# Additive combinatorics and its applications

organized by
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Workshop Summary

Goals of the workshop. The theory of additive combinatorics is aimed at developing approprite notions of "weak structure" which can be used to define approximate notions of various algebraic objects. These techniques have found applications in a wide variety of areas and has also been motivated by structures and applications arising in these areas including: number theory, ergodic theory, discrete geometry, algorithm design, coding theory and complexity theory. The goals of this workshop were:

- To gather experts from many of these areas to consider new approaches informed by multi-disciplinary perspectives for resolving some of the long-standing problems in additive combinatorics. In particular, the participants focussed on the Polynomial Freiman-Rusza conjecture which connects various notions of an approximate group structure. Several variants of conjecture, as well as approaches and potential counterexamples, each of these inspired by perspectives in different application areas, were explored.
- To consider new applications of the polynomial method introduced by Croot-Lev-Pach and Ellenberg-Gijswijt, which enabled the solution of a long-standing open problem on the sizes of 3-AP free sets in finite fields. The workshop included a discussion of potential applications and extensions to the method, motivated by applications in combinatorics and theoretical computer science.

Structure of the workshop. The morning on each of the first three days had two lectures each to cover many of the relevant ideas from different areas. On Monday, Shachar Lovett spoke on several variants of the the Polynomial Freiman-Ruzsa (PFR) conjecture and Tom Bloom gave an introduction to the polynomial method of Croot-Lev-Pach for obtaining (significantly) improved bounds on the density of subsets of $\mathbb{F}_p^n$ avoiding length-3 arithmetic progressions. On Tuesday, Olof Sisask spoke about the Bogolyubov-Ruzsa lemma and its variants (via almost-periodicity results). This was followed by Kaave Hosseinis talk outlining several applications to theoretical computer science, and related open problems. Two talks on Wednesday morning, by Freddie Manners and Nets Katz, discussed approaches for proving the PFR conjecture via energy-incrementation arguments.

The afternoons on each of the days, and part of morning on Thursday and Friday were devoted to problem solving sessions. During an open problem session on Monday afternoon (moderated by Yufei Zhao) several participants contributed problems related to several applications and variations of the polynomial method and the PFR conjecture, as well related areas in additive combinatorics. During the following afternoons, some of these directions were selected the organizers and the participants divided themselves into working groups on these problems. While both the list of problems and the composition of the groups varied daily, some of the questions received continual attention and saw interesting progress during the workshop. Some of the questions considered by the working groups included the following:

(1) **Finding examples/counterexamples of nontrivial additive growth**: This group considered the problem of finding potential counterexamples to the Polynomial Bogolyubov-Ruzsa conjecture. Another goal was to construct interesting examples of subsets with mild additive growth, such as dense subsets of $\mathbb{F}_2^n$ (which are far from subspaces) such that $A+A$ does not (essentially) cover all of $\mathbb{F}_2^n$. The group also considered the application of the polynomial method to analyze the structure of $A+A$ and $A+A+A$.

(2) **Sumset structure in** $A+A$: This group considered a strengthening of a result of Sanders used in the proof of the Quasipolynomial Bogolyubov Ruzsa theorem. While Sanders proved that a set $A$ of density $\delta$ must satisfy that $A+A$ contains $1-\epsilon$ fraction of a subspace with co-dimension $(1/\epsilon^2)\cdot(\log(1/\delta))^{O(1)}$, this group considered if the dependence on $\epsilon$ can also be made polylogarithmic. This would have immediate consequences for the communication complexity of XOR functions.

(3) **Applications of polynomial method**: The goal of this group was to novel applications of the polynomial method, for problems in additive combinatorics. One of the problems considered was to find a short proof of a theorem of Frankl and Rodl, which shows that a subset $S$ of $\mathbb{F}_2^n$, which contains no pairs $x, y \in S$ at Hamming distance exactly $(1-\gamma)n$, must have size at most $(2-\Omega(\gamma^2))^n$. The group discovered a very short polynomial method proof of a weaker result (due to Solymosi), which establishes a bound of the above form, when $S$ contains no pairs with distances *in a small interval* (of size $o(\gamma n)$) centered at $(1-\gamma)n$. This version of the result is useful for several applications in theoretical computer science.

(4) **Modeling**: If $|A+A| \leq K|A|$, must $A$ contain a $poly(K)$-dense subset that has a good Freiman model? That is, must there exist $A' \subset A$ with $|A'| \geq CK^{-C}|A|$ and a group $H$ with $|H| \leq CK^C|A'|$ such that $A'$ is Freiman-isomorphic to a subset of $H$? This would follow from the Polynomial Freiman-Ruzsa conjecture, but is a priori weaker. The latest on this problem, during the workshop, came from Freddie Manners and Fernando Shao, who thought it seemed likely that the problem is equivalent to PFR: even the case of solving this problem for the groups $(\mathbb{Z}/4\mathbb{Z})^n$ (or $\mathbb{F}_4^n$) might imply PFR for $\mathbb{F}_2^n$.

(5) **Higher additive energies and additive non-smoothing**: For a set $A$ in a finite Abelian group, the energy $E_{2k}(A)$ is defined as the number of solutions in $A$ to the equation $a_1+\cdots+a_k = a_{k+1}+\cdots+a_{2k}$. The goal of the group was to discover structural consequences when (for some $k$) $E_{2k+2}$ is not significantly greater then $E_{2k}$. A paper of Bateman and Katz gives consequences of additive non-smoothing between $E_4$ and $E_8$ and structural consequences for higher $k$ would have direct applications to the PFR conjecture.

(6) **Tensor rank**: This group considered the problem of finding alternative notions of rank for tensors over finite fields, which behave well under tensoring. Such a notion would have applications to obtaining upper bounds on the size of 4AP-free sets using the polynomial method. The group considered several candidate notions, of both algebraic and analytic nature.

(7) **Rudin's problem in $\mathbb{F}_p^n$**: This group considered an analog of Rudin's problem in $\mathbb{F}_p^n$. Suppose $A \subset \mathbb{F}_p[x]$ is finite and that each element of $A$ is a square. The goal is to show that $|A+A| \geq |A|^{1+\varepsilon}$ for some $\varepsilon > 0$. This is related to the problem of finding a version of the Croot-Lev-Pach polynomial method that works with small doubling hypotheses rather than density hypotheses. The group considered a combination of tools in both analytic number theory and additive combinatorics.

(8) **van der Waerden numbers**: The number $W(3,\ell)$ is defined to be the smallest number $n$ such that any red/blue coloring of the set $\{1,\ldots,n\}$ contains a red 3-AP or a blue AP of length $\ell$. The group started working on a conjectured bound of $\Theta(\ell^{2+o(1)})$ for $W(3,\ell)$. While they soon proved a lower bound of $\Omega(\ell^2)$, they also came up with a candidate approach to strengthen the lower bound to quasi-polynomial in $\ell$.

Conclusion. The workshop was successful in bringing together mathematicians working in a diverse set of areas, leading to an interesting collection of problems at the interface of these areas, as well as partial progress on some of them. The workshop also helped establish new collaborations between the participants. The organizers are grateful to the AIM staff for their care and help with organization, as well as suggesting a workshop structure leading to many informal work sessions and discussions.