Analytic, arithmetic, and geometric aspects of automorphic forms

organized by
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Workshop Summary

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Workshop objectives.
The main objective of the workshop was to bring together two communities of number theorists working in the area of automorphic forms, automorphic periods and associated L-functions[1]:

– People studying automorphic forms, periods and L-functions from an arithmetic/geometric viewpoint aiming at applications to the structure of Shimura varieties, Galois representation in connection to the Birch–Swinnerton-Dyer/Bloch-Kato conjectures and using tools such as Iwasawa theory.
– People studying automorphic forms, periods and L-functions from a more analytical perspective with problems and conjecture coming from analytic number theory such as equidistribution problems, the subconvexity problem or the study of moments of L-function and by using different sets of tools such as harmonic analysis on homogenous spaces, sieve methods and/or homogenous dynamic.

In the past, these two communities have found fruitful common interests on numerous occasions (for instance in the use of analytic non-vanishing results for central value of L-functions to start an Euler system argument); this workshop was aimed at updating one another on the most recent techniques from both sides with the hope to identify problems of interest to both communities and to foster new collaborations.

Morning lectures.

Following the traditional format of an AIM workshop, the mornings were constituted of two one hour lectures. Below is the list of lectures along with a rough description of their content

In choosing the speaker we tried, as must as possible, to keep a balance between arithmetic/analytic oriented lectures. The overall recommendation for the speakers was to deliver a not too technical lecture (survey type) that could be grasped, for the most part, by the whole audience and to leave the more technical aspect for discussions during free time or during the working group sessions.

Most lectures are accessible at

[1]Very much in the spirit of the research conducted by Dinakar Ramakrishnan through his whole career.

https://vimeo.com/showcase/aagaautomorphic

Monday Jan. 29.

– Chris Skinner (Princeton): *Arithmetic, and geometric aspects of automorphic forms.* This introductory lecture focused on the arithmetic aspects of automorphic periods and L-functions for low rank groups such as $_1$, $_2$ or its inner forms; it notably introduced p-adic L-functions and their interpolation property with respect to critical L-values and explained how these can be constructed via periods.
– Philippe Michel (EPFL): *Analytic aspects of automorphic forms.* This introductory lecture focused on the analytic aspects of automorphic periods and L-functions for low rank groups such that $_1$, $_2$ or its inner forms. It described basic methods to establish non-vanishing of L-functions, including equidistribution which can be achieved either by subconvexity or via homogenous dynamics.

Tuesday, Jan. 30.

– Mladen Dimitrov (Lille): *Constructing automorphic forms of low level.* The talk presented an ongoing joint work with Dinakar Ramakrishnan on uniform irreducibility of Galois action on the ℓ-primary part of Abelian 3-folds of Picard type, generalizing Manin's theorem on uniform bound for the order of cyclic ℓ-power isogeny between two non-CM elliptic curves over a number field. The key part of the proof is the contruction of certain low level residual forms for $U(3)$.
– Paul Nelson (Aarhus): *Nonvanishing questions in higher rank.* This talk discussed analytic aspect of automorphic L-functions and periods for Rankin-Selberg pairs $(_{n+1}, _n)$ as well as for Gan-Gross-Prasad pairs like (U_{n+1}, U_n). Paul Nelson discussed how averaging techniques over families in either the small or large group allow to exhibit non-vanishing periods and hence central values of L-functions in the archimedean aspect. The techniques were mainly based on his recent work on the spectral theory of automorphic periods in higher rank.

Wednesday, Jan. 31.

– Wei Zhang (MIT): *Arithmetic GGP and RTF.* This talk discussed the arithmetic/geometric relations between automorphic periods (viewed as geometric cycles on Shimura varieties) and values of L-functions and their derivatives for Gan-Gross-Prasad pairs in the case of cohomological representations (especially for unitary for signature $(n, 1)$, $(n-1, 1)$). Wei Zhang explained the basic principle underlying the Arithmetic Relative Trace formula approach to connect periods to first derivatives of L-functions.
– Ashay Burungale (Austin):*Horizontal non-vanishing of L-values modulo p.* This talk introduced the problem of establishing non-vanishing mod p for critical L-values in horizontal families, with an emphasis on results in the case of Dirichlet and Rankin–Selberg L-values. The methods rely on integral representation of L-values, pertinent Fourier analysis and distribution of integers with certain multiplicative structures in a given interval as well as equidistribution of special points on definite Shimura sets. As was explained, much room remains for improvement especially regarding the quantitative aspects.

Thursday Feb. 1st.

– Andrei Jorza (Notre Dame): *p-adic L-functions, p-adic families of automorphic forms and non-vanishing questions.* This talk expanded Chris Skinner lecture describing the theory of p-adic L-functions attached to algebraic cuspidal automorphic representations on $_{2n}$, for $n \geq 1$. Andrei Jorza explained various techniques to construct p-adic L-functions and described in greater details a geometric construction by Glenn Stevens based on Shalika's periods.

– Abhishek Saha (London): *Siegel cusp forms of degree 2 and special values of L-function.* This talk discusses some analytic aspects, especially the questions of non-vanishing and of size, of Fourier coefficients (aka Bessel periods in the adelic language) of holomorphic Siegel cuspforms of degree 2 (on $_4$) and their L-functions (which are related through the recent Formula of Furusawa-Morimoto).

Friday Feb. 2nd.

– Min-Lee (Bristol): *Non-vanishing of* $L(\mathrm{sym}_3 f, 1/2)$. In her lecture, Min Lee described her joint work with Jeff Hoffstein and Junehyuk Jung establishing the non-vanishing of central values of symmetric cube L-functions of a $_2$-modular form (over the field of Eisenstein numbers) when the later varies in suitable families using the Petersson trace formula.

– Vinayak Vatsal (UBC): *Congruences between special values.* This talk discussed the question of when two congruent holomorphic forms have their twisted central values (suitably normalized) congruent. Vinayak Vatsal notably explained why this natural looking problem is not as obvious as it might seem and then discussed three cases for which this question admit reasonable answers: the cases of Hecke, Symmetric square and Rankin-Selberg L-functions.

Afternoon activities.

Problem session.
The problem session took place on Monday January 29 in the afternoon and was scribed by Robin Zhang. One can find the edited list of problems/questions that have emerged from this session at

http://aimpl.org/aagaautomorphic/

Research project.
The problem session permitted to identify some problems of interest to many people in the audience. Via a voting process, some working groups were formed around some of the most popular ones. Progresses (or absence thereof) were discussed the following afternoon before a new vote took place (with some projects being abandoned and replaced by others).

At the end of the week the five remaining problem were:

(1) **Connections between values of L-functions and their derivatives** (M. Lee, Ph. Michel with help and advices from W. Zhang). The objective was to connect more closely the central values of Rankin-Selberg L-functions $L(f_K \times \chi, 1/2)$ occurring in Waldspurger's formula (when the sign of the functional equation is $+1$) and the central values of their derivative $L(g_K \times \chi, 1/2)$ (when the sign is -1) as they occur in the Gross-Zagier formula. Such a connection exists for instance in the form of the Jochnowitz congruences and was notably exploited by Bertolini-Darmon. The possibility of such a connection is also suggested by Jacquet's proof of Waldspurger's

formula via the relative trace formula. We have now a (rough) conjectural formula relating some averages of of twisted central L-values to averages of twisted central values of the derivatives, unfortunately there are convergence issues that remain to be resolved. The project is expected to be continued after the workshop.

(2) **Nonvanishing of Derivatives of Rankin-Selberg Central L-values for $U(n+1)\times U(n)$** (R. Rodrigez, M. Zanarella, T. Hammonds, L. Yang, R. Zhang, W. Zhang). Let $G = \mathrm{U}(n+1)$ and $H = \mathrm{U}(n)$. Given a unitary cuspidal automorphic representation π of G, the aim of the project was to find a unitary cuspidal automorphic representation σ of H such that the derivative $L'(1/2, \mathrm{BC}(\pi) \times \mathrm{BC}(\sigma)) \neq 0$. This has applications via the arithmetic Gan-Gross-Prasad conjecture to the nonvanishing of algebraic cycles (if π and σ are both cohomological). Using the Jacquet-Rallis relative trace formula, with a carefully chosen test function and arguments on the root number, the group established the existence of a pair (π, σ) satisfying $L'(1/2, \mathrm{BC}(\pi) \times \mathrm{BC}(\sigma)) \neq 0$. This needs to be refined to yield results for algebraic cycles. The challenge is to construct test functions to isolate cohomological representations with a computable geometric side.

(3) **Quantitative non-vanishing of Dirichlet L-values modulo p.** (A. Burungale, L. Beneish, S. Lai, G. Oh). The project was a follow-up of A. Burungale lecture and its aim was to improve the quantitative mod p non-vanishing results of Dirichlet L-values in horizontal families obtained by Burungale and Sun. The group studied closely two strategies presented in the paper, and reformulated one of them as a distribution problem for random matrices in characteristic p. The reformulation seems flexible, yet it remains to be seen if it can lead to any improvement.

(4) **Automorphic forms invariant under some special compact subgroups**. (M. Dimitrov, T. Hammonds, M. Roy, C.-Y. Hsu, P. Nelson, C. Sorensen) The objective of this project was to study the existence of vectors of an admissible representation of $G(\mathbb{Q}_p)$ which are invariant under compact subgroups of the shape $K_H(p^n) = H({}_p) + p^n G({}_p)$ where $H \subset G$ a suitable subgroup (usually (G, H) forming a reductive pair). More precisely, one would like to find a finite a sequence of congruent subgroups K_i with the following property: for any r there exists C such that any subgroup of index at least C of the standard maximal compact subgroup of $G(\mathbb{Q}_p)$, is contented (up to conjugation) in at least one amongst $K_1, \ldots, K_r$. This questions is motivated by studying Galois images attached to families of abelian varieties. Typically those having Galois image contained in an open compact subgroup K are parametrised by a Shimura variety S_K. Suppose we want to pin down those abelian varieties for which the index of the image is bigger than some C. The natural way would be to say that they yield points on one amongst $S_{K_1}, \ldots, S_{K_r}$.

In the case of GL(2) over ${}_p$ the tentative question is the following: given r find C such that any open subgroup of $\mathrm{GL}_2({}_p)$ of index at least C is contained (up to conjugation) either in the r-th Iwahori subgroup $K_B(p^r)$ or in $K_T(p^r)$, for a for a non-split torus T. The starting case is $r = 1$ was discussed in greater detail. The case of U(3) is relevant to a collaborative project with Dinakar Ramakrishnan.

(5) **Beilinson Conjectures for standard L-functions** (F. Castella, M. Dimitrov, K. T. Do, M. Emory, C.-Y. Hsu, J. Johnson-Leung, A. Jorza, C.-H. Kim, M. Roy, C. Skinner, C. Sorensen) The objective of this project was to elaborate on a framework for associating extensions of rational Hodge-structures to trivial zeros of twists of

standard L-functions for low-rank symplectic groups. The specific Hodge structures are found in relative cohomology of Shimura varieties of larger-rank symplectic groups and related to L-functions via integral representations. During the workshop some of the initial real representation theory was worked, and the group members expressed interest in continuing this project, especially over the summer.

Known developments that have occurred after the workshop.

The workshop has inspired a number of research project currently ongoing.

- Ph. Michel and D. Ramakrishnan are incorporating in their ongoing study of Waldspurger's periods and the central values of the related Rankin-Selberg L-functions some of the ideas displayed in the talk of Ashay Burungale on the non-vanishing of L-functions modulo p.
- V. Blomer, A. Burungale, J.-H. Min and Ph. Michel are collaborating on a project entitled *On the generation of joint Hecke fields by products of modular L-values* whose aim is to investigate links between the Hecke field generated by the Fourier coefficients of a pair $\{f, g\}$ of holomorphic cuspforms, and the field generated by product of the algebraic part of the associated L-values in the spirit of the work of Luo and Ramakrishnan. So far, complete results have been obtained in the case $f = g$, based on recent results due to Blomer-Michel on the distribution of modular symbols (the resolution of the unipotent mixing conjecture) mentioned during the workshop. One paper is currently being completed. In the near future, they hope to consider the case $f \neq g$.
- M. Dimitrov and D. Ramakrishnan are finalizing a paper for which a crucial ingredient was produced in the days immediately following the workshop.
- In the continuation of Project (2). In the case $n = 2$ L. Yang, instead of using Jacquet-Rallis trace formula, is now considering the possibility of using the Jacquet-Ye's Kuznetsov-type relative trace formulas together with a Dirichlet series expression (approximate functional equation) for the Rankin-Selberg L-functions in the spirit of the work of Templier and Templier-Tsimerman in the context of the Gross-Zagier formula.