Limiting measures in some simple examples

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Here are some simples examples that motivate questions about QUE. They come from considering the eigenfunctions for the Laplacian on the line, square, and disc. In these cases we do not have chaotic billiard ball motions. As we will see below, in one dimension it is easy to show that QUE holds, while in two dimensions, some interesting things happen.

1 Line

Here is what happens in the one-dimensional setting. Consider the eigenvalue problem for the Laplacian,

$$-\frac{d^2u}{dx^2} = \lambda u$$

where u is defined on $[0,1]$ and we have boundary conditions, $u(0) = u(1) = 0$.

The solutions to this are linear combinations of $u_k(x) = \sin 2\pi kx$. We think of u_k as an eigenfunction or state and a highly excited state if k is large.

Now let $\phi(x)$ be a smooth test function. The integral

$$\int_0^1 \phi(x)|u_k(x)|^2 dx = \int_0^1 \phi(x)|\sin 2\pi kx|^2 dx = \int_0^1 \phi(x)\frac{(1-\cos 4\pi kx)}{2}dx.$$

By the Riemann-Lebesgue lemma, this tends to $\frac{1}{2}\int_0^1 \phi(x)dx$.

Hence the sequence (and thus every subsequence) of measures $|u_k(x)|^2 dx$ converges weakly to $\frac{1}{2}dx$ and QUE holds!

2 Square

Now consider the $2-$dimensional Laplacian defined on the square $[0,1]\times[0,1]$ with zero boundary conditions. Then the eigenfunctions are $u_k(x)u_n(y)$ with eigenvalues m^2+n^2. If both m and n tend to infinity then by the same argument as above the sequence of measures

$$|u_m(x)u_n(y)|^2 dxdy$$

converges weakly to $\frac{1}{4}dxdy$. However, if $m = 1$ and n tends to infinity then we have convergence to $\frac{1}{2}\sin^2(y)dxdy$. Clearly this does not converge to a constant times Lebesgue measure. This is not inconsistent with QUE since for a square we do not have chaotic billiard ball motion with unstable orbits. However, the measure here with the factor of $\sin^2 y$ term is not so unlike the bouncing ball modes of the Bunimovich stadium and gives a hint as to the behavior of the eigenstates for the stadium.

For the square, it also happens that sometimes we do not have convergence for every subsequence of eigenfunctions and their corresponding measures. Just modify the above example so that the values of m vary between 1 and 2 and n tends to infinity.

3 Disc

For a disc of radius one, the corresponding Laplacian has eigenfunctions in polar coordinates

$$C_{n,m} \sin n\theta J_n(\rho_{n,m} r) \quad \text{or} \quad C_{n,m} \cos n\theta J_n(\rho_{n,m} r)$$

where J_n is the Bessel function of order n and the parameters $\rho_{n,m}$ are roots of J_n and $C_{n,m}$ is a normalizing constant so that these eigenfunctions have constant norm. (We did not need to do this in the line and square cases since the norms were already constant.) Let $\phi(r, \theta)$ be a test function. Then, for example, consider the integral

$$C_{n,m}^2 \int_0^{2\pi} \int_0^1 \phi(r, \theta) |\cos(n\theta) J_n(\rho_{n,m} r)|^2 r dr d\theta.$$

Let us look at the case where $n = 0$ and where we indicate $\rho_{0,m}$ by ρ_m, and $C_{0,m}$ by C_m. Then we have

$$C_m^2 \int_0^{2\pi} \int_0^1 \phi(r, \theta) |J_0(\rho_m r)|^2 r dr d\theta.$$

From the asymptotics of the Bessel function we know that if x is large then

$$J_0(\rho r) \sim \sqrt{\frac{2}{\pi x}} (\cos(x - \frac{\pi}{4}))$$

.

If we proceed heuristically and substitute this into the above integral, then we have

$$\frac{2C_m^2}{\pi \rho_m} \int_0^{2\pi} \int_0^1 \phi(r, \theta) (\cos^2(\rho_m r - \frac{\pi}{4})) dr d\theta.$$

Now it is known that $C_m^2 = \frac{2}{J_1(\rho_m)^2}$ and using the similar asymptotics for J_1 and the Riemann-Lebesgue lemma we can conclude that if m tends to infinity this sequence of measures converges weakly to a constant times $dr d\theta$, which is not Lebesgue measure.

The above argument can be made precise by a more careful analysis of the integral. For other values of n fixed we will have convergence to $\cos^2 n\theta dr d\theta$ and in the case of both n and m tending to infinity, the argument must be modified at bit, but we will have a constant times $dr d\theta$ for our limiting measure. Just as in the square case sequences can be constructed that do not converge at all.

It is quite interesting that for the examples where we have convergence for the disc case, the measure is singular at the origin. Thus somehow the Bunimovich stadium gives measures that are a hybrid of the square and the disc, which perhaps should not be too surprising.