Weak subconvexity

An L-function $L(s) = \sum_{n=1}^{\infty} \lambda(n) n^{-s}$ is a Dirichlet series which has an Euler product and a functional equation which relates its value at s with its value at $1-s$; the Riemann zeta-function is the prototypical L-function.

Of great interest is the size of $L(1/2)$ at the point of symmetry, or more generally the growth of $L(1/2+it)$ for large t. Good bounds for these quantities can be used to deduce equidistribution of arithmetic quantities. For example, Ramanujan asked about integer values represented by the quadratic form $x^2 + y^2 + 10z^2$; the pioneering works of Iwaniec, Duke and Schulze-Pillott on subconvexity shows that all large odd numbers can be written in this form. Another example where subconvexity is the key is the work of Cogdell, Piatetski-Shapiro, and Sarnak on Hilbert's 11th problem about the representation of integers in number fields by quadratic forms.

The so-called 'convexity' bound is the bound for the L-function at the center of the critical strip that one obtains by applying the Phragmen-Lindelöf theorem to simple bounds for the L-function at the edges of the critical strip and interpolating convexly. In terms of the analytic conductor $c(L)$ of the L-function, the convexity bound asserts that $L(1/2) \ll c(L)^{1/4+\epsilon}$; this follows, very roughly speaking, from the trivial bounds $L(1) \ll c(L)^{\epsilon}$ and $L(0) \ll c(L)^{1/2+\epsilon}$ at the edges of the critical strip. If one can prove that there is a $\delta > 0$ such that $L(1/2) \ll c(L)^{1/4-\delta}$, then one has achieved 'subconvexity' and a corollary is often an equidistribution theorem.

Soundararajan realized the possibility that equidistribution theorems might be obtained using a weaker form of convexity, basically where one replaces the bound above by

$$L(1/2) \ll \frac{c(L)^{1/4}}{(\log c(L))^{1-\epsilon}};$$

Soundararajan can prove this bound for a very general class of L-functions.

His method is very subtle, and grew out of ideas of Halasz, Hildebrand, Elliott, and joint work that he had previously done with Granville. The starting point is that bounds for L-functions are equivalent to bounds for partial sums of the coefficients. The basic idea is to try to achieve cancelation in the coefficient sums

$$\sum_{n \le x} \lambda(n)$$

for x as small as possible in terms of the conductor $c(L)$. Using fairly standard methods one can obtain cancelation as soon as x is as large as $c(L)^{1/2}(\log c(L))^{A}$ for any $A > 0$. In this new paper Soundararajan shows that

$$\sum_{n \le x} \lambda(n) \ll \frac{x}{(\log x)^{1-\epsilon}}$$

i.e. there is cancelation, for x as small as $c(L)^{1/2}(\log c(L))^{-A}$ for all $A > 0$.

The idea, which goes back to Hildebrand is that mean values of multiplicative functions cannot vary too quickly. Indeed, for real valued multiplicative functions taking values between -1 and 1, it is the case that

$$\frac{1}{x}\sum_{n\le x}\lambda(n)=\frac{w}{x}\sum_{n\le \frac{x}{w}}\lambda(n)+o(1)$$

for any w with $1\le w\le \sqrt{x}$. The example $\lambda(n)=n^{it}$ shows that this theorem is not true for complex valued λ. Elliott saw how to handle complex coefficients, namely by allowing a possible factor of w^{it} into the factor in front of the second sum above. Soundararajan here extends these results to the situation where the λ are not only complex, but also may take values outside the unit circle. In this more complicated situation, the original mean value is seen to be a linear combination of shorter mean values of the sort Elliott introduced. Finding the right linear combination that works builds on the "pretentiousness" ideas developed by Granville and Soundararajan; this is the main new idea.

The connection with mass distribution and QUE is through Watson's formula:

$$\left(\int_{\mathcal{F}}\phi(z)F_k(z)d\mu\right)^2=C_{k,\phi}\frac{L(1/2,f\times f\times\phi)}{L(\mathrm{sym}^2 f,1)^2}$$

where the integral is over a fundamental domain $\mathcal{F}$ for the upper half-plane modulo $SL(2,\mathbf{Z})$; f is a weight k Hecke eigenform for the full modular group; $F_k(z)=y^k|f(z)|^2$ and ϕ is a Maass cusp-form. Here $C_{k,\phi}\approx_\phi k^{-1}$. This is the starting point for Soundararajan's approach. The goal is to prove that the integral on the left tends to 0 as $k\to\infty$. The L-function in the numerator on the right-hand-side is the triple product L-function of Garrett evaluated at the center of the critical strip and the L-function in the denominator is the symmetric square L-function associated with the cusp form f evaluated at the edge of the critical strip. Soundararajan's weak-subconvexity ideas can be applied to bound the numerator L-function in such a way as to achieve this goal for all but k^ϵ all forms f.

Despite its inherent complications, Soundararajan's argument is quite elegant and is within the reach of a wide audience of mathematicians. His paper is remarkably complete, containing full details from first principles.