Thursday afternoon discussion

Geometric perspectives in mathematical quantum field theory
American Institute of Mathematics, Palo Alto

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1 Alice: BRST

Alice: Here is a meta problem. I seem to be one of the few people with her feet still in the 50s, working on the relationship between Hamiltonian and path integral pictures. I work with BRST methods, supermanifold methods, and generalization of stochastic calculus on such things, and I get my hands dirty. I will begin with a pedagogical discussions: things happen when you integrate over anticommuting variables (the Berezinian rules, which were invented to give the relationship between kernels and traces and so on) scale the other way as they do for ordinary variables, so you can often get results from scaling arguments. This is linked to theories with symmetries.

To begin, let me say just a little bit about the non-BRST setting that I will try to handle via BRST. This is the situation in which you have quantum mechanics on a symplectic manifold with symmetries, and so you should do the Marsden–Weinstein reduction. The actual key thing is the supertrace, which has the ability to project onto the cohomology.

The basic setting that you start with is a symplectic manifold P of dimension $2n$ ($\mathbb{R}^{2n}$ or T^*M), which you do know how to quantize — how to represent your ps and qs. Suppose also that you have a Lie group G of dimension $m \leq n$, and for pedagogical reasons let's suppose that it acts freely by symplectomorphisms on P.

The *moment map*, or rather its transpose, is a function $T : \mathrm{Lie}(G) \to \mathrm{Fun}(P)$, and we demand that $\mathcal{L}_{\tilde{\rho}} f = \{T_\rho, f\}$ for $\rho \in \mathrm{Lie}(G)$. The other thing is that $T_\rho(gy) = T_{\mathrm{Ad}_g(\rho)}(y)$ for $y \in P$ and $\rho \in \mathrm{Lie}(G)$ and $g \in G$. This is given to you by Noether's theorem.

Define C to be the vanishing locus of T_ρ, with ρ ranging over $\mathrm{Lie}(G)$. This C is not symplectic — it is of dimension $2n-m$ — but G still acts on C, and the quotient C/G, called the *Marsden–Weinstein reduction* $P//G$, is symplectic of dimension $2(n-m)$.

The problem is that $P//G$ is generally very complicated, and so you don't know how to quantize it. The idea will be to take the quotient cohomologically, and then you still know how to quantize.

Recall Lie algebra homology: You take $\mathrm{Fun}(P) \otimes \bigwedge(\mathrm{Lie}\, G)$, and you will try to define a differential δ of degree -1. It will be defined by $\delta f = 0$ for $f \in \mathrm{Fun}(P)$, and for $\rho \in \mathrm{Lie}\, G$ you set $\delta\rho = T_\rho$.

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You extend this by declaring that δ must be an (odd super)derivative: $\delta(ab) = \delta a\, b + (-1)^{|a|} a\, \delta b$. You can check directly that $\delta^2 = 0$.

Jonathon: This is not the Weil complex. Alice: No, it isn't. G still acts everywhere, and you can do this equivariantly, but I am not an expert.

If we look at $\mathrm{H}_0(\delta) = \ker_0 \delta / \operatorname{im}_0 \delta = \mathrm{Fun}(P)/\{f^a T_a\}$, where $\{\rho_a\}$ is a basis of $\mathrm{Lie}(G)$ and $T_a = T_{\rho_a}$. Optimistically, $\delta(\mathrm{Lie}\, G) = \{f \in \mathrm{Fun}(P) \text{ s.t. } f|_C = 0\}$, and so $\mathrm{H}_0 = \mathrm{Fun}(C)$.

Let us furthermore introduce a differential d on $L \otimes \bigwedge(\mathrm{Lie}\, G)^*$ for any $\mathrm{Lie}\, G$ space K, by defining $dk : \xi \mapsto \xi \triangleright k$ for $k \in K$, $\xi \in \mathrm{Lie}\, G$, and $\triangleright$ is the action. Now, we can think of $(\mathrm{Lie}\, G)^*$ as left-invariant one-forms on G, and so we make d act via the exterior derivative. Tim: So d acts by precomposing with the bracket (perhaps up to some numerical factor). Alice: Yes.

Moreover, d and δ supercommute on $\mathrm{Fun}(P) \otimes \bigwedge(\mathrm{Lie}\, G) \otimes \bigwedge(\mathrm{Lie}\, G)^*$, and so you can form a double complex, which we can totalize by putting $\mathrm{Lie}\, G$ in degree -1 and $(\mathrm{Lie}\, G)^*$ in degree 1, so that we will work in cohomological degrees.

What is this cohomology? There are a number of nontrivial algebraic conditions that must be satisfied, but what you find is that H^0_d are the G-invariant elements of K. What would we like to use for K? We could use the whole complex $\mathrm{Fun}(P) \otimes \bigwedge(\mathrm{Lie}\, G)$, or we could use its $\mathrm{Fun}(C) = \mathrm{H}_0(\mathrm{Fun}(P) \otimes \bigwedge(\mathrm{Lie}\, G), \delta)$. The argument that we know must perform is a spectral sequence convincing you that the double cohomology $\mathrm{H}^i_d(\mathrm{H}^j_\delta(\mathrm{Fun}(P) \otimes \bigwedge(\mathrm{Lie}\, G) \otimes \bigwedge(\mathrm{Lie}\, G)^*))$ is the cohomology of the total complex.

This all has an interpretation in supermanifolds. We have introduces already a basis $\{\rho_a\}$ for $\mathrm{Lie}\, G$, and let's take the dual basis η^a for $(\mathrm{Lie}\, G)^*$. Then you can think of $\mathrm{Fun}(P) \otimes \bigwedge(\mathrm{Lie}\, G) \otimes \bigwedge(\mathrm{Lie}\, G)^*$ as $P \times \mathbb{R}^{0,2m}$, and give it a symplectic form $\omega = \mathrm{d}p_i \wedge \mathrm{d}q^i + \mathrm{d}\rho_a \wedge \mathrm{d}\eta^a$. This gives the Poisson brackets $\{\rho_a, \eta^b\} = \delta_a^b = \{\eta^b, \rho_a\}$ (the sign is correct).

Finally, the miracle is that $\Omega = d + \delta$ we can think of as a function on the extended phase space $P \times \mathbb{R}^{0,2m}$. What you find is that $\{\Omega, f\} = (d + \delta) f$. In coordinates,

$$\Omega = \eta^a T_a - \frac{1}{2} C^c_{ab} \eta^a \eta^b \rho_c$$

Now, because we have a simple symplectic form, we can declare that our states will be functions $\psi(q, \eta)$, and represent $p_i = -i\hbar \frac{\partial}{\partial q^i}$ and $\rho_a = -i\hbar \frac{\partial}{\partial \eta^a}$, when for example $P = \mathbb{R}^{2n}$. It is essentially just as easy when $P = \mathrm{T}^* M$.

Finally, let's talk a bit about dynamics. In ordinary settings, we want to work with $\exp(-iHt)$, and we will quickly drop the i and look at the kernel $\exp(-Ht)(x_F, x_I)$. Perhaps we then want to integrate over $x_F = x_I$, which will calculate a trace. In the super situation, we can do the same, but now we end up computing a supertrace over the space of states.

Let $\mathcal{H}$ be the space of states, and split it as $\mathcal{H} = G \oplus F \oplus E$ where $G = \operatorname{im} \Omega$ and $G \oplus E = \ker \Omega$. Then E is the cohomology, and $\Omega : F_\bullet \to G_{\bullet+1}$ is an isomorphism. The condition on the Hamiltonian is

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that it respects the symmetry, which is that $\{H, \Omega\} = 0$, and it follows from this that the supertrace of (any function of) H is its supertrace on E.

The last thing to do is to *gauge fix*, which is when you add to the Hamiltonian H something that looks homologically trivial of the form $[\Omega, \chi]$, to improve the analytic properties of the heat kernel. What happens practically is that you add in extra quadratic terms, but fermionic gaussians scale the other way than ordinary gaussians.

In particular, there are various results, in particular in Witten's susy qm and Morse theory, where you can button up every mathematical detail. In QFT, this builds on what Reshetikhin was saying yesterday. Another example is when you extend your phase space by a new parameter t and its conjugate E, and then $H - E$ is the infinitesimal generator of paths, so that Ht is part of your BRST extended action. This turns out to be a powerful technique when the standard BRST techniques do not work because the symmetry has a secondary constraint.

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