AIM workshop counterexample to Wall's conjecture

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Wall's conjecture, that the number of maximal subgroups of any given finite group is less than the group order, was first published by G.E. Wall in 1961. Serious progress on it was made in 2007 by Martin Liebeck, Laszlo Pyber, and Aner Shalev, who proved, if the group in question was simple, then it satisfied Wall's conjecture in all but finitely many cases. This largely directed attention to composite groups, where Wall in his original paper had at least shown the conjecture to be true for finite solvable groups. The key remaining cases were known to be semidirect products of a vector space V with a nearly simple finite group G acting faithfully and irreducibly on it. Here the maximal subgroups can be elegantly described using the first cohomology group $H^1(G, V)$. In particular, the number of maximal subgroups (in the semidirect product) that do not contain V is precisely the order of this cohomology group multliplied by the order of V, and the remaining maximal subgroups are in bijection with those of G. The order of the semidirect product group itself is the product of the orders of G and V. This leads to issues of comparison between the orders of the groups $H^1(G, V)$ and G.

In 1986 Bob Guralnick made the very relevant conjecture that $\dim H^1(G, V) < C$, for some universal constant C. Here G could be any finite group, acting faithfully and absolutely irreducibly on V, though the conjecture did reduce to the case of finite simple groups. The "absolute" irreducibility could always be assumed by changing coefficients, assuming only an irreducible action, as in the cases relevant to Wall's conjecture.

It was this conjecture of Guralnick, and recent dramatic progress on it (and related questions), that had inspired our AIM workshop. This progress said, in particular, that "yes" there was such a bound, provided one made a restriction on (the so-called Lie rank of) the group G. One goal of our AIM conference was to see if this recent progress, suitably supplemented by efforts initiated at the conference, might be sufficient to establish the Wall conjecture, and several early sessions were devoted to this task.

Bob Guralnick gave a talk noting that it was not necessary to prove his conjecture about a universal constant bound, and that, instead, it would be sufficient–and was definitely necessary–to show $\dim H^1(G, V)$ was bounded by what might be called the Lie dimension, the dimension of the underlying algebraic group as a variety, in the the case of a finite simple group G of Lie type. (The order of such a group can always be computed exactly, but is well approximated by the number obtained by raising the size of its field of definition to a power equal to the Lie dimension. So, in most cases, size comparisons of $H^1(G, V)$ and G reduce to size comparisons of $\dim H^1(G, V)$ with the Lie dimension of G.) This was quite consistent with what might be called "conventional wisdom" at the time, that, even if Guralnick's conjecture proved false, that the bound C should grow relatively slowly with the Lie rank. (The square of the dimension of the Lie rank is, in most cases, larger than the Lie dimension.)

A possible value, consistent with known calculations, for the proposed universal constant bound C had been 2 until 2003, when a paper of Scott presented computer calculations of his undergraduate students for groups $PSL(n, F)$ with F a finite field. The $n = 6$ calculations of related Kazhdan-Lusztig polynomials, gave infinitely many cases with $\dim H^1(G, V) = 3$. (This required also quoting powerful theoretical advances: the theory of rational and generic cohomology theory, which related cohomology to finite groups of Lie type to those of ambient algebraic groups, and the validity over fields of large characteritics of the Lusztig conjecture for algebraic group irreducible modules. The latter gave not only the dimension of these modules, but–as realized later–also their cohomology, in many cases, all in terms of recursively defined polynomials.) For the groups $G = PSL(6, F)$ group the Lie dimension is 25, much greater than 3, so, in 2003, the Wall conjecture was quite safe from cohomology theory. Let us mention here that for the next cases $n = 7$ and $n = 8$, the Lie dimensions of $PSL(n, F)$ are 37 and 63, respectively.

The value 3 as a possible universal constant bound stood until our AIM conference. On the third day of the conference, a Wednesday, Scott reported that a new undergraduate student, Tim Sprowl, had been trying to run one of the old programs for the case $n = 7$. It had crashed when it was 80 per cent finished, but had in the process produced examples with $\dim H^1(G, V) = 4$ and 5. Though many conference participants felt this made any universal constant bound unlikely to exist (or had already felt that way), the values of 4 and 5 were still well below the Lie dimension, and Wall's conjecture was, briefly, still safe. However, Frank Luebeck, the director of the long-productive GAP computer project, was in the audience, and knew that he had a program on his desktop computer in Aachen, that, though not yet fully tested, could similarly compute the relevant Kazhdan-Lusztig polynomials, even for other types (of groups of Lie type). Overnight, he made electronic connection with his computer and produced an amazing array of calculations. These included completing the remaining 20 per cent of Sprowl's calculations, and new calculations for all Lie types. For the cases in groups G of Lie type $F4$ and $E6$, the calculations led to values of $\dim H^1(G, V)$ greater than the Lie dimension.

Bob Guralnick immediately noticed this and announced the demise of Wall's conjecture to the audience. There was a clear impact, and some people in the audience took flash photos of Luebeck with their cell phones. Somewhat taken aback by all the attention, Luebeck spent most of the rest of the day thinking through the logic of his program. After the talk, Scott privately asked Luebeck to also try the group $PSL(n, F)$ for $n = 8$, and Luebeck made some calculations overnight (Thursday). He obtained some values greater than the Lie dimension, and promised to e-mail Scott a list. At that point, both Luebeck and Scott realized the potential for confirmation of Luebeck's computer results for this case. This was carried out several weeks later, when Scott and his student (who had now got his program to complete for $n = 7$ and also run for $n = 8$) exchanged e-mails lists, piecemeal, to insure independence. All answers agreed perfectly, producing twelve Kazhdan-Lusztig polynomials, each of which corresponded to examples of $\dim H^1(G, V) > 63$, the Lie dimension for $PSL(8, F)$, and, thus, each corresponding to infinitely many counterexamples to Wall's conjecture (for F a field of p elements, p any sufficiently large prime). The largest obtained was $\dim H^1(G, V) = 469$, easily exceeding 63, and of the general order of magnitude for calculations Luebeck had made at the conference in the $F4$ and $E6$ cases.

The effect of Luebeck's calculations at the conference had been, in Bob's later words "revolutionary". Everyone had at least believed that these 1-cohomology groups were really fairly small, and if their dimension grew, it would be a gradual thing. This could no longer be believed. On the positive side, some of the bounds previously obtained for $\dim H^1(G, V)$, originally thought ridiculously large, began to look more realistic.. This was also true for bounds for higher cohomology groups, in research sparked by the $n = 1$ case. Perhaps more importantly, it became clear from results presented at the conference that the Guralnick and Wall conjectures had helped lead the subject of finite group cohomology to a genuinely higher level, where it became plausible to begin thinking about what a general theory of degree 1 and higher degree cohomology with irreducible coefficients might look like, and how far modern methods had gone to making calculations possible.

This account leaves out some things, such as the progress and discussions at the AIM workshop on the corresponding picture for Ext groups with (pairs of) irreducible coefficients, and the discussion of a conjecture of Aschbacker and Guralnick, not made at the conference, but earlier, that would now rise to be the main conjecture in maximal subgroup theory. (The conjecture states that it is the number of conjugacy classes of maximal subgroups that is bounded, less than the number of conjugacy classes of elements in the group.) Aschbacher, of course, was among the conference participants. It is also noteworthy that a weaker version of Wall's conjecture is true, as Bob Guralnick remarked in a recent e-mail to Tim Wall. Namely, the number of maximal subgroups of a group G is at most some (universal) constant times the order of the group raised to a power $1+a$, with $a = 1/2$ sufficient, and the actual value of a likely to be much smaller. We can get some feel for this in the current examples, taking the field F to have order p for some large prime p. Then the number of maximal subgroups in one of the examples is $p^{(469+\dim V)}$, whereas the group order is roughly $p^{(63+\dim V)}$. The exact value of $\dim V$ hasn't been calculated (though there is a recursive formula for it). However, judging from similar modules for the $n = 6$ case, $\dim V > 410,000$ (and the true value is likely closer to 100,000,000). So the number of maximal subgroups is approximately the group order raised to the power $1 + a$ with $a < 1/1000$, and likely $a < 1/100,000$. From this point of view, Wall's conjecture wasn't so far off.