# Wednesday afternoon discussion

Geometric perspectives in mathematical quantum field theory
American Institute of Mathematics, Palo Alto

10 April 2013

## 1 Malek: QFT in 30 minutes

There is an article in the Bulletin of the AMS called "QFT in 90 minutes" — we know a lot more about QFT by now.

QFT involves two pieces: a differential geometry aspect, and a question about integrating over fields. This morning, Reshetikhin explained some of the differential geometry portion. For the next 30 minutes, we will talk about the other part: integrating over fields. We will work in the context of renormalization.

We will focus on scalar fields $\phi : \mathbb{K}^d \to \mathbb{R}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{Q}_p$ or any local field. The basic problem is to define a probability measure $\mathrm{d}\nu$ on these $\phi$. I.e. there is a space of distributions $\mathcal{S}'(\mathbb{K}^d)$, and by definition, $\mathrm{QFT} = \mathrm{d}\nu = \{\langle \Phi(f_1) \cdots \Phi(f_n)\rangle = \int f_1(\phi) \cdots f_n(\phi)\, \mathrm{d}\nu(\phi)\}$.

We are interested in QFT in the case where

$$\mathrm{d}\nu = \frac{1}{Z} \exp\left[ - \int \frac{g}{4!}\phi^4 + \frac{1}{2}\mu\phi^2 + \frac{1}{2}a(\nabla\phi)^2 \right] \mathcal{D}\phi$$

where $\mu = m^2 + \dots$ and $a = 1 + \dots$, and the idea is to break the measure into a Gaussian part and a correction.

So, what we will do to regularize this is, for example, to choose a UV cutoff $r \ll 0$ and an IR cuttoff $s \gg 0$, which control the lattice spacing and the size of the box, respectively. With this, we can get a sequence of measures on the space of fields, and the question is convergence, and to what?

Generally, you should let $g, \mu, a$ depend on $r, s$, lest the limit be a $\delta$ function at $\phi = 0$, or a Gaussian measure, or something else boring. We make a "bare ansatz" that $g, \mu$ depend on $r$ alone, and ignore $s$.

The standard way of understanding this is via the Wilson renormalization group, of which there are various flavors: $\{$rigorous, nonrigorous$\} \times \{$perturbative, nonperturbative$\}$.

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Let's focus on the perturbative part, for simplicity. What we will try to do is to construct $\lim_{r\to-\infty}$ of the $n$-point functions not valued in $\mathbb{R}$, but rather in $\mathbb{R}[\![g_R]\!]$, by means of Feynman diagrams. Actually, keeping $\frac{1}{2}(\mu-m^2)+\frac{1}{2}(a-1)(\nabla\phi)^2$ as "interaction", and putting $(m^2,1)$ into the moments of the lattice measure, we a priori get a sequence of Feynman diagrams that try to live in:

$$\mathbb{R}[\![g_r, \mu_r - m^2, a_r - 1]\!]$$

and are distributional, where for now we imagine that $g_r, \mu_r, a_r$ are constants depending on the scale $r$.

Let's focus on $d = 4$, which is an example of "renormalizable" and not "superrenormalizable". Let's remove the box; then we expect translation invariance, and after factoring out a $\delta$ function, we can set

$$\hat{S}_4^T(0,0,0,0) = -g_R, \quad \hat{S}_2^T(0,0) = \frac{1}{m^2}, \quad \frac{\mathrm{d}}{\mathrm{d}g_R}\hat{S}_2^T(k,-k)|_0 = -\frac{\Delta}{m^4} \qquad (*)$$

where $\hat{S}^T$ is the Fourier transform of $S^T$, and $S^T$ is the value of some Feynman diagrams after factoring out a $\delta$ function.

Said another way, we are considering $g_r = g_R + \dots$ and so on as formal power series in $g_R$, and then we will solve $(*)$ for $g_r, \dots$ as functions of $g_R$, and it is a theorem that $(*)$ uniquely determine $g_r, a_r - 1, \mu_r - m^2 \in g_R\mathbb{R}[\![g_R]\!]$.

There is also an analysis part of the theorem. If you look at all other correlation functions, they are of the form $\sum \gamma_{r,i} g_R^i$, and the fact is $\lim_{r\to-\infty}\gamma_{r,i}$ exists.

## 2 Sylvie: prolegomenon renormalization

Sylvie: My goal is to understand the abstract structure of renormalization, and apply it in non-QFT contexts.

Jonathan: In a former paper, I banned my coauthors from using the word "renormalization." Sylvie: I claim that it is. Jonathan: Then we have an argument. Doug: That's a good start.

Sylvie: Here are two questions to which I do not have answers:

1. What is an abstract version of the renormalization group? Doug: Semigroup? Sylvie: I don't have an answer.
2. What is canonical about renormalization?

To begin, here is a no-go statement. One of the characteristics of Lebesgue measure is that it is translation-invariant. Can it be extended beyond $\mathrm{L}^1$ functions? I won't stick to integrals, but also to sums, e.g. the $\zeta$ function. Look for example as $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$, which is related via the


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Euler–Maclaurin formula to $\int_1^\infty \frac{1}{x^s}\mathrm{d}x$, and let me also consider $\int_{\mathbb{R}^d} \frac{1}{(1+k^2)^{s/2}}\mathrm{d}k$. What is in common among all three of these is that sometimes they converge, and sometimes they diverge, depending on $s$ — notice that I have UV (large $k, x, n$) divergence, but not IR divergence (none of the sums go to 0).

In these cases, we have various pseudodifferential symbols, e.g. $\sigma_s(x) = \frac{1}{|x|^s}\chi(x)$, where $\chi$ is a cut-off keeping you away from 0, or $\tau_s(x) = \frac{1}{(1+x^2)^{s/2}}$. In the last example, we want $\int_{\mathbb{R}^d} \tau_s(x)\mathrm{d}x$, and in the first example we want $\sum_{\mathbb{Z}\cap\mathbb{R}_+} \sigma_s$.

In any case, we want this where it does not converge. And that's what will not exist, via the no-go theorem. Jonathan: Let me try. To Euler, every sum made sense, the question was just how to make sense of it, keeping as many properties as we can. But you will tell us that a reasonable list rules out any possibility, and so you have to drop some reasonable condition. Sylvie: You will have to drop all of them.

We are going to allow symbols that are polyhomogeneous (slightly weaker than quasihomogeneous) at $\infty$ (e.g. we disallow $\log|x|$ and $\exp(x)$, but $\frac{1}{x^2+1} = x^{-2}(1+\dots)$, so that's OK — you'll have finitely many problemmatic orders for any fixed $s$). Note that these are not always $\mathrm{L}^1$; the intersection are those symbols that have degree $< -d$. Jonathan: And "symbol" means enough derivatives gets you into there? Sylvie: I don't intend to define "symbol". Finally, there is a very small class of symbols, namely the Schwartz functions:

$$\mathscr{S}(\mathbb{R}^d) \subseteq \mathrm{L}^1(\mathbb{R}^d) \subseteq \mathrm{CS}(\mathbb{R}^d)$$

The theorem says that $\int_{\mathbb{R}^d}$ does not extend linearly from $\mathscr{S}(\mathbb{R}^d)$ to $\mathrm{CS}(\mathbb{R}^d)$ in a way that preserves *any* of:

1. translation invariance (and continuity),
2. $\mathrm{GL}_d$ covariance (and continuity),
3. Stokes' property: the extension should vanish on total derivatives.

What you can do is to build an extension, and measure how much it fails these. Those are anomalies.

On the other hand, here is a positive statement: There is a unique (up to multiplicative constant) linear form on $\mathrm{CS}(\mathbb{R}^d)$, with properties 1,2,3 above — in fact, any one of them characterizes it — namely the *Wodzicki residue*

$$\mathrm{res}(\sigma) = \int_{S^{d-1}} \sigma_{-d}(x)\,\mathrm{d}_S x.$$

This *vanishes* on $\mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{CS}(\mathbb{R}^d)$. For example, look at $\frac{1}{(1+x^2)^{s/2}}$, and extract the $-d$ part: near $\infty$, $\frac{1}{(1+|x|^2)} = |x|^{-2}(1 - |x|^{-2} + |x|^{-4} - \dots)$, and so $\sigma_{-4} = -|x|^{-4}$.

Finally, here is a good theorem. Let us consider $\mathrm{CS}^{\notin\mathbb{Z}}$ to be those symbols whose order is not an integer. When I say there is an asymptotic expansion, then what I'm saying is that at infinity,


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$\sigma \sim \sum_{j=0}^{\infty} \sigma_{a-j}$, where $\sigma_{a-j}(tx) = t^{a-j}\sigma(x)$ for $t > 0$, and in this case $a$ is called the *order* of $\sigma$. So $\mathrm{CS}^{\notin \mathbb{Z}}$ is spanned by things like this. Then:

Theorem: There exists a unique linear extension from $\mathbb{R}$ to $\mathrm{CS}^{\notin \mathbb{Z}}$ satisfying 1,2,3, and determined by any one of them. It is completely canonical, so use any method physicists use, e.g. by cutoffs and taking a finite part.

Tim: Is this like a dimensional regularization? Sylvie: All of this has to do with two things. Uniqueness of meromorphic extensions, and uniqueness of extensions of polyhomogeneous symbols a la Hormander.

So it seems that there is no situation where we have both the Wodzicki residue and this canonical integration, and moreover we have no actual examples of non-integer symbols. Dirac operator, Green's functions, etc. all have integer symbols.

So what do you do? Replace your symbol $\sigma$ by some symbol $\sigma(z)$ in which you perturb the integrand to have noninteger orders, e.g. by $\sigma(z) = H(z)\sigma|x|^z$ for $H$ some holomorphic function $H(0) = 1$. This is a precise version of "dimensional regularization".

This changes the problem to the following: How to make sense of meromorphic functions at poles? The old theorem is that $z \mapsto \int \sigma(z)$ is a meromorphic function with poles in $\mathbb{Z}$.

Jonathan: Why not use the Wodzicki integral? Sylvie: You can do all of QFT with residues. You could use $\det_{\mathrm{res}}(\Delta)$, or whatever. But if $\prod \lambda_n$ converges, then the residue determinant is trivial. I've never met a physicist who likes this idea, but I've never tried.

Oh, I should remark: There is a parallel theory for sums.

Finally, we had one integral and one sum, but for a more interesting example, consider

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{(k_1^2+1)(k_2^2+1)((k_1+k_2)^2+1)} \mathrm{d}k_1 \mathrm{d}k_2 = \int (\sigma \otimes \sigma \otimes \sigma)(A(k_1,k_2))$$

where $A(k_1, k_2) = (k_1, k_2, k_1 + k_2)$. Another example: the multiple zeta values.

More generally, let's write $\vec{\sigma}$ for a tensor product of $\sigma$s, and $A(\vec{x})$ for some matrix $A$, and we want:

$$\int_{(\mathbb{R}^d)^L} \vec{\sigma} \circ A(\vec{x}) \, \mathrm{d}\vec{x}$$

You should think of $L$ as the number of loops.

Unfortunately, tensor products of symbols are not symbols, but that's not much of a problem. What's more an issue is the $A$. If $A = \mathrm{id}$, then we can decide that the finite part of a product is a product of finite parts — it is a decision, of course.

Finally, the questions. There is literature about sums on cones and a generalized Euler–Maclauren formula relating this to integrals on cones; and we can try to integrate tensor products of symbols — this is combinatorics on cones. For abstract Feynman integrals, one way is to work with integrals of tensors of symbols over intersections of hyperplanes. This leads to the questions I began with:


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1. What is the higher-dimensional generalization of the canonical sum/integral? Jonathan: Or of the Wodzicki residue?

2. We can, at least in examples, with Guo, Zhang, and Manchon, put in a $z$, and mathematically it makes sense to do this with different $z$s. There are many ways to decide about the finite part, and so we ask: What is a good "finite part" of a meromorphic function in multiple variables? It should be a valuation, but that's easy to lose in multiple variables. Multiplicativity for disjoint graphs is also easy to lose.

   Anyway, I know at least five methods. What is the renormalization group that relates them?


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