

ϕ_k: global polynomial BFs in $\mathbb{P}^\nu(0,1)\times\mathbb{P}^\nu(0,1)$, $\quad k = 1:(\nu+1)^2$

- Use **full Galerkin** in space – gives system of Volterra IEs:

$$\int_0^t Q(s)\,\underline{U}(t-t')dt' = \underline{a}(t), \quad t>0$$

- **Result:** scheme appears stable for space degrees $\nu = 0,\ldots,36$ and DG0 in time.
- **Collocation** in space, DG0 in time is unstable

EM – perfect conductor

- **Rynne** (1999) well-posedness result for EFIE for smooth closed Γ
- **Terrasse** (PhD 1993) introduced functional framework for time dependent BIEs in EM problems
- **Michielssen** fast plane wave methods; singular edge behaviour; ve impressive simulations; . . .
- **Simulation:** scattering from a perfectly conducting narrow strip (1 model)

No dispersion at large distances

EM simulation

Scattering from a perfectly conducting narrow strip (1D model)

Shows 1D surface current

run

No dispersion at large distances.

Why time domain BIE?

- Can be implemented **more efficiently** than
  - full time domain PDE discretisation:

$$O(N^3 \log N) \quad \text{versus} \quad O(N^4)$$

  on a $O(N\times N\times N)$ space domain over $O(N)$ time steps (Michielssen + team)
  - frequency domain BIE (Nédélec)
- Mesh: only on the **surface**
- **No dispersion** in scattered field computation
- **But** theory and numerics much less well developed . . .

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